11

ELEMENTARY MULTIPLE-VALUED FUNCTIONS

53. Single-Valued Branches. Univalent Functions

Except for the Möbius transformations, which are one-to-one mappings of the extended plane onto itself, the entire and meromorphic functions $w = f(z)$ studied so far are such that the equation $f(z) = A$ has multiple roots, which are distinct except for certain special values of A. This means that the mapping $w = f(z)$ is not one-to-one, i.e., that the inverse function $z = f^{-1}(w)$ is multiple-valued. Before the concepts and results obtained for single-valued functions can be applied to the multiple-valued function $f^{-1}(w)$, we must find domains on which $f^{-1}(w)$ is no longer multiple-valued, thereby constructing so-called *single-valued branches* of $f^{-1}(w)$. This has already been done in certain special cases, e.g., for the inverses of the functions

$$w = (z - a)^n, \quad w = e^z, \quad w = \cos z$$

(see Secs. 37, 39, 42).[1] The general procedure goes as follows:

Suppose $w = f(z)$ is a single-valued function which is defined and wide-sense continuous on a domain G of the extended z-plane, but which is not one-to-one on G. Suppose we can find a countable family of disjoint subdomains $G_1 \subset G, G_2 \subset G, \ldots$ such that every point of G is either a point of one of the subdomains $G_1, G_2, \ldots$ or a common boundary point of at least two subdomains G_k, G_l, and such that the function $w = f(z)$ is one-to-one on every subdomain G_k. Thus, if E is the set of all points of G which are common boundary points of at least two subdomains G_k, G_l, we have the decomposition

$$G = E \cup G_1 \cup G_2 \cup \cdots. \tag{11.1}$$

Then every image $\mathscr{G}_1 = f(G_1), \mathscr{G}_2 = f(G_2), \ldots$ is also a domain (see Theorem 6.1), and

$$\mathscr{G} = f(G) = f(E) \cup \mathscr{G}_1 \cup \mathscr{G}_2 \cup \cdots.$$

By hypothesis, the function $z = f^{-1}(w)$ is multiple-valued on $\mathscr{G}$, i.e., $f^{-1}(w)$ can take any of a whole set of values $E_w \subset G$ at any point $w \in \mathscr{G}$. However, suppose we now define on each domain $\mathscr{G}_k$ a function $f_k^{-1}(w)$ such that

$$f_k^{-1}(\{w\}) = E_w \cap G_k \qquad (w \in \mathscr{G}_k).$$

Since the set $E_w \cap G_k$ consists of one and only one point, $f_k^{-1}(w)$ is single-valued and (wide-sense) continuous on $\mathscr{G}_k$ (see Theorem 6.1), and obviously $G_k = f_k^{-1}(\mathscr{G}_k)$. Each of the functions $f_k^{-1}(w)$, $k = 1, 2, \ldots$, of which there may be infinitely many, is called a *single-valued branch* of the function $f^{-1}(w)$.

Remark 1. It should be noted that the character of the domains $\mathscr{G}_k$ and of the single-valued branches $f_k^{-1}(w)$ depends in an essential way on just how the domain G is decomposed into subdomains G_k. In the simplest cases, a decomposition of G can be found such that all the domains $\mathscr{G}_k$ are the same.

Remark 2. For an arbitrary wide-sense continuous function $w = f(z)$, the decomposition (11.1) is not possible. However, considerations which will not be given here show that if the function $f(z) \not\equiv$ const is wide-sense continuous on a domain G, and analytic on G except possibly on a set $I \subset G$ consisting entirely of isolated points (see Problem 3.16), then the decomposition (11.1) is always possible (actually, in infinitely many ways). A function $f(z)$ which is wide-sense continuous on a domain G, and analytic on G except possibly on a set $I \subset G$ consisting entirely of isolated points, is said to be *univalent* (synonymously, *schlicht* or *simple*) on G if $f(z_1) \neq f(z_2)$ whenever $z_1, z_2 \in G$ and $z_1 \neq z_2$, i.e., if $f(z)$ is one-to-one on G. On the other hand, a function $f(z)$ which is wide-sense continuous on a domain G, and analytic on G except possibly on a set $I \subset G$ consisting entirely of isolated points, is said to be *multivalent* on G if there exists at least one pair of points $z_1, z_2 \in G$, $z_1 \neq z_2$, such that $f(z_1) = f(z_2)$. With this terminology, the result just mentioned takes the following form: If the function $f(z) \not\equiv$ const is multivalent on a domain G, then G has a decomposition (11.1) such that $f(z)$ is univalent on every subdomain G_k. The domains G_k $(k = 1, 2, \ldots)$ are called *domains of univalence* for the function $w = f(z)$. Moreover, the inverse function $z = f^{-1}(w)$ is single-valued, in fact univalent, on each of the domains $\mathscr{G}_k = f(G_k)$, $k = 1, 2, \ldots$ (cf. Rule 5, p. 109).

[1] See also the preliminary discussion of the concept of a single-valued branch in Sec. 30.

In this chapter, we shall illustrate the above considerations by applying them to certain elementary multiple-valued functions. However, we shall not have to rely on the result cited in Remark 2, since in every case the decomposition of the domain G into domains of univalence can be obtained by using known properties of elementary functions.

54. The Mapping $w = \sqrt[n]{z}$

Let $n > 1$ be an integer, and consider the function

$$w = \sqrt[n]{z}, \tag{11.2}$$

which is the inverse of the function $z = w^n$. For every value of z except 0 and ∞, (11.2) takes n different values, given by the formula

$$w = \sqrt[n]{|z|}\left(\cos\frac{\operatorname{Arg} z}{n} + i\sin\frac{\operatorname{Arg} z}{n}\right). \tag{11.3}$$

For $z = 0$ or $z = \infty$, the function $w = \sqrt[n]{z}$ takes just one value, i.e., $w = 0$ or $w = \infty$. The n numbers (11.3), representing the points of the w-plane at which w^n takes the same value z, correspond to the vertices of a regular n-gon, inscribed in the circle $|w| = \sqrt[n]{|z|}$. Conversely, the vertices of any regular n-gon with center at the origin of coordinates represent n possible values of the function (11.2), for a suitable complex number z. Therefore, a domain in the w-plane will be a domain of univalence for the function $z = w^n$ if and only if it contains no more than one vertex of every regular n-gon with center $w = 0$. Obviously, this condition is satisfied by the interior of every angle of $2\pi/n$ radians with vertex at $w = 0$.

As already noted, the inverse of (11.2) is the multivalent function $z = w^n$, defined on the whole w-plane. Suppose we draw any n rays from the point $w = 0$ such that the angles between adjacent rays all equal $2\pi/n$. Then the interiors $\mathscr{G}_1, \ldots, \mathscr{G}_n$ of the n angles of $2\pi/n$ radians formed by these rays are all domains of univalence for the function $z = w^n$. The image (under $z = w^n$) of each of these domains $\mathscr{G}_k$ is the same domain G in the z-plane, whose boundary is some ray drawn from the point $z = 0$. In fact, if the boundary of $\mathscr{G}_k$ consists of the rays with slopes

$$\varphi_0 + \frac{2k\pi}{n} \quad \text{and} \quad \varphi_0 + \frac{2(k+1)\pi}{n},$$

the boundary of G consists of the single ray L with slope $n\varphi_0$. In this way, we obtain n single-valued branches

$$(\sqrt[n]{z})_1, \ldots, (\sqrt[n]{z})_n \tag{11.4}$$

of the function $\sqrt[n]{z}$, all defined on the same domain G, where $(\sqrt[n]{z})_k$ denotes the branch which maps G onto $\mathscr{G}_k$. Moreover, since $w = (\sqrt[n]{z})_k$ is a one-to-one continuous mapping of G onto $\mathscr{G}_k$, and since $z = w^n$ has a nonzero derivative nw^{n-1} on $\mathscr{G}_k$, the branches $(\sqrt[n]{z})_k$ all have nonzero derivatives on G, i.e.,

$$\frac{d}{dz}(\sqrt[n]{z})_k = \frac{1}{nw^{n-1}} = \frac{1}{n(\sqrt[n]{z})_k^{n-1}} \qquad (k = 1, \ldots, n)$$

(cf. Rule 5, p. 109).

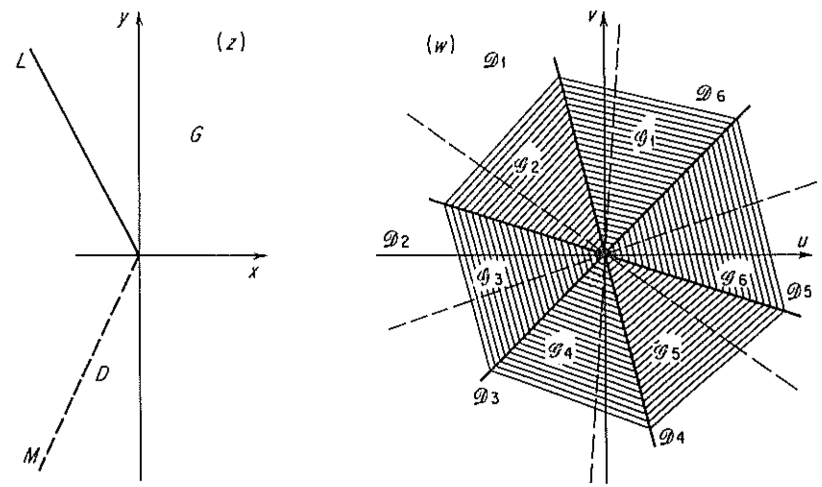

FIGURE 11.1

Now suppose we rotate our family of n rays through an angle α about the origin, where $0 < \alpha < 2\pi/n$, thereby obtaining a new family of rays, which divides the w-plane into a new family of domains $\mathscr{D}_1, \ldots, \mathscr{D}_n$. Each domain $\mathscr{D}_k$ intersects two domains $\mathscr{G}_k$ and $\mathscr{G}_{k+1}$, with $\mathscr{G}_{n+1} = \mathscr{G}_1$ by definition (see Figure 11.1 illustrating the case $n = 6$, where boundaries of the domains $\mathscr{G}_k$ are indicated by solid lines, and boundaries of the domains $\mathscr{D}_k$ by dashed lines). The inverse image in the z-plane of each of the domains $\mathscr{D}_k$ is the same domain D, whose boundary is the single ray M drawn from the origin with inclination $n\varphi_0 + n\alpha$. As before, we can define n single-valued branches [2]

$$\{\sqrt[n]{z}\}_1, \ldots, \{\sqrt[n]{z}\}_n \tag{11.5}$$

of the function $w = \sqrt[n]{z}$, where now $\{\sqrt[n]{z}\}_k$ is the branch mapping D onto

[2] Note the vital distinction between the parentheses in (11.4) and the braces in (11.5).

$\mathscr{D}_k$. Again, each of the branches (11.5) is differentiable on D, and

$$\frac{d}{dz}\{\sqrt[n]{z}\}_k = \frac{1}{n\{\sqrt[n]{z}\}_k^{n-1}} \qquad (k = 1, \ldots, n).$$

Moreover, it is clear that $\{\sqrt[n]{z}\}_k$ coincides with $(\sqrt[n]{z})_k$ on the set $\mathscr{D}_k \cap \mathscr{G}_k$ and with $(\sqrt[n]{z})_{k+1}$ on the set $\mathscr{D}_k \cap \mathscr{G}_{k+1}$. Thus, when we go from one family of domains of univalence to another such family, each new single-valued branch is obtained by combining two of the old single-valued branches, where on the part of $\mathscr{D}_k$ belonging to the common boundary Γ of $\mathscr{G}_k$ and $\mathscr{G}_{k+1}$, $\{\sqrt[n]{z}\}_k$ is the appropriate limit of either $(\sqrt[n]{z})_k$ or $(\sqrt[n]{z})_{k+1}$. More precisely, we have

$$\begin{aligned} \{\sqrt[n]{z}\}_k &= (\sqrt[n]{z})_k && \text{if } z \in \mathscr{D}_k \cap \mathscr{G}_k, \\ \{\sqrt[n]{z}\}_k &= (\sqrt[n]{z})_{k+1} && \text{if } z \in \mathscr{D}_k \cap \mathscr{G}_{k+1}, \\ \{\sqrt[n]{z}\}_k &= \lim_{\zeta \to z} (\sqrt[n]{\zeta})_k = \lim_{\zeta \to z} (\sqrt[n]{\zeta})_{k+1} && \text{if } z \in \mathscr{D}_k \cap \Gamma, \end{aligned}$$

where $\Gamma = \bar{\mathscr{G}}_k \cap \bar{\mathscr{G}}_{k+1}$.

Remark. If the angle of rotation α is zero, then $D = G$ and $\mathscr{D}_k = \mathscr{G}_k$ $(k = 1, \ldots, n)$, while if $\alpha = 2\pi/n$, $D = G$ again but $\mathscr{D}_k = \mathscr{G}_{k+1}$ $(k = 1, \ldots, n)$, where $\mathscr{G}_{n+1} = \mathscr{G}_1$. As α increases continuously from 0 to $2\pi/n$, the domain $\mathscr{D}_k$ overlaps the domain $\mathscr{G}_{k+1}$ more and more, until it finally coincides with $\mathscr{G}_{k+1}$, and the ray M representing the boundary of D undergoes a counterclockwise rotation of 2π radians, where its initial and final positions coincide with the ray L representing the boundary of G. At the same time, the branch $\{\sqrt[n]{z}\}_k$, which originally coincides with $(\sqrt[n]{z})_k$, shares more and more of its domain of definition with the branch $(\sqrt[n]{z})_{k+1}$, until it finally coincides with $(\sqrt[n]{z})_{k+1}$. In this sense, we can say that as α increases continuously from 0 to $2\pi/n$, the branch $(\sqrt[n]{z})_k$ changes continuously into the branch $(\sqrt[n]{z})_{k+1}$.

We can also keep track of the way one branch $(\sqrt[n]{z})_k$ changes into another branch $(\sqrt[n]{z})_{k+1}$ by making the point z describe a complete circle with center at the point $z = 0$. Suppose that at the point $z_0 \in G$ we choose a value of $\sqrt[n]{z}$ belonging to the branch $(\sqrt[n]{z})_k$ and represented by the point

$$w_0 = \sqrt[n]{|z_0|}\left(\cos\frac{\theta_0}{n} + i \sin\frac{\theta_0}{n}\right),$$

belonging to the domain $\mathscr{G}_k$. Then, as the point z moves continuously around the circle $|z| = |z_0|$ in the counterclockwise direction, starting from the point z_0, the value of

$$w = \sqrt[n]{|z_0|}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right) \tag{11.6}$$

varies continuously with θ, and when z returns to its original value z_0, (11.6) goes into the value

$$w_1 = \sqrt[n]{|z_0|}\left(\cos\frac{\theta_0 + 2\pi}{n} + i \sin\frac{\theta_0 + 2\pi}{n}\right),$$

obtained by rotating w_0 through the angle $2\pi/n$ about the point $w = 0$.[3] Therefore w_1 belongs to the domain $\mathscr{G}_{k+1}$ adjacent to $\mathscr{G}_k$, and w_1 is the value of the branch $(\sqrt[n]{z})_{k+1}$ at the point z_0. Since the point $z_0 \in G$ is arbitrary, we can say that one circuit around the origin $z = 0$ in the counterclockwise direction causes the branch $(\sqrt[n]{z})_k$ to change continuously into the branch $(\sqrt[n]{z})_{k+1}$.[4] Moreover, it is easy to see that in this sense n circuits around the origin in the counterclockwise sense cause the branch $(\sqrt[n]{z})_k$ to undergo the sequence of transformations

$$(\sqrt[n]{z})_k \to (\sqrt[n]{z})_{k+1}, (\sqrt[n]{z})_{k+1} \to (\sqrt[n]{z})_{k+2}, \ldots,$$
$$(\sqrt[n]{z})_n \to (\sqrt[n]{z})_1, \ldots, (\sqrt[n]{z})_{k-1} \to (\sqrt[n]{z})_k,$$

which carry it continuously into itself after "going through" all the other branches in succession. Since $(\sqrt[n]{z})_k$ is arbitrary, n circuits around the origin carry any branch into itself.

Given a multiple-valued function $w = f(z)$ with continuous single-valued branches defined on a domain G, we say that the point $\zeta \in \bar{G}$ is a *branch point* of $f(z)$ if there exists a neighborhood $\mathscr{N}(\zeta)$ such that one complete circuit around an arbitrary closed Jordan curve $\gamma \subset \mathscr{N}(\zeta)$ with $\zeta \in I(\gamma)$, carries every branch of $f(z)$ into another branch of $f(z)$. If a finite number of circuits around γ (in the same direction) carries every branch of $f(z)$ into itself, and if n is the smallest such number, we say that ζ is a branch point *of finite order*, specifically, *of order* $n - 1$. In this case, the point ζ is also called an *algebraic branch point* of $f(z)$, provided that $f(z)$ has a limit (finite or infinite) at ζ. Thus we have just shown that the point $z = 0$ is an algebraic branch point of order $n - 1$ of the function $w = \sqrt[n]{z}$.

Remark 1. It is clear that the point $z = \infty$ can also be regarded as an algebraic branch point of order $n - 1$ of the function $w = \sqrt[n]{z}$, since every circuit around the point at infinity along a circle of arbitrarily large radius with center at the origin is simultaneously a circuit around the origin. Therefore the multiple-valued function $w = \sqrt[n]{z}$ has two branch points in the z-plane, i.e., $z = 0$ and $z = \infty$, both of order $n - 1$.

Remark 2. The single-valued branches described above were constructed

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[3] Of course, in making the circuit around the circle $|z| = |z_0|$, we allow z to pass through the ray L, which is excluded from the domain G.

[4] More precisely, every value of $\sqrt[n]{z}$ on the branch $(\sqrt[n]{z})_k$ changes continuously into the corresponding value of $\sqrt[n]{z}$ on the branch $(\sqrt[n]{z})_{k+1}$.

for a domain like G or D, whose boundary is a rectilinear ray joining the two branch points 0 and ∞. More generally, let γ be any Jordan curve in the extended z-plane joining the points 0 and ∞, and this time let G be the domain with boundary γ. As the point z traces out the curve γ from its initial point 0 to its final point ∞, the n points corresponding to the n values of $w = \sqrt[n]{z}$ trace out n Jordan curves $\Gamma_1, \ldots, \Gamma_n$ joining 0 and ∞. These curves have no points in common other than 0 and ∞, and each set $\Gamma_k \cup \Gamma_{k+1}$ (where $\Gamma_{k+1} = \Gamma_1$) represents a closed Jordan curve in the extended w-plane. Of the two domains with boundary $\Gamma_k \cup \Gamma_{k+1}$, let $\mathscr{G}_k$ be the domain which does not contain the other curves $\Gamma_1, \ldots, \Gamma_{k-1}, \Gamma_{k+2}, \ldots, \Gamma_n$. By construction, when the w-plane is rotated through the angle $2\pi/n$ about the origin, Γ_k goes into Γ_{k+1} and Γ_{k+1} goes into Γ_{k+2}, and hence the domain $\mathscr{G}_k$ goes into the domain $\mathscr{G}_{k+1}$ ($\mathscr{G}_{n+1} = \mathscr{G}_1$). Since

$$\mathscr{G}_k \cap \mathscr{G}_{k+1} = 0 \qquad (k = 1, \ldots, n),$$

the rotation cannot carry any point of $\mathscr{G}_k$ into another point of $\mathscr{G}_k$. Therefore the domains $\mathscr{G}_1, \ldots, \mathscr{G}_n$ are all domains of univalence for the function $w = z^n$, and we obtain n single-valued branches of the function $w = \sqrt[n]{z}$, all defined on the domain G, by requiring that the kth branch take its values in the domain $\mathscr{G}_k$ ($k = 1, \ldots, n$). To specify a branch, it is sufficient to indicate the value of $\sqrt[n]{z}$ at some point $z_0 \in G$; if this value is w_0, there is a unique domain $\mathscr{G}_k$ containing w_0, and a unique branch of $\sqrt[n]{z}$ taking the value w_0 at the point z_0.

Now let $[\sqrt[n]{z}]_k$ and $[\sqrt[n]{z}]_l$ be two single-valued branches of the function $\sqrt[n]{z}$, which are defined on the domain G and take values w_0' and w_0'', respectively, at a point $z_0 \in G$. Since

$$w_0' = [\sqrt[n]{z}]_k = \sqrt[n]{|z|}\left(\cos\frac{\theta_0 + 2m'\pi}{n} + i\sin\frac{\theta_0 + 2m'\pi}{n}\right),$$
$$w_0'' = [\sqrt[n]{z}]_l = \sqrt[n]{|z|}\left(\cos\frac{\theta_0 + 2m''\pi}{n} + i\sin\frac{\theta_0 + 2m''\pi}{n}\right),$$

where $\theta_0 = \arg z$, and m', m'' are integers, it follows that w_0'' equals w_0' multiplied by

$$\eta = \cos\frac{2(m'' - m')\pi}{n} + i\sin\frac{2(m'' - m')\pi}{n},$$

i.e., by a value of $\sqrt[n]{1}$. But $\eta[\sqrt[n]{z}]_k$ is obviously a single-valued continuous function on G, such that $(\eta[\sqrt[n]{z}]_k)^n = z$, i.e., $\eta[\sqrt[n]{z}]_k$ is one of the single-valued branches of $\sqrt[n]{z}$ defined on G, in fact, the branch $[\sqrt[n]{z}]_l$ containing the point $\eta[\sqrt[n]{z_0}]_k = [\sqrt[n]{z_0}]_l = w_0''$. In other words, any single-valued branch of $\sqrt[n]{z}$ defined on G can be obtained by multiplying any other single-valued branch defined on G by an appropriate nth root of unity.

Remark 3. The conclusions of this section apply (with certain obvious modifications) to the somewhat more general functions

$$w = \sqrt[n]{z - a} \quad \text{and} \quad w = \sqrt[n]{\frac{z - a}{z - b}}, \tag{11.7}$$

which are the inverses of the functions

$$z = w^n + a \quad \text{and} \quad z = \frac{bw^n - a}{w^n - 1},$$

respectively. The first of the functions (11.7) has branch points a and ∞, while the second has branch points a and b. Moreover, single-valued branches of each of the functions (11.7) can be defined on any domain whose boundary is a Jordan curve joining the appropriate branch points.

55. The Mapping $w = \sqrt[n]{P(z)}$

To gain a deeper insight into the concept of a branch point, we now study the multiple-valued function

$$w = f(z) = \sqrt[n]{P(z)},$$

where $P(z)$ is an arbitrary polynomial of degree N. Let $P(z)$ have zeros $a_1, \ldots, a_m$, of orders $\alpha_1, \ldots, \alpha_m$, respectively, where $\alpha_1 + \cdots + \alpha_m = N$. Then, according to Sec. 35, $P(z)$ can be written in the form

$$P(z) = A(z - a_1)^{\alpha_1} \cdots (z - a_m)^{\alpha_m},$$

and hence

$$f(z) = \sqrt[n]{A(z - a_1)^{\alpha_1} \cdots (z - a_m)^{\alpha_m}}. \tag{11.8}$$

Consider an arbitrary closed Jordan curve γ which does not pass through any of the points $a_1, \ldots, a_m$, and suppose z traverses γ once. At some point $z_0 \in \gamma$ we choose definite values $\varphi_1^{(0)}, \ldots, \varphi_m^{(0)}$ of the arguments of the complex numbers $z_0 - a_1, \ldots, z_0 - a_m$, thereby selecting a certain single-valued branch of the function $f(z)$.

As the point z goes around the curve γ once, starting from z_0 and returning to z_0, the argument φ_k of the vector $z - a_k$ varies continuously; if a_k belongs to $E(\gamma)$ [the exterior of γ], φ_k returns to its original value $\varphi_k^{(0)}$, while if a_k belongs to $I(\gamma)$ [the interior of γ], φ_k acquires an increment $\pm 2\pi$ (see Figure 11.2).[5] The sign of the increment depends only on the direction

[5] These facts, which are easily verified in the simplest cases (for example, when γ is a circle, an ellipse or a polygon), can be proved in complete generality. See e.g., P. S. Aleksandrov, *op. cit.*, Chap. 2.

in which γ is traversed, and if the appropriate angles *increase*, we say that the direction is *positive* (this is always the counterclockwise direction). Suppose, for definiteness, that the point z describes γ in the positive direction. Then, if none of the points $a_1, \ldots, a_m$ lie inside γ, all the angles φ_k return to their original values, and as a result, the function $f(z)$ also returns to its original value (11.8). It follows that *a point ζ of the finite z-plane different from the zeros $a_1, \ldots, a_m$ of the polynomial $P(z)$ cannot be a branch point of $\sqrt[n]{P(z)}$*. In fact, for any such point ζ we can find a neighborhood $\mathcal{N}(\zeta)$ containing none of the points $a_1, \ldots, a_m$, and then a complete circuit around any closed Jordan curve $\gamma \subset \mathcal{N}(\zeta)$ with $\zeta \in I(\gamma)$ does not change the branch of $f(z)$ which has been chosen.

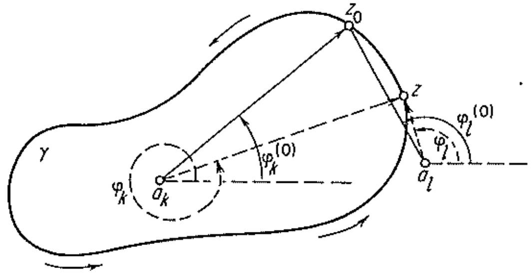

FIGURE 11.2

Next let $\mathcal{N}(a_k)$ be a neighborhood of the point a_k which is small enough not to contain any of the other points $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m$. Then a complete circuit around any closed Jordan curve $\gamma \subset \mathcal{N}(a_k)$ with $a_k \in I(\gamma)$ changes φ_k by 2π, while all the other angles $\varphi_1, \ldots, \varphi_{k-1}, \varphi_{k+1}, \ldots, \varphi_n$ return to their original values. It follows that the right-hand side of (11.8) is multiplied by the factor

$$\cos\frac{2\pi\alpha_k}{n} + i\sin\frac{2\pi\alpha_k}{n}, \tag{11.9}$$

which is different from unity if and only if α_k is not a multiple of n. Therefore *every zero a_k of the polynomial $P(z)$ whose order is not a multiple of n is a branch point of $\sqrt[n]{P(z)}$*. To determine the order of such a branch point, suppose δ_k is the greatest common divisor of α_k and n ($\delta_k < n$). Then, setting $\alpha_k = \delta_k\alpha'_k$ and $n = \delta_k\nu_k$ ($\nu_k > 1$), we see that (11.9) equals

$$\cos\frac{2\pi\alpha'_k}{\nu_k} + i\sin\frac{2\pi\alpha'_k}{\nu_k}.$$

Therefore, as a result of p circuits around γ (in the same direction), $f(z)$ is multiplied by the factor

$$\cos\frac{2\pi\alpha'_k p}{\nu_k} + i\sin\frac{2\pi\alpha'_k p}{\nu_k},$$

which is equal to unity if and only if p is a multiple of ν_k. Since ν_k is obviously the smallest multiple of ν_k, *the branch point a_k is of order $\nu_k - 1$.*

Finally, let $\mathcal{N}(\infty)$ be a neighborhood of the point at infinity which contains none of the points $a_1, \ldots, a_m$, and let γ be a closed Jordan curve such that $\gamma \subset \mathcal{N}(\infty)$ and $\infty \in E(\gamma)$. Since $I(\gamma)$ contains all the points $a_1, \ldots, a_m$, a complete circuit around γ in the positive direction changes all the angles $\varphi_1, \ldots, \varphi_m$ by 2π. Therefore the right-hand side of (11.8) is multiplied by the factor

$$\cos\frac{2\pi(\alpha_1 + \cdots + \alpha_m)}{n} + i\sin\frac{2\pi(\alpha_1 + \cdots + \alpha_m)}{n} = \cos\frac{2\pi N}{n} + i\sin\frac{2\pi N}{n},$$

which is different from unity if and only if N is not a multiple of n. Therefore *∞ is a branch point of $\sqrt[n]{P(z)}$ if and only if N is not a multiple of n*. Suppose N is not divisible by n, so that ∞ is a branch point of $\sqrt[n]{P(z)}$. Let δ be the greatest common divisor of N and n ($\delta < n$), and let $n = \delta\nu$. Then *the branch point ∞ is of order $\nu - 1$.*

Remark. Let γ be any closed Jordan curve lying in the finite z-plane. As we have just seen, a circuit around γ does not change the values of $f(z) = \sqrt[n]{P(z)}$ if either of the following two conditions is met:

1. $a_k \in I(\gamma)$, $a_j \in E(\gamma)$ for $j \neq k$, and α_k is a multiple of n;
2. $a_k \in I(\gamma)$ for $k = 1, \ldots, m$, and $N = \alpha_1 + \cdots + \alpha_m$ is a multiple of n.

More generally, let $a_{k_1}, \ldots, a_{k_q}$ ($q \leqslant m$) be any set of zeros of $P(z)$, such that $\alpha_{k_1} + \cdots + \alpha_{k_q}$ is a multiple of n. Then a circuit around any closed Jordan curve γ, such that $I(\gamma)$ contains $a_{k_1}, \ldots, a_{k_q}$ and $E(\gamma)$ contains all the other zeros, does not change the values of $f(z) = \sqrt[n]{P(z)}$.

Now let G be a domain such that every closed Jordan curve γ lying in G has the property that either $I(\gamma)$ contains no zeros of $P(z)$ at all, or else $I(\gamma)$ contains a set of zeros the sum of whose orders is divisible by n. Then on every such domain G we can define single-valued branches of the function $f(z) = \sqrt[n]{P(z)}$. In fact, let $z_0 \in G$ and let w_0 be one of the n values of the function $f(z)$ at z_0. The single-valued branch $f(z)$ which takes the value w_0 at z_0 is constructed as follows: To find the value of this branch at any other point $z_1 \in G$ we draw a Jordan curve $L \subset G$ joining z_0 and z_1 (see Theorem 4.12), and we move along z_0 to z_1, making sure that the corresponding values of $f(z)$ vary continuously, starting from the initial value w_0 at z_0. As a result, we arrive at z_1 "accompanied" by one of the n values of $f(z)$, which we denote by w_1. It remains to show that w_1 is unique, i.e., that w_1 depends only on z_0, w_0 and z_1, and not on the particular Jordan curve joining z_0 to z_1. Suppose that by going from z_0 to z_1 along another Jordan curve $L' \subset G$ we arrive at z_1 accompanied by a value w_1' of $f(z)$, where $w_1' \neq w_1$. Without loss of generality, we can assume that L and L' have only the points z_0 and z_1 in common (see Problem 11.3). Then $\gamma = L \cup L'$ is a closed Jordan curve such that $\gamma \subset G$ and $z_1 \in \gamma$, but such that one circuit around γ beginning and ending at z_1 changes the value of $f(z)$ at z_1 from w_1 to w_1'. But this is impossible, since by hypothesis γ either contains no zeros of $P(z)$ or a set of zeros the sum of whose orders is divisible by n. This contradiction establishes the uniqueness of w_1.

Example 1. The function

$$w = f(z) = \sqrt{(1 - z^2)(1 - k^2z^2)} \qquad (0 < k < 1) \qquad (11.10)$$

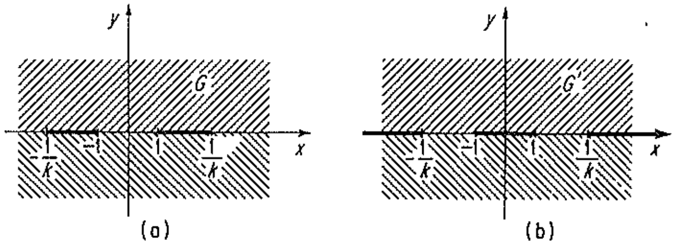

FIGURE 11.3

is a double-valued function with four branch points ± 1, $\pm 1/k$. Here $N = 4$ is a multiple of $n = 2$, and hence ∞ is not a branch point. Since ± 1, $\pm 1/k$ are all simple zeros of the expression under the radical, the numbers α_k all equal 1. Therefore a circuit around any closed Jordan curve γ containing only two branch points in its interior does not change the values of the function. Thus, for example, we can define two single-valued branches of $f(z)$ on the domain G with boundary consisting of the two segments

$$-\frac{1}{k} \leqslant x \leqslant -1, \qquad 1 \leqslant x \leqslant \frac{1}{k}$$

[see Figure 11.3(a)], or on the domain G' with boundary consisting of the segment $-1 \leqslant x \leqslant 1$ and the infinite segment of the real axis joining the points $-1/k$ and $1/k$ through the point at infinity [see Figure 11.3(b)]. On the domain G, the two branches $f_1(z)$ and $f_2(z)$ of the function (11.10) can be distinguished by the values they take at the origin, i.e.,

$$f_1(0) = 1, \qquad f_2(0) = -1.$$

Example 2. Consider the function

$$w = \sqrt{4z^3 - g_2z - g_3}, \qquad (11.11)$$

where g_2 and g_3 are complex numbers satisfying the condition

$$g_2^3 - 27g_3^2 \neq 0,$$

which means that the discriminant of the cubic polynomial

$$4z^3 - g_2z - g_3$$

is nonzero, so that the zeros e_1, e_2, e_3 of (11.11) are all distinct.[6] In this case, $N = 3$ is not divisible by $n = 2$, and hence the point at infinity is also a branch point. As before, a circuit around any pair of branch points along any closed Jordan curve does not change the value of the function. Therefore, joining e_1 to e_2 and e_2 to ∞ by Jordan curves γ_1 and γ_2, we obtain a domain G with boundary consisting of γ_1 and γ_2 on which we can define single-valued branches of the function (see Figure 11.4).

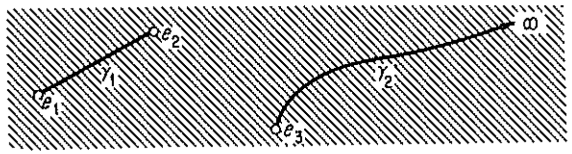

FIGURE 11.4

Example 3. Consider the function

$$w = f(z) = z + \sqrt{z^2 - 1}, \qquad (11.12)$$

which is the inverse of the Joukowski function

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right). \qquad (11.13)$$

The function $f(z)$ is double-valued, and has the same branch points ± 1 as the function $\sqrt{z^2 - 1}$. To obtain a domain G on which single-valued branches of (11.12) can be defined, we join the points -1 and 1 by a finite segment of the real axis. As we know from Sec. 51, this gives a domain which is mapped in a one-to-one fashion by the function (11.12) onto each of the two domains $I(\gamma)$ and $E(\gamma)$, where γ is the unit circle. To

[6] See e.g., G. Birkhoff and S. MacLane, *op. cit.*, p. 113.