Dido's problem

- This is probably the oldest problem in the Calculus of Variations.
- Dido founded the city of Carthage, in Tunisia.
- According to legend, she arrived at the site with her entourage, a refugee from a power struggle with her brother in Tyre in the Lebanon.
- She asked the locals for as much land as could be bound by a bull's hide.
- She cut the hide into a long thin strip and bounded the maximum possible area with this.
- The maximum possible area bounded by a curve of fixed length is a circle. So the city of Carthage is circular in shape.
- Dido's problem is an example of what is called an *isoperimetric problem*.
- These are problems about enclosing areas with the same length of perimeter, or enclosing volumes by surfaces of the same area.
- A very simple example is to find the rectangle with maximum area, given that the perimeter is L – that is, maximize $x(\frac{1}{2}L - x)$ for $0 \leq x \leq \frac{1}{2}L$.
- These are important mathematical problems, usually not easy, and not the most elementary in the Calculus of Variations.
- The Greek mathematician Zenodorus managed to show that the area of a circle is large than the area of any regular polygon with a perimeter of the same length.
- James and John, the Bernoulli brothers, competed over the solution to an isoperimetric problem, each claiming that their solutions were correct, and that the other's were wrong.
- James' work on this and other problems developed the technique of Calculus of Variations and John later improved some of his solutions.
- But it seems that the first complete solution to Dido's problem was given somewhat later, by the geometer Steiner, in the nineteenth century.

Dido's problem is about maximizing the area bounded by a closed curve $(x(t), y(t))$ for $a \leq t \leq b$ subject to the curve having fixed length. The area is

$$\int_a^b (xdy/dt - ydx/dt)dt$$

and the length of the boundary is

$$\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}dt.$$

Newton's ship design

- In Principia Book 2 Newton talks about the best design of a ship, to minimize the resistance.
- This is perhaps the first application of calculus to the Calculus of Variations.
- The shape of the hull is the rotation about the x axis of the curve $y(x)$ for $a \leq x \leq b$.
- The problem is to minimize the resistance

$$\int_{y(b)}^{y(a)} \frac{cyy'^2}{1+y'^2}dy = \int_a^b \frac{-cy(x)y'(x)^3}{1+y'(x)^2}dx$$

  over all functions $y(x)$ defined for $a \leq x \leq b$, and for fixed values of $y(a)$ and $y(b)$

The Brachistochrone Problem

- This was the problem solved by John Bernoulli, and on being challenged by him, also by James Bernoulli, Leibniz, Newton and l'Hopital.
- All the solutions were published in *Acta Eruditorum* in 1697.
- The problem is to find the curve joining (a, c) to (b, d) ($d < c$, $a \neq b$) along which a particle slides in the shortest possible time.
- The time is given for a curve $x(y)$ (or $y(x)$) by the formula given in the problem of the Tautochrone, that is

$$T = \int_c^d \sqrt{\frac{1+(dx/dy)^2}{2g(c-y)}}dy = \int_a^b \sqrt{\frac{1+(dy/dx)^2}{2g(c-y(x))}}dx.$$

- The solution found by James was the closest to the modern technique of the Calculus of Variations.

The cable problem

- This is the problem of finding the curve taken by a cable of uniform density when it is at rest, and hung from two fixed points in a vertical plane.
- Galileo said that the curve looks like a parabola.
- The solution to the original cable problem is a catenary

$$C\cosh(\alpha x + \beta).$$

- Leibniz, Huygens and John Bernoulli all published solutions.
- This was John Bernoulli's first main result independent of his brother.
- His proof and Leibniz' used calculus. Huygens' proof was geometric, and less rigorous.

The mathematical formulation

- The sum of the potiential and kinetic energy of a uniform cable in shape $y(x)$ or $a \leq x \leq b$ is constant, in the absence of external forces.
- If the cable is at rest then the kinetic energy is 0 and the potential area is maximized.
- So the problem is to maximize the potential energy

$$\rho \int_a^b gy\sqrt{1+(dy/dx)^2}dx$$

(where ρ is the density) subject to the cable being of fixed length, that is

$$\int_a^b \sqrt{1+(dy/dx)^2}dx = L.$$

- Like Dido's problem, this is a maximising problem subject to a constraint.
- Calculus of Variations Problems are about finding minimising or maximising *functions* y for a function of the function which is an integral of the form

$$\int_a^b F(x,y,y')dx$$

where $y(a)$ and $y(b)$ are specified.

- In some problems there is also a constraint: we only consider y such that, for some fixed G,

$$\int_a^b G(x,y,y')dx = L.$$

- Minimising and maximising problems were a major impetus for the development of calculus. The solutions of some problems about maximising and minimising real-valued functions pre-date the formal introduction of calculus.
- As we know, a necessary condition for a point x_0 to be a minimum or maximum of a differentiable real-valued function f is that

$$f'(x_0) = 0.$$

- So one can ask what the generalisation of this is, in the Calculus of Variations problems.
- The man who formalised this was Leonhard Euler (1707-1783), possibly the most prolific mathematician who ever lived.

Leonhard Euler (1707-1783)

- He was born in Basel, and brought up not far from there, in Riehen.
- His father, Paul was a protestant minister who had studied theology at Basel University, and had attended James Bernoulli's lectures there.
- He taught his son a bit of mathematics and passed on his interest in the subject.
- When Leonhard himself went to Basel University, where he, too, was supposed to be studying theology, he made contact with John Bernoulli, who had now inherited his brother's chair. Although John Bernoulli did not agree to give Leonhard Euler private lessons, as he requested, he did give him valuable direction in his studies.
- In 1726, at the age of nineteen, Euler accepted a chair in St Petersburg, which became vacant on the death of Nicholas Bernoulli II, son of John.
- Leonhard Euler delayed travelling to St Petersburg because there was a chance of a chair in Basel also. But in 1727 he travelled to St Petersburg.
- He worked there for much of his life, although he spent 25 years in Berlin, from 1741 to 1766 – after which he returned to St Petersburg.
- He married Katharina Gsell, also of Swiss origin. Together they had thirteen children, although only five survived infancy.
- At about the time that he returned to St Petersburg, when he was 59, he went totally blind. Despite this, almost half of his mathematical works were produced after this time.
- His vast mathematical output is still being edited. The Dartmouth College archive at http://www.math.dartmouth.edu/ euler/tour has 866 published papers.
- Key publication seems to be no 65, *Methodus inveniendi lineas curvas* which was published in 1740, which the mathematican Carathéodory considered "one of the most beautiful mathematical works ever written".
- The basic problem, as described in another paper 296 *Elementa Calculi variationum*, is to find maximum and minimum values of an integral

  $$\int Z dx$$

  where $Z = Z(x, y, p)$ is a function of x and of a function $y = y(x)$ of x and of the first derivative $p(x) = dy/dx$.
- The integral with respect to x is therefore a function of y and p, and the problem is maximise or minimise a function of a function.

- As always in maximising and minimising problems, the method is to differentiate the function (of y and p) and find the zero of the derivative.

To compute the derivative, if $y+h\eta$ is a small perturbation of y with $(y+h\eta)(a) = y(a)$ and $(y+h\eta)(b) = y(b)$ for all small h then $\eta(a) = \eta(b) = 0$ and

$$\int_a^b Z(y+h\eta, p+h\eta', x)dx - \int_a^b Z(x,y,p)dx$$

$$= \int_a^b \left(hN\eta + hP\eta' + O(h^2)\right) dx$$

where

$$N = \frac{\partial Z}{\partial y}(x,y,p), \quad P = \frac{\partial Z}{\partial p}(x,y,p)$$

But

$$\int_a^b P\eta' dx = [P\eta]_a^b - \int_a^b \frac{dP}{dx}\eta dx = -\int_a^b \frac{dP}{dx}\eta dx$$

For this to be ≤ 0 for all small h and all η, (for a local maximum) or, similarly, always ≥ 0 (for a local minimum) then it must be the case that

$$N = \frac{dP}{dx}.$$

This is a second order differential equation for y, and is the basic differential equation in the Calculus of Variations

The solution of the Brachistochrone problem

The differential equation is rather simpler if we consider x as a function of y. Recall that in this case the integral to minimised is

$$\int_c^d \sqrt{\frac{1+(dx/dy)^2}{2g(c-y)}}dy$$

and the differential equation for $x(y)$ becomes

$$\frac{d}{dy}\left(\sqrt{\frac{1+(dx/dy)^2}{2g(c-y)}}\right) = 0,$$

which means that

$$\frac{1+(dx/dy)^2}{g(c-y)}$$

is constant. This is precisely the same condition as we had for the problem of the tautochrone. So the solution curve must be the same — a cycloid.

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