MATH348. Harmonic Analysis. Problems 1.

Lecture times for this module are : Mondays at 10, Wednesdays at 10, Fridays at 2. Problem sheets will be set once a week. The hand-in day will be Wednesday -preferably hand in at the lecture on Wednesday, but some lee-way will be given. At any event, work will be handed back in the lecture on Friday, and late work will not be formally marked - although I am happy to look over work which is done late for reasons such as illness. There is no problem class but I run two office hours a week. Looking at the timetabled times of other honours modules, and taking the hand-in day into consideration, I suggest Mondays at 11 and Tuesdays at 4 in my office (506 in Week 1, 515 after that). In some weeks it may be necessary to change these times. In Week 1 I shall need to change the 10 a.m. Wednesday lecture time, just for that week. The changed time will be decided in the monday lecture, and an announcement put on the VITAL page for this module. I am happy to arrange other times on an individual basis.

All available materials for this module can be accessed either through VITAL or from my homepage www.liv.ac.uk/Maths/People/

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This assignment is due on *Wednesday 6th October.*

1. For any piecewise continuous f on $[-\pi,\pi]$ and $n\in\mathbf{N}$, let

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\cos(ny)dy\,,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\sin(ny)dy\,,\ \widehat{f}(n)=\int_{-\pi}^{\pi}f(y)e^{-iny}dy.$$

Show that

$$\frac{1}{\pi}(\widehat{f}(0)=a_0,$$

$$\frac{1}{2\pi}(\widehat{f}(n)e^{inx}+\widehat{f}(-n)e^{-inx})=a_n\cos(nx)+b_n\sin(nx)\ \ (n>0).$$

2. Find $\widehat{f}(n)$ for all $n\in\mathbf{Z}$, and the Fourier series of f, where
a) $f:[-\pi,\pi]\to\mathbf{R}$ is given by $f(x)=x^2$ for all $x\in[-\pi,\pi]$,
b) $f:[0,2\pi)\to\mathbf{R}$ is given by $f(x)=xe^x$ for all $x\in[0,2\pi)$.

3. Find the limits of the following sequences of functions, where they exist, and determine whether convergence is pointwise or uniform on the given set.
a) $f_n:[0,\frac{1}{2}]\to\mathbf{R}$ given by $f_n(x)=x^n$,
b) $g_n:[0,\frac{1}{2}]\to\mathbf{R}$ given by $g_n(x)=e^{-nx}$,
c) $s_n:(0,\pi]\to\mathbf{R}$ given by

$$s_n(x)=\frac{\sin(n+\frac{1}{2})x}{2\pi\sin\frac{1}{2}x},$$

d) $\sigma_n:[-\pi,\pi]\to\mathbf{R}$ given by $\sigma_n(0)=0$ and for $x\neq 0$,

$$\sigma_n(x)=\frac{\sin^2(\frac{n+1}{2}x)}{2\pi(n+1)\sin^2\frac{1}{2}x}.$$

Hint for d): Try $x_n=\pi/(n+1)$. You may use $|\sin y|\leq|y|$ for all $y\in\mathbf{R}$.

MATH348. Harmonic Analysis. Solutions 1.

1. $\cos(0x) = e^{i.0.x} = 1$ for all x, so $\frac{1}{pi}\widehat{f}(0) = a_0$. Since $e^{inx} = \cos nx + i\sin nx$ we have $\frac{1}{2\pi}\widehat{f}(n) = \frac{1}{2}(a_n - ib_n)$.
$\cos(-nx) = \cos(nx)$, $\sin(-nx) = -\sin(nx)$, so $\frac{1}{2\pi}\widehat{f}(-n) = \frac{1}{2}(a_n + ib_n)$. So, for $n > 0$,

$$\frac{1}{2\pi}(\widehat{f}(n)e^{inx} + \widehat{f}(-n)e^{-inx}) = \tfrac{1}{2}(a_n - ib_n)(\cos(nx) + i\sin(nx)) +$$
$$\tfrac{1}{2}(a_n + ib_n)(\cos(nx) - i\sin(nx)) = a_n\cos(nx) + b_n\sin(nx).$$

2.

a)
$$\widehat{f}(0) = \int_{-\pi}^{\pi} x^2 dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}.$$

For $n \neq 0$,

$$\widehat{f}(n) = \int_{-\pi}^{\pi} x^2 e^{-inx} dx = \left[\frac{x^2 e^{-inx}}{-in}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2xe^{-inx}}{in} dx$$
$$= 0 + \left[\frac{2xe^{-inx}}{n^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2e^{-inx}}{n^2} dx$$
$$= \frac{4\pi(-1)^n}{n^2} - \left[\frac{2e^{-inx}}{-in^3}\right]_{-\pi}^{\pi} = \frac{4\pi(-1)^n}{n^2}.$$

So the Fourier series of f is

$$\frac{\pi^2}{3} + \sum_{n\neq 0} \frac{2(-1)^n e^{inx}}{n^2}$$
$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos nx}{n^2}.$$

b)
$$\widehat{f}(n) = \int_0^{2\pi} xe^{x-inx} dx = \left[\frac{xe^{x(1-in)}}{(1-in)}\right]_0^{2\pi} - \frac{1}{(1-in)}\int_0^{2\pi} e^{x(1-in)} dx$$
$$= \frac{2\pi e^{2\pi}}{(1-in)} - \left[\frac{1}{(1-in)^2} e^{x(1-in)}\right]_0^{2\pi}$$
$$= \frac{2\pi e^{2\pi}}{(1-in)} - \frac{(e^{2\pi}-1)}{(1-in)^2}.$$

So the Fourier seris of f is

$$\sum_{n=-\infty}^{\infty} \frac{e^{2\pi} e^{inx}}{(1-in)} - \sum_{n=-\infty}^{\infty} \frac{(e^{2\pi}-1)e^{inx}}{2\pi(1-in)^2}$$

This can be written in terms of cos and sin as

$$e^{2\pi}(1-(2\pi)^{-1})+(2\pi)^{-1})+\sum_{n=1}^{\infty} 2e^{2\pi}\frac{\cos nx - n\sin nx}{1+n^2}$$
$$-\sum_{n=1}^{\infty}\frac{2(e^{2\pi-1}((1-n^2)\cos nx+2n\sin nx)}{\pi(1+n^2)^2}.$$

3a) $|f_n(x)| \leq 2^{-n} \to 0$ as $n \to 0$. So $f_n(x) \to 0$ as $n \to \infty$. The convergence is uniform. For let $\varepsilon > 0$ be given. Choose N so that $1/2^N < \varepsilon$. Then for all $n \geq N$ and all $x \in [0, \frac{1}{2}]$,

$$|f_n(x) - 0| = x^n \leq (1/2)^n \leq (1/2)^N < \epsilon.$$

b) $g_n(x) = e^{-nx} \to 0$ for $x \neq 0$, but for $x = 0$, $g_n(0) = 1$ for all n, so $g_n(0) \to 1$ as $n \to \infty$. So $g_n(x) \to g(x)$ pointwise as $n \to \infty$ where $g(x) = 0$ for $0 < x \leq \frac{1}{2}$ and $g(0) = 1$. The convergence is not uniform because (for example) $g_n(1/n) = e^{-1}$. So given $\varepsilon = \frac{1}{2}e^{-1}$, for any n, there is $x = 1/n \in [0, \frac{1}{2}]$ with $|g_n(x) - g(x)| = e^{-1} > \varepsilon$.
c) For most x, $\lim_{n\to\infty} s_n(x)$ does not exist. For example if $x = \pi$, $\sin(n + \frac{1}{2})\pi = (-1)^n$, so $s_n(\pi) = (-1)^n/2\pi$. Similarly if $x = \pi/2$, $s_n(\pi/2)$ alternates between two different values, and usually the situation is much more complicated than this.
d) For any fixed $x \neq 0$,

$$|\sigma_n(x)| \leq \frac{1}{2\pi(n+1)\sin^2\frac{1}{2}x} \to 0 \text{ as } n \to \infty.$$

So $\sigma_n(x) \to 0$ pointwise as $n \to \infty$. But if we take $x_n = \pi/(n+1)$, $\sin^2 x_n \leq x_n$, so

$$(n+1)^{-1}\sin^{-2}(x_n) \geq (n+1)^{-1}x_n^{-2} = (n+1)\pi^{-2}.$$

Moreover, $\sin((n+1)x_n/2) = 1$. So $\sigma_n(x_n) \geq \frac{1}{2}(n+1)\pi^{-3} \to \infty$ as $n \to \infty$. Take $\varepsilon = 1$ (for example). Then for any $n \geq 4$ there is an $x \in [-\pi, \pi]$ (in fact $x = x_n$) with $|\sigma_n(x) - 0| = (n+1)\pi^2 > \varepsilon$. So the convergence to 0 is not uniform.