MATH348. Theory of Integration.

It takes time to define integration properly and derive properties of the integral. In these notes, important basic properties of the integral will be collected together, mostly without proofs.

There are two main theories of integration: *Riemann integration* and *Lebesgue integration.* Lebesgue integration theory is larger than Riemann integration theory, in that all functions that are Riemann integrable are also Lebesgue integrable, and for such functions the two definitions of

$$\int f$$

given by the different theories coincide. Many more functions are Lebesgue integrable than are Riemann integrable.

Both theories basically start with the integration of *step functions* on $\mathbf{R}$. The function g on $\mathbf{R}$ is a *step function* if for some integer n and $a_0 < a_1 \cdots a_n$,

$$g(x) = c_i \text{ if } x \in (a_{i-1}, a_i),$$

$$g(x) = 0 \text{ if } x \notin [a_{i-1}, a_i] \text{ for any } 1 \leq i \leq n.$$

The definition of $g(a_i)$ can be anything. Then we define

$$\int g = \sum_{i=1}^{n} c_i(a_i - a_{i-1}).$$

If $c_i \geq 0$ for all i, then this is intuitively the area bounded by the x-axis and the graph of g. Then integrals of other functions are computed by taking sups and infs.

Sup and Inf

Given any set A of real numbers, the quantity $\sup A \in \mathbf{R}$ exists if and only if A is bounded above, that is, there is at least one number M such that $x \leq M$ for all $x \in A$. $\sup A$ is a number such that

$$x \leq \sup A \text{ for all } x \in A, \text{ and :}$$

$$\text{if } x \leq M \text{ for all } x \in A \text{ then } \sup A \leq M.$$

These two conditions determine $\sup A$ uniquely. The most important property of the real numbers is that $\sup A$ does exist if A is bounded above.

Similarly, $\inf A$ exists if and only if A is bounded below, and is uniquely determined by the properties that

$$\inf A \leq x \text{ for all } x \in A, \text{ and :}$$

$$\text{if } M \leq x \text{ for all } x \in A, \text{ then } M \leq \inf(A).$$

Here are some examples.

$$\sup[0,1] = 1 = \sup[0,1) = \sup(0,1) = \sup(0,1],$$

$$\inf[0.1] = 0 = \inf[0,1) = \inf(0,1) = \inf(0,1],$$

$$\text{If } A = \{1/n : n \in \mathbf{Z},\ n > 0\} \text{ then } \sup A = 1,$$

$$\text{If } A = \{1/n : n \in \mathbf{Z},\ n > 0\}, \text{ then } \inf A = \lim_{n \to \infty} 1/n = 0.$$

To see the last one, note that $0 < 1/n$ for all $n \in \mathbf{Z}$ with $n > 0$. So $0 \leq \inf A$. But if $M > 0$ then there is some $n \in \mathbf{Z}$, $n > 0$ with $1/n < M$. So we must have $\inf A \leq 0$ and $\inf A = 0$.

If a set A is not bounded above we sometimes write

$$\sup A = +\infty.$$

This may seem a strange thing to do when $\sup A$ does not exist. It is similar to the way we write

$$\lim_{x \to 0}(-\log x) = +\infty,$$

although strictly speaking this limit does not exist.

Definition of the Riemann Integral Let I be any interval in $\mathbf{R}$ (finite or infinite). A function $f : I \to \mathbf{R}$ is *Riemann integrable on I* if

$$\sup\left\{\int g : g \leq f,\ g \text{ is step, } g = 0 \text{ off } I\right\}$$

$$= \inf\left\{\int h : f \leq h,\ h \text{ is step, } h = 0 \text{ off } I\right\}.$$

We then define

$$\int_I f = \sup\left\{\int g : g \leq f,\ g \text{ is step, } g = 0 \text{ off } I\right\}$$

$$= \inf\left\{\int h : f \leq h,\ h \text{ is step, } h = 0 \text{ off } I\right\}.$$

If $I = [a, b]$ or $[a, b)$ or $(a, b]$ or (a, b) (including $a = -\infty$ in the last two cases and $b = \infty$ in the second and fourth case) then we also write

$$\int_I f = \int_a^b f = \int_a^b f(x)dx = \int_I f(x)dx.$$

We can replace x by any other variable. If $I = \mathbf{R}$ then we also write

$$\int_{\mathbf{R}} f = \int_{-\infty}^{\infty} f = \int f.$$

Again, we may write $f(x)dx$ instead of f, or use any other variable. If f is a step function, then this definition of the integral agrees with the previous one. Lots of functions are Riemann integrable. For example, if f is a continuous function on $[a, b]$ then it is Riemann integrable on $[a, b]$. For any continuous function f on $[a, b]$ there are many sequence of step functions $\{f_n\}$ such that $f_n = 0$ off $[a, b]$ and $f_n \to f$ *uniformly* on $[a, b]$. For any such sequence

$$\int_a^b f = \lim_{n \to \infty} \int f_n.$$

One does not usually use this definition to compute the integral - we shall come back to this later. A continuous function on $[a, b)$ or $(a, b]$ or (a, b) may not be integrable. This includes the possibility $a = -\infty$ in the last two cases, and $b = \infty$ in the first and third cases. It is often possible to determine whether or not the function is integrable, and to compute the integral when it does exist - we shall come back to this later. If $f : I \to \mathbf{C}$ then we say that f is Riemann integrable on I if the two functions $\mathrm{Re}(f) : I \to \mathbf{R}$ and $\mathrm{Im}(f) : I \to \mathbf{R}$ are Riemann integrable. We then define

$$\int_I f = \int_I \mathrm{Re}(f) + i \int_I \mathrm{Im}(f).$$

Lebesgue-measurable sets and functions

First we define a function χ_A for any $A \subset \mathbf{R}$ by

$$\chi_A(x) = 1 \text{ for } x \in A,$$

$$\chi_A(x) = 0 \text{ otherwise}.$$

There is a certain class of sets, called *Lebesgue measurable* sets. I am not going to give the definition but there are a lot of such sets. They include all intervals, and all countable unions and intersections of intervals. For such sets, it is possible to define a quantity

$$\lambda(A) = \int_A \in [0, +\infty].$$

If A is an interval with endpoints a and b then, coinciding with the previous definition for finite intervals

$$\int_A = b - a,$$

where for any finite a, or $a = -\infty$,

$$+\infty - a = +\infty$$

and for any finite b, or $b = +\infty$,

$$b - (-\infty) = +\infty.$$

Simple Functions

A *simple function* is then any function of the form

$$g = \sum_{i=1}^{n} c_i \chi_{A_i}.$$

where the sets A_i are all Lebesgue measurable and $c_i \in \mathbf{C}$. If the c_i are all positive real then g is a *positive* simple function and we define

$$\int g = \sum_{i=1}^{n} c_i \lambda(A_i) = \sum_{i=1}^{n} c_i \int_{A_i}.$$

In particular

$$\int \chi_A = \lambda(A) = \int_A.$$

The value of the integral may be $+\infty$. For any simple function g and any set A we also define

$$\int_A g = \int \chi_A g.$$

If $g = 0$ off A then this is just the same as $\int g$. If A is an interval with endpoints a and b (which van be $\pm\infty$) then we also write

$$\int_A g = \int_a^b g.$$

Lebesgue-measurable functions

A real-valued function f is *Lebesgue measurable* if, for all $a \in \mathbf{R}$, $\{x : f(x) < a\}$ is Lebesgue measruable. A complex valued function is defined to be Lebesgue measuable if its real and imaginary parts are. Any simple function is Lebesgue measurable. So is any continuous function. So is any Riemann integrable function.

The integral of positive Lebesgue measurable functions: positive Lebesgue-integrable functions

For a positive Lebesgue-measurable function f on $\mathbf{R}$, we define

$$\int f = \sup\left\{\int g : g \text{ is simple}, 0 \le g \le f\right\}$$

Here, the sup is taken to be $+\infty$ if any element of the set of $\int g$ is $+\infty$, or if the set is not bounded above. In fact, if we do not mind $\int f$ taking the value $+\infty$, then we can define $\int f$ for any Lebesgue measurable function $f : \mathbf{R} \to [0, +\infty]$. (We can do this using simple functions that are finite- valued, so that we are always taking a sup over numbers in $[0,\infty)$ - but the sup such a set might be $+\infty$.) We say f is *(Lebesgue) integrable* if

$$\int f < +\infty.$$

The integral of Lebesgue-integrable real- and complex-valued functions

Now let $f : \mathbf{R} \to \mathbf{R}$ be Lebesue measurable. Then we can write f as the difference of two integrable positive functions, $f = f_1 - f_2$. For example, we can take f_1 to be f where $f \geq 0$ and 0 otherwise, and can take f_2 to be $-f$ where $f \leq 0$ and 0 otherwise. Then we define

$$\int f = \int f_1 - \int f_2.$$

This is well-defined: we get the same answer whatever the choice of positive integrable f_1, f_2 with $f = f_1 - f_2$. We can also do this if just one of f_1, f_2 is integrable, if we allow $\int f$ to take values $\pm\infty$. But we cannot do it if both f_1 f_2 are not integrable, because it is not clear what the value of $+\infty - (+\infty)$ should be. We say that f is *integrable* if $|f|$ is integrable, that is,

$$\int |f| < +\infty.$$

In this case, it is certainly possible to write $f = f_1 - f_2$ for positive integrable f_1 and f_2, for example with the choices above. If $f : f \to \mathbf{C}$ and $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are Lebesgue measurable and integrable then we say that f is *integrable*, and we define

$$\int f = \int \mathrm{Re}(f) + i \int \mathrm{Im}(f).$$

We define

$$\int_A f = \int \chi_A f.$$

If f is positive real-valued, then

$$\int_A f = \sup \left\{ \int_A g : g \text{ is simple, } 0 \leq g \leq f \text{ on } A \right\}.$$

Again, if A is an interval with endpoints a and b, we write

$$\int_A f = \int_a^b f.$$

We shall usually drop the word Lebesgue because any Riemann integrable function is also Lebesgue integrable, and the two definitions of the integral coincide.

Basic Properties of the (Riemann or Lebesgue)Integral

The following basic properties hold for both Riemann and Lebesgue integral.

Positivity

If $f : \mathbf{R} \to [0,\infty)$ is (Riemann or Lebesgue) integrable or Lebesgue measurable, then

$$\int f \geq 0.$$

This is actually clear from the definition of the integral.

Linearity

For any integrable f, g, and any λ, $\mu \in \mathbf{C}$,

$$\int (\lambda f + \mu g) = \lambda \int f + \mu \int g.$$

Modulus Bounds For any $f : \mathbf{R} \to \mathbf{C}$, f is integrable if and only if f is Lebesgue-measurable and $|f|$ is integrable, and

$$\left|\int f\right| \leq \int |f|.$$

Also, if f is integrable and g is Lebesgue measurable and $|g| \leq |f|$, then g is integrable and

$$\left|\int g\right| \leq \int |f|.$$

This is actually not obvious obvious for complex-valued functions. It is very useful for estimating integrals. For example take a function like $\sin(x^2)$. We cannot compute

$$\int_a^b \sin(x^2)dx$$

exactly, but we can estimate it, because, for example

$$\left|\int_0^1 \sin(x^2)dx\right| \leq \int_0^1 |\sin(x^2)|dx \leq \int_0^1 1dx = 1.$$

Similarly consider the function $x^{-1}\sin x$ on $(0,1]$. This function is well-defined, and bounded, because $|\sin x| \leq |x|$ for all x. So

$$\left|\int_0^1 \frac{\sin x}{x}dx\right| \leq \int_0^1 1dx = 1.$$

Fundamental Theorem of Calculus

This is very important for computing integrals. Let $f : [a,b] \to \mathbf{R}$ (or $[a,b] \to \mathbf{C}$) integrable (for eaxmple, continuous), and let $F : [a,b] \to \mathbf{R}$ (or $[a,b] \to \mathbf{C}$) be continuous on $[a,b]$ and differentiable on (a,b) with $F'(x) = f(x)$ for all $x \in (a,b)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This enables us to compute very many integrals, for example, if $n \neq -1$,

$$\int_a^b x^n = \left[\frac{x^{n+1}}{n+1}\right]_a^b,$$

$$\int_a^b \cos x dx = [\sin x]_a^b.$$

It is also the reason why *integration by parts* works. If u, v are continuous on $[a,b]$ and differentiable on (a,b) and the derivatives u', v' are integrable on $[a,b]$ (for example if they extend to continuous functions on $[a,b]$) then uv' $u'v$ are integrable on $[a,b]$, and

$$\int_a^b uv' = \left(-\int_a^b u'v\right) + u(b)v(b) - u(a)v(a).$$

This uses the Fundamental Theorem of Calculus with the functions $f = uv' + u'v$ and $F = uv$.

The *Change of Variable Formula* also follows from the Fundamental Theorem of Calculus. Let $u : I \to J$ be a continuous differentiable function between intervals I and J. Then if f is integrable on J and $x \mapsto f(u(x))u'(x)$ is integrable on J,

$$\int_I f(u(x))u'(x) = \int_J f(u)du$$

If u' has constant sign we do not need to have $f(u(x))u'(x)$ integrable - it automatically will be. We also do not need any condition on either f or $f(u(x)u'(x)$ being integrable if I has endpoints a and b, J has endpoints c and d and f and $f(u(x))u'(x)$ extend to continuous functions on $[c,d]$ and $[a,b]$.

Monotone and Dominated Convergence Theorems. Monotone Convergence.

Let f_n be Lebesgue integrable for all n, $f_n(x) \leq f_{n+1}(x)$ for all n and $f_n(x) \to f(x)$ as $n \to \infty$ for all x and n. Then

$$\int f = \sup_n \int f_n = \lim_{n\to\infty} \int f_n, \tag{1}$$

where the righthand side might be $+\infty$ - but if it is finite, then f is Lebesgue integrable. A similar statement holds if $f_{n+1}(x) \leq f_n(x)$ for all x and n, and if sup is replaced by inf. This statement does hold for Riemann integrable functions f_n too, if we know that the limit $f(x)$ is Riemann integrable, that is, if we know this, then we can use (1) to compute $\int f$.

Dominated Convergence Let f_n be Lebesgue integrable for all n, let g be Lebesgue integrable with $|f_n(x)| \leq |g(x)|$ for all x and n. Let $\lim_{n\to\infty} f_n(x) = f(x)$ for all x. Then f is Lebesgue integrable and

$$\int f = \lim_{n\to\infty} \int f_n. \tag{2}$$

Again, if the functions f_n, g, f are just Riemann integrable, then we can use (2) to compute $\int f$.

Examples

The Monotone Convergence Property is very useful for showing that functions on infinite intervals are integrable and for computing these integrals, sometimes in conjunction with the Fundamental Theorem of Calculus on finite intervals. The Dominated Convergence Property is also useful, but in practice, on occasions when one might use Dominated convergence, it is often more efficient to use Monotone Convergence and Positivity.

1. Take $f(x) = x^{-2}$ on $[1,\infty)$ and $f_n(x) = \chi_{[1,n]}(x)f(x)$, that is, $f_n(x) = f(x)$ if $x \in [1,n]$ and $= 0$ otherwise. Then the fundamental Theorem of Calculus gives

$$\int f_n = \int_1^n x^{-2}dx = [-x^{-1}]_1^n = 1 - n^{-1} \to 1 \text{ as } n \to \infty.$$

But $f_n(x) \leq f_{n+1}(x)$ for all x and n, $f_n(x) \to f(x)$ as $n \to \infty$ for all x. So by Montone Convergence

$$\int_1^\infty x^{-2}dx = \lim_{n\to\infty}(1 - n^{-1}) = 1,$$

and x^{-2} is integrable on $[1,\infty)$.

2. Take $f(x) = x^{-1}$ on $(0,1)$. Intuitively we feel that f is not integrable on $(0,1)$ because x^{-1} is the derivative of $\log x$ and $\log 0$ is not defined, and $\log x \to -\infty$ as $x \to 0$. If we take $f_n = \chi_{[1/n,1)}f$, then $f_n(x) \leq f_{n+1}(x)$ for all x and n $f_n(x) \to f(x)$ as $n \to \infty$ for all x and by Monotone Convergence

$$\int_0^1 x^{-1} = \lim_{n\to\infty} \int_{1/n}^1 x^{-1}dx$$

$$= \lim_{n\to\infty}[\log x]_{1/n}^1 = \lim_{n\to\infty}(\log 1 - \log(1/n)) = \lim_{n\to\infty} \log n = +\infty,$$

and so x^{-1} is not integrable on $(0,1)$.

3. Let $f = \chi_A$, where A is the set of rational numbers in $[0,1]$. Then A is *countable*, which simply means we can find a sequence $\{a_n : n \in \mathbf{Z},\ n \geq 1\}$ such that

$$A = \{a_n : n \in \mathbf{Z},\ n \geq 1\}.$$

Then for each integer $N \geq 1$, write

$$A_N = \{a_n : 1 \leq n \leq N\}.$$

Then A_N is finite and $A_N \subset A_{N+1}$ for all N. Then if we write

$$f_N = \chi_{A_N},$$

then for all N we have

$$0 \leq f_N \leq f_{N+1} \leq 1.$$

We also have, for all x,

$$\lim_{N \to \infty} f_N(x) = f(x),$$

because if $x \notin A$ then $f_N(x) = 0 = f(x)$ for all N and if $x \in A$ then $x \in A_N$ for some N and then $f_k(x) = 1 = f(x)$ for all $k \geq N$. Now we want to apply the Monotone Convergence Theorem. The function f_N is actually a step function (if the steps are allowed to be of 0 width, which they are), so we have

$$\int \chi_{A_N} = \sum_{n=1}^{N} a_n - a_n = 0.$$

So by the Monotone Convergence Theorem,

$$\int \chi_A = 0$$

and χ_A is Lebesgue integrable. It is not Riemann integrable, however, and this is one of the simplest examples of a function which is Lebesgue integrable but not Riemann integrable. If we take the sup integrals of step functions $\leq f$ in the definition Riemann integral then we get 0, but if we take the inf integrals of step functions $\geq f$ we get 1. So the sup and inf do not coincide, as they have to for Riemann integrable functions.

4. $f(x) = x^{-2}(1 - \cos x)$ on $(-\infty, \infty)$. Althoouh this function is not defined at 0, using l'Hopital twice (or Taylor series) gives

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

So f is continuous, and bounded on any finite interval. In fact it can be checked that $|1 - \cos x| \leq \frac{x^2}{2}$ for all $|x| \leq 1$. Also $|1 - \cos x| \leq 2$ for all x. So we have

$$\left| \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx \right| \leq$$

$$\int_{-1}^{1} \frac{1}{2} 1 dx + 2 \left(\int_{-\infty}^{-1} + \int_{1}^{\infty} \right) x^{-2} dx \leq 1 + 2 \left[-x^{-1} \right]_{-\infty}^{-1} + 2 \left[-x^{-1} \right]_{1}^{\infty} = 5.$$

We can use Monotone Convergence as in 1 to show that x^{-2} is integrable on $(-\infty, -1]$, with integral 1.

Almost Everywhere.

If a Lebesgue measurable set A has

$$\int_A = 0 (= \lambda(A))$$

then it is said to be of *(Lebesgue) measure* 0. The set A in Example 3 above has measure 0. If $B \subset A$ and A is Lebesgue measurable of measure 0, then B is also Lebesgue measurable of measure 0. A property holds *almost everywhere* (or a.e. for short) if it holds except on a Lebesgue measurable set of measure 0. For example we say

$$f = 0 \ a.e.$$

if $f = 0$ except on a Lebesgue measurable set of measure 0. Such a function is automatically Lebesgue measurable and integrable with

$$\int f = 0.$$

Positivity Property Extended

An important extension of the Positivity Property holds for Lebesgue-measurable functions:

$$\int |f| = 0 \text{ if and only if } f = 0 \ a.e..$$

L^p spaces

For $1 < p < +\infty$, we define L^p (or $L^p(\mathbf{R})$) to be the space of Lebesgue-measurable functions $f : \mathbf{R} \to \mathbf{C}$ such that

$$\int |f|^p < \infty.$$

So L^1 is precisely the set of functions which are integrable. Similarly, if $A \subset \mathbf{R}$ is Lebesgue measurable (usually an interval), then we define $L^p(A)$ to be the set of Lebesgue-measurable functions $f : A \to \mathbf{C}$ such that

$$\int_A |f|^p < +\infty$$

So, again, $L^1(A)$ is precisely the set of functions which are integrable on A. We can also define L^∞. Basically, this is the set of bounded functions, but the formal definition is a little more free. A Lebesgue-measurable function $f : \mathbf{R} \to \mathbf{C}$ is in L^∞ (or $L^\infty(\mathbf{R})$) if there is a set E of measure 0 such that

$$\sup\{|f(x)| : x \notin E\} < +\infty.$$

We define $L^\infty(A)$ similarly.

Example 4. We saw above (Example 1) that $x^{-2}\chi_{[1,\infty)}$ is integrable It follows that $x^{-2} \in L^1[1,\infty))$ and $x^{-2}\chi_{[1,\infty)} \in L^1(\mathbf{R})$. Also $x^{-1} \in L^2[1,\infty)$ and $x^{-1}\chi_{[1,\infty)} \in L^2(\mathbf{R})$ because $|x|^{-2} = |x^{-2}|$.

If f, $g \in L^p$ and λ, $\mu \in \mathbf{C}$ then $\lambda f + \mu g \in L^p$.

The L^p Norm

The *L^p norm* is the number $\|f\|_p$ defined by

$$\|f\|_p = \left(\int |f|^p\right)^{1/p}$$

if $1 < p < \infty$ and

$$\|f\|_\infty = \inf\{\sup\{|f(x)| : x \notin E\} : E \text{ has measure } 0\}.$$

This is also called the *essential sup* of f, written ess $\sup(f)$. We have, for all p,

$$\|f\|_p \geq 0, \;\; \|f\|_p = 0 \text{ if and only if } f = 0 \; a.e.,$$

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

$$\|\lambda f\|_p = |\lambda|\|f\|_p.$$

The L^2 norm is especially important, and related to the L^2 norm there is a quantity $< f, g >$, for any f, $g \in L^2$, defined by

$$< f, g >= \int f\overline{g},$$

where $\overline{g}$ is the complex conjugate of the finction g. Of course, if g is real-vlaued, this function is just g itself. The integral above is finite because of the

Cauchy-Schwartz Inequality

$$\left|\int f\overline{g}\right| \leq \|f\|_2\|g\|_2.$$

Note also that

$$\|f\|_2 = \sqrt{< f, f) >}.$$

Integration in $\mathbf{R}^2$.

The proper definition of the integral in $\mathbf{R}^2$ follows the same lines as the definition in $\mathbf{R}$, with step functions (for the Riemann integral) replaced by linear combinations of functions χ_A where each A is a rectangle $(a, b) \times (c, d)$ and simple functions (for the Lebesgue integral) being linear combinations of functions

χ_A where A is *Lebesgue measurable in* $\mathbf{R}^2$. Such sets include rectangles. there are smilar definitions of the integral in $\mathbf{R}^n$ for all n.

In practise, the integral in $\mathbf{R}^2$ is computed using double integrals. If $f : \mathbf{R}^2 \to [0, +\infty]$ is Lebesgue measurable then, for a.e. y,

$$x \mapsto f(x, y)$$

is Lebesgue measurable, and, for a.e. x,

$$y \mapsto f(x, y)$$

is Lebesgue measurable. Also,

$$x \mapsto \int_{-\infty}^{+\infty} f(x, y)dy : \mathbf{R} \to [\mathbf{0}, +\infty]$$

is Lebesgue measurable, as is

$$y \mapsto \int_{-\infty}^{+\infty} f(x, y)dx.$$

We can therefore integrate these two functions, obtaining the double integrals

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dydx,$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dxdy,$$

which take values in $[0, +\infty]$

The following theorem is very useful. It only works for Lebesgue integrable functions, not Riemann integrable ones.

Tonelli's Theorem. Let $f : \mathbf{R}^2 \to \mathbf{R}$ *be Lebesgue measurable (continuous, for example) and suppose that either*

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x, y)|dydx < +\infty$$

or

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x, y)|dxdy < +\infty.$$

Then both the functions

$$x \mapsto \int_{-\infty}^{+\infty} f(x, y)dy, \ y \mapsto \int_{-\infty}^{+\infty} f(x, y)dx$$

are finite a.e. and integrable and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dydx = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dxdy.$$

Double integrals of continuous functions on closed bounded rectangles.

If $f : [a, b] \times [c, d] \to \mathbf{C}$ is continuous then

$$x \to \int_c^d f(x, y)dy,$$

$$y \to \int_a^b f(x, y)dx$$

are continuous functions on $[a, b]$ and $[c, d]$ respectively, and

$$\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

This does not follow from Tonelli's Theorem. If $[a,b]$ or $[c,d]$ are replaced by infinite intervals, or non-closed intervals, even if f is integrable, the single integrals with respect to dx or dy may not be continuous, and may not even exist everywhere. The best we can say, by Tonelli's theorem, is that the singles integrals exist a.e. and are integrable.

Application to Convolutions.

The *convolution* $f*g$ of two integrable functions f and g on $\mathbf{R}$ is defined as

$$(f*g)(x)=\int_{-\infty}^{+\infty} f(x-y)g(y)dy.$$

This function is defined and integrable by Tonelli's Theorem, because

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x-y)g(y)|dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x)|dx|g(y)|dy$$

by the Change of Variable formula on the inner integral, and this is equal to

$$\int_{-\infty}^{+\infty}|f(x)|dx\int_{-\infty}^{+\infty}|g(y)|dy=\|f\|_1\|g\|_1<+\infty,$$

since both f and g are integrable. So $f*g$ is defined a.e.. It is also integrable, because

$$\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty} f(x-y)g(y)dy\right|dx\le\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x-y)g(y)|dydx$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x-y)g(y)|dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x-y)|dx|g(y)|dy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|f(x)|dx|g(y)|dy$$

$$=\|f\|_1\|g\|_1.$$

Improper Integrals

There are a number of important functions f with the following properties. The function f is integrable on $[-R,R]$ for any $R>0$,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

exists and yet

$$\lim_{R\to\infty}\int_{-R}^{R}|f(x)|dx=+\infty.$$

For such functions f, by Monotone Convergence

$$\int_{-\infty}^{+\infty}|f(x)|dx=\lim_{R\to\infty}\int_{-R}^{R}|f(x)|dx=+\infty.$$

In such cases, strictly speaking,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

is known as the *improper integral* of f, and is sometimes written as

$$\int_{-\infty}^{+\infty} f(x)dx$$

even though *strictly speaking, this integral does not exist.* The most important example of such a function is

$$f(x) = \frac{\sin x}{x}.$$

By contour integration it can be shown that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x} dx = \pi.$$

But for $n \geq 0$

$$\int_{n\pi}^{(n+1)\pi} |\frac{\sin x}{x}| dx \geq \int_{(n\pi+(\pi/4)}^{n\pi+(3\pi/4)} \frac{1}{\sqrt{2}x} > \frac{\pi}{\sqrt{2}(n+1)\pi}.$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = +\infty$$

we get

$$\lim_{n\to\infty} \int_{0}^{n} |\frac{\sin x}{x}| dx = +\infty,$$

and hence

$$\int_{-\infty}^{+\infty} |\frac{\sin x}{x}| dx = +\infty.$$

The Riemann Lebesgue Lemma

The Riemann lebesgue Lemma is a result which is used in the proofs of several Inverse Fourier theorems, such as the Pointwise Fourier Series Theorem for piecewise smooth 2π-periodic functions. Here is a statement.
Theorem Let $-\infty \leq a < b \leq +\infty$. Let f be integrable on (a, b). Then

$$\lim_{\lambda \to \pm\infty} \int_a^b f(x) e^{i\lambda x} dx = 0,$$

$$\lim_{\lambda \to \pm\infty} \int_a^b f(x) \sin \lambda x dx = 0,$$

$$\lim_{\lambda \to \pm\infty} \int_a^b f(x) \cos \lambda x dx = 0.$$

The proof of this is very easy if f is a step function. For example take f to be the constant function 1 and $-\infty < a < b < +\infty$ then we can prove the limits are 0 just by integration:

$$\int_a^b e^{i\lambda x} = \left[\frac{e^{i\lambda x}}{i\lambda} \right]_a^b = \frac{e^{i\lambda b} - e^{i\lambda a}}{i\lambda} \to 0 \text{ as } \lambda \to \pm\infty.$$

The proof for a general f uses the following theorem.
Theorem Let f be integrable. Then for any $\varepsilon > 0$ there is a step function g such that

$$\int |f - g| < \varepsilon.$$

Once we have this we see that for all λ,

$$\left| \int f(x) e^{-\lambda x} dx - \int g(x) e^{i\lambda x} dx \right| \leq \int |f - g < \varepsilon.$$

Hence

$$\left| \lim_{\lambda \to \pm\infty} \int f(x) e^{i\lambda x} dx - \lim_{\lambda \to \pm\infty} \int g(x) e^{i\lambda x} dx \right| < \varepsilon.$$

So

$$\left| \lim_{\lambda \to \pm\infty} \int f(x) e^{i\lambda x} dx \right| < \varepsilon.$$

This is true for all $\varepsilon > 0$. So

$$\lim_{\lambda \to \pm\infty} \int f(x) e^{i\lambda x} dx = 0,$$

and similarly if $e^{i\lambda x}$ is replaced by $\cos \lambda x$ or $\sin \lambda x$.