MATH348. Harmonic Analysis. Problems 2

Work due on *Wednesday 13th October.*

1. Let $f : (-\pi, \pi] \to \mathbf{R}$ be defined by

$$f(x) = \begin{matrix} 1 \text{ if } 0 \le x \le \pi \\ 0 \text{ if } -\pi < x < 0 \end{matrix}$$

Extend to a 2π-periodic functions, and call this f also. a) Give the values of

$$\frac{f(0+) + f(0-)}{2}, \ \frac{f(\pi+) + f(\pi-)}{2}, \ \frac{f((\pi/2)+) + f(\pi/2)-)}{2}.$$

b) Compute the Fourier coefficients $\hat{f}(n)$. Use the pointwise Fourier Series Theorem at $\pi/2$ to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

c) Use Parseval's equality applied to f to show that

$$\sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

2. Determine which of the following functions are integrable.

a) $f(x) = x^{-3/4}$ on $(0, 2\pi)$.

b) $f(x) = x^{-4/3}$ on $(1, \infty)$.

c) $f(x) = x^{-3/4}$ on $(0, \infty)$.

d) $f(x) = x^{-4/3}$ on $(0, \infty)$.

e) $f(x) = (\sin^3 x)x^{-3}$ on $(0, \infty)$.

3. Determine which of the functions in question 2 are in (i) L^1, (ii) L^2, (iii) L^∞.

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1a). We have $f(0+) = 1 = f(\pi-)$, $f(0-) = 0 = f(-\pi-) = f(\pi-)$. So

$$\frac{f(0+)+f(0-)}{2} = \frac{1}{2} = \frac{f(\pi+)+f(\pi-)}{2}.$$

Also $1 = f(\pi/2) = f((\pi/2)+) = f((\pi/2)-)$. So

$$\frac{f((\pi/2)+)+f((\pi/2)-)}{2} = 1.$$

b) We have

$$\hat{f}(0) = \int_0^\pi dx = \pi.$$

If $n \neq 0$,

$$\hat{f}(n) = \int_0^\pi e^{-inx}dx = \frac{(-1)^n - 1}{-in}.$$

So $\hat{f}(n) = 0$ if n is even, $n \neq 0$ and $2/in$ if n is odd. So if $n = 2p+1$ or $2p+2$

$$S_n(f)(x) = \frac{1}{2} + \frac{1}{2\pi}\sum_{m=1}^{n}(\hat{f}(m)e^{imx} + \hat{f}(-m)e^{-imx})$$

$$= \frac{1}{2} + \frac{1}{2\pi}\sum_{k=0}^{p}\left(\frac{2e^{i(2k+1)x}}{i(2k+1)} - \frac{2e^{-(2k+1)ix}}{-i(2k+1)}\right)$$

$$= \frac{1}{2} + \frac{1}{2\pi}\sum_{k=0}^{p}\frac{4\sin(2k+1)x}{2k+1}.$$

So then by the Fourier Series Theorem at $x = \pi/2)$, using the value of $f(x+) + f(x-))/2$ calculated in a),

$$1 = \frac{1}{2} + \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{4\sin(2k+1)(\pi/2)}{2k+1}.$$

Now $\sin(2k+1)(\pi/2) = (-1)^k$. So

$$\frac{\pi}{4} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

as required.

c) Also

$$< f, f >= \int_0^\pi dx = \pi.$$

So by Parseval's equality, since $|\hat{f}(n)| = |\hat{f}(-n)|$,

$$\pi = \frac{\pi^2}{2\pi} + \frac{2}{2\pi}\sum_{k=0}^{\infty}\frac{4}{(2k+1)^2},$$

which gives

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

2a) The function $f(x) = x^{-3/4}$ is continuous on $(0, 2\pi]$ and hence certainly integrable on $[1/n, 2\pi]$ for any integer $n > 0$. So put $f_n = \chi_{[1/n,2\pi]}x^{-3/4}$. Then $0 \le f_n \le f_{n+1} \le f$ By the Fundamental Theorem of Calculus,

$$\int_{1/n}^{2\pi} x^{-3/4}dx = \left[4x^{1/4}\right]_{1/n}^{2\pi} = 4((2\pi)^{1/4} - n^{-1/4}) \to 4(2\pi)^{1/4} \text{ as } n \to \infty.$$

So by the Monotone Convergence Theorem

$$\int_0^{2\pi} x^{-3/4}dx = \lim_{n\to\infty} \int f_n = 2(2\pi)^{1/4}.$$

So f is integrable.

b) Again by the Fundamental Theorem of Calculus, for all $N > 1$,

$$\int_1^N x^{-4/3}dx = \left[-3x^{-1/3}\right]_1^N = 3 - 3N^{-1/3} < 3.$$

So, again, f is integrable. In fact

$$\int_1^\infty x^{-4/3}dx = \lim_{N\to\infty} 3 - 3N^{-1/3} = 3.$$

c) For all $N > 0$,

$$\int_{1/N}^N x^{-3/4} = \left[4x^{1/4}\right]_{1/N}^N = 4(N^{1/4} - N^{-1/4}) \to \infty \text{as} N \to \infty.$$

So f is not integrable.

d) For all $N > 0$,

$$\int_{1/N}^N x^{-4/3}dx = \left[-3x^{-1/3}\right]_{1/N}^N = -3N^{-1/3} + 3N^{1/3} \to \infty \text{as} N \to \infty$$

So again, f is not integrable.

e) The easiest way to do this one is to use the fact that if $f(x)$ is a Lebesgue measurable function (for example, a continuous function) and $|f(x)| \le |g(x)|$ for all x, and g is integrable, then f is integrable also.

In this case we can apply this with

$$g(x) = \chi_{[0,1]}(x) + \chi_{(1,\infty)}(x)\frac{1}{x^3},$$

as follows.

For all x, $|\sin(x)| \le 1$ and $\le |x|$. So $|\sin^3(x)x^{-3}| \le 1$, and $\le x^{-3}$. So

$$\int_0^\infty \frac{|\sin^3(x)|dx}{x^3} \le \int_0^1 1dx + \int_1^\infty x^{-3}dx = \int_0^\infty g(x)dx = 1 + \lim_{N\to\infty}\left[-2x^{-2}\right]_1^N$$
$$= 1 + \lim_{N\to\infty}(2 - 2N^{-2}) = 3.$$

So f is integrable.

3a) This function is positive, has already been shown to be integrable, and is in L^1. $f^2 = x^{-3/2}$ on $(0, 2\pi)$ and

$$\int_{1/N}^{2\pi} x^{-3/2}dx = \left[-2x^{-1/2}\right]_{1/N}^{2\pi} = N^{1/2} - (2\pi)^{-1/2} \to \infty \text{as} N \to \infty.$$

So $f \notin L^2$. Also, f is not bounded on $(0, 2\pi)$ and not in L^∞ because $f(x) \to \infty$ as $x \to 0$.

b) Again, this function is positive and integrable, and so is in L^1. We have $f(x) \le 1$ for all x, so $f \in L^\infty$. Also $(f(x))^2 \le f(x)$. So f^2 is integrable and $f \in L^2$. Alternatively one can show directly that $x^{-8/3}$ is integrable on $(1, \infty)$, using the Fundamental Theorem of Calculus and Monotone Convergence as in 2b)

c) f is positive and not integrable so $f \notin L^1$. We have seen in a) that f^2 is not integrable even on $(0, 2\pi)$, so f^2 is not integrable on $(0, \infty)$ and $f \notin L^2$. Similarly, from a) $f \notin L^\infty$.

d) Again, f is positive and not integrable by question 1d), so $f \notin L^1$. Also $f(x) \to \infty$ as $x \to 0$ so $f \notin L^\infty$. We have $(f(x))^2 = x^{-8/3}$, and

$$\int_{1/N}^{N} x^{-8/3} dx = \left[(-3/5)x^{-5/3}\right]_{1/N}^{N} = (-3/5)(N^{-5/3} - N^{5/3}) \to \infty \mathrm{as} N \to \infty.$$

So $f \notin L^2$. In fact, it would be enough to look at the integral over $(1/N, 1)$, which would still $\to \infty$ as $N \to \infty$.

e) We have $f \in L^1$ since f is integrable (which, by definition, is the same as $|f|$ being integrable). Also since $|f(x)| \le 1$ (as was show in 1e)), we have $|f(x)|^2 \le |f(x)| \le 1$ for all x. and hence $|f(x)|^2$ is integrable and $f \in L^2$. Since f is bounded, in fact ≤ 1, $f \in L^\infty$ also.