Gottfried Wilhelm Leibniz (1646-1716)

- His father, a professor of Philosophy, died when he was small, and he was brought up by his mother.
- He learnt Latin at school in Leipzig, but taught himself much more and also taught himself some Greek, possibly because he wanted to read his father’s books.
- He studied law and logic at Leipzig University from the age of fourteen – which was not exceptionally young for that time.
- His Ph D thesis “De Arte Combinatoria” was completed in 1666 at the University of Altdorf. He was offered a chair there but turned it down.
- He then met, and worked for, Baron von Boineburg (at one stage prime minister in the government of Mainz), as a secretary, librarian and lawyer – and was also a personal friend.
- Over the years he earned his living mainly as a lawyer and diplomat, working at different times for the states of Mainz, Hanover and Brandenburg.
- But he is famous as a mathematician and philosopher.
- By his own account, his interest in mathematics developed quite late.
- An early interest was mechanics.
  - He was interested in the works of Huygens and Wren on collisions.
  - He published Hypothesis Physica Nova in 1671. The hypothesis was that motion depends on the action of a spirit (a hypothesis shared by Kepler—but not Newton).
  - At this stage he was already communicating with scientists in London and in Paris. (Over his life he had around 600 scientific correspondents, all over the world.)
  - He met Huygens in Paris in 1672, while on a political mission, and started working with him.
  - At Huygens suggestion he started reading the works of St Vincent (a Flemish Jesuit, another key figure in the early development of calculus).
- He also produced a calculating machine in 1670-1, which could carry out the four basic arithmetic operations.
- (In the next decade he developed binary arithmetic.)
- The diplomatic mission to France failed. In 1673 he accompanied von Boineburg’s nephew on a related mission to London.
• He came into contact with mathematicians and scientists, including Huygens, while working as an ambassador for the Elector of Mainz, first in Paris, in 1672, and then in London in 1673.

• He visited the Royal Society, was elected a fellow, and talked to a number of scientists there, including Robert Hooke, Boyle and Pell.

• Pell told him that his work on series had been done by a mathematician called Mouton (which was correct).

• Hooke later spoke slightingly of his calculating machine.

• Leibniz returned home and redoubled his efforts in mathematics

Leibniz and calculus

• His notes on calculus date from 1673.

• Many of these were never published. They include original ideas and also his reinterpretation of the works of others.

• Even in his Ph D thesis he was interested in successive differences of sequences, and sums of successive differences, that is,

$$a_n = a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})$$

• This is the discrete version of the Fundamental Theorem of Calculus.

• In a manuscript in October 1675 he had a statement of the Fundamental Theorem of Calculus:

“just as $\int f$ will increase, so $d$ will diminish the dimensions”

• This was also the manuscript in which he introduced the notation $\int$ for integral – using both this and the $\int$ that he had previously used.

Here is an excerpt from this manuscript

• Throughout the 1670’s, Leibniz developed his calculus

• By 1676 he had the derivative and integral of $x^n$.

• In 1677 he had the correct rules for differentiation of sums, products, quotients.

• By 1680 he had the notation $dx$, $dy$ for differentials.

• His first publications on calculus was in 1684: Novus Methodus pro maximis et minimis, itemque tangentibus.

• Newton heard about Leibniz’ work and wrote to him, at least twice, around 1676, to tell him about his own results.
Both times, Leibniz replied later than Newton expected, simply because the letters took a long time to reach him.

Newton, however, interpreted this tardiness as meaning that Leibniz wanted to steal his results.

This was the start of the Newton-Leibniz controversy.

In a letter to James Bernoulli in 1703, Leibniz describes how his studies in calculus progressed. He mentions many names: Descartes, Cavalieri, Vieta, Huygens, Pascal, Gregory St Vincent, Roberval, James Gregory (but not Newton).

In 1711 Leibniz was accused of plagiarism in the Transactions of the Royal Society. When he protested, the Royal Society set up a committee to determine priority, but did not ask Leibniz to give evidence. The committee decided in favour of Newton, who wrote the report.

Leibniz went off to work for the Duke of Hanover (the uncle of George I, later king of Great Britain)

Among many other activities, he did pioneering work in geology, through planning projects concerning mines in the Harz Mountains.

He died in obscurity.

The Bernoulli brothers

- Jakob (James) Bernoulli (1655-1705)
- Johann (John) Bernoulli (1667-1748)

These brothers were both important mathematicians in their own right and also important correspondents of Leibniz.

They were among the first readers of Leibniz’ work on calculus, and among the first to use the calculus.

The Bernoulli family produced mathematicians over three generations whose work is still known today.

They were all called James or John or Daniel or Nicholas. (Since they were Swiss, various versions of their names are used.)

Although James initially taught John mathematics – in the face of opposition from their father – the brothers were very competitive – and also competitive with Leibniz.

James had a professorship in Basel.

John had a chair in Groningen – but in earlier years was paid handsomely by his friend, the mathematician l’Hopital, for teaching him calculus.
• James solved the problem of the tautochrone which was also solved by Leibniz.

• John found the solution of the brachistochrone problem and issued a challenge to others to find a solution.

• Solutions were found by James Bernoulli, Leibniz, l’Hopital and Newton.

The Tautochrone

• A tautochrone or isochrone is a monotone curve with a minimum, which can be taken at $y = 0$, such that the time take for a bead to slide along the curve to the bottom is always the same, no matter what the starting point.

• Huygens found that an inverted cycloid is such a curve.

![Image of a cycloid]

• He tried to make a mechanism to illustrate this but – not surprisingly – it was not possible to eliminate friction, and so he could not do it.

• Some time later James Bernoulli used calculus to verify Huygen’s result that the cycloid is the only solution.

How is this done?

• We assume there is no friction.

• So the potential energy of a bead of mass $m$ at height $y$ is $mgy$ and the kinetic energy is $\frac{m}{2}((dx/dt)^2 + (dy/dt)^2)$ and the sum of these:

$$\frac{m}{2}(2gy + (dx/dt)^2 + (dy/dt)^2)$$

is constant.

• If the bead starts at height $y_0$ then the bead is at rest when $y = y_0$, which we can take to happen at $t = 0$, meaning that $x'(0) = y'(0) = 0$ and $y(0) = y_0$. So

$$(x'(t))^2 + (y'(t))^2 = 2g(y_0 - y)$$

• Writing $dx/dt = (dx/dy)(dy/dt)$, we have

$$-\frac{dy}{dt}\sqrt{\frac{(dx/dy)^2 + 1}{2g(y_0 - y)}} = 1.$$  

(Clearly $y$ decreases with $t$ so $dy/dt \leq 0$)
• So
\[ \int_{y_0}^{y_0} \sqrt{\frac{(dx/dy)^2 + 1}{2g(y_0 - y)}}\,dy = \int_{0}^{T} dt \]
where \( T \) is the time taken to slide to the bottom \( y = 0 \).

• The time \( T \) is supposed to be the same no matter what the choice of \( y_0 \).

• In this integral \( x \) is a function of \( y \) (not \( t \)) so the curve \( x(y) \) is the same for all \( y_0 \).

• In the integral, write \( y = y_0 u \) and write \( (dx/dy)(y) = x'(y) \). Then the integral becomes
\[ I = \int_{0}^{1} \sqrt{\frac{y_0 - ((x'(y_0 u))^2 + 1)}{2g(1 - u)}}\,du, \]
which has to be equal to \( T \) for all choices of \( y_0 \).

• Since \( y = 0 \) is a minimum of \( y(x) \), we expect \( dy/dx = 0 \) at \( y = 0 \), and therefore we do not expect \( dx/dy \) to exist at \( y = 0 \) — and it does not.

• If
\[ x'(y))^2 = \frac{A}{y} - 1 \]
then
\[ 1 + (x'(y_0 u))^2 = \frac{A}{y_0} \]
which makes \( I \) independent of \( y_0 \).

• In fact this is the only way that \( I \) can be independent of \( y_0 \) (at least if \( y(x) \) has a Taylor series expansion).

• So
\[ \frac{dx}{dy} = -\sqrt{\frac{A}{y} - 1} = -\sqrt{\frac{A - y}{y}}. \]
So
\[ x = -\int \sqrt{\frac{A - y}{y}}\,dy \]
Making the change of variable \( y = A(1 + \cos \theta)/2 = A \cos^2(\theta/2) \) gives
\[ dy = -(A/2) \sin \theta \,d\theta, \quad x = A - A \cos^2(\theta/2) = A \sin^2(\theta/2) \]
and
\[ x = \int \frac{A}{2} \sqrt{\tan^2(\theta/2)} \sin \theta \,d\theta = \int A \sin^2(\theta/2)\,d\theta \]
\[ = \int \frac{A}{2} (1 - \cos \theta)\,d\theta = \frac{A}{2} (\theta - \sin \theta) \]
This is indeed the inverted cycloid.
References


- http://www-history.mcs.st-andrews.ac.uk/history/

- Child, J.M., The early mathematical manuscripts of Leibniz translated from the Latin ..., with critical and historical notes, Open Court, Chicago, 1920 QA37.L52