39. The Mapping $w = e^z$

It follows from (9.22) that e^z is nonzero for all z and

$$|e^z| = e^x, \qquad \operatorname{Arg} e^z = y + 2k\pi.$$

For $z = iy$ $(x = 0)$ we obtain *Euler's formula*

$$e^{iy} = \cos y + i \sin y \tag{9.23}$$

(see p. 8). Using (9.23), we can replace the trigonometric form of a complex number

$$z = r(\cos \Phi + i \sin \Phi)$$

by the more concise *polar form*

$$z = re^{i\Phi}.$$

It is apparent from (9.22) that the exponential is *periodic* in z with *period* $2\pi i$. In other words, if z is changed by $2\pi i$, so that y is changed by 2π, the value of e^z does not change:

$$e^{z+2\pi i} = e^z.$$

We now show that $2\pi i$ is the *fundamental* (or *primitive*) *period* of the function e^z, i.e., that any other period ω of e^z must be of the form $2k\pi i$, where k is an integer. To see this, let $\omega = \alpha + i\beta$. Then

$$e^{z+\omega} = e^z$$

for any z, and in particular,

$$e^\omega = e^{\alpha+i\beta} = e^\alpha(\cos\beta + i\sin\beta) = 1$$

for $z = 0$. But this means that $|e^\omega| = e^\alpha = 1$ which implies $\alpha = 0$, and hence $\cos\beta + i\sin\beta = 1$ which implies $\beta = 2k\pi i$, so that

$$\omega = \alpha + i\beta = 2k\pi i,$$

as asserted.

The expression e^∞ will be regarded as meaningless, since

$$\lim_{z\to\infty} e^z$$

does not exist. This can be seen from the fact that $e^x \to \infty$ as $x \to +\infty$, whereas $e^x \to 0$ as $x \to -\infty$. In particular, it follows that e^z cannot coincide with any polynomial, i.e., e^z is actually an entire *transcendental* function, since any polynomial (excluding the trivial case of a constant) approaches infinity as $z \to \infty$.

Next we study the geometric behavior of the mapping $w = e^z$. As already noted, e^z is nonzero for all z. This means that the origin of coordinates in the w-plane does not belong to the image of the finite z-plane under the mapping $w = e^z$. However, as we now show, any other finite point of the w-plane does belong to this image. In fact, from the equation $w = e^z$, where $w \neq 0$ is given and $z = x + iy$ is unknown, we obtain

$$|w| = e^x \quad \text{or} \quad x = \ln|w|$$

and

$$\operatorname{Arg} w = y + 2k\pi \quad \text{or} \quad y = \operatorname{Arg} w.$$

Therefore the inverse images of the point w can only be points of the form

$$z = \ln|w| + i \operatorname{Arg} w. \tag{9.24}$$

Obviously there are infinitely many points (9.24), since $\operatorname{Arg} w$ takes infinitely many values, all differing by integral multiples of 2π. Moreover, each of these points is actually an inverse image of w, since

$$\begin{aligned}\exp[\ln|w| + i\operatorname{Arg} w] &= e^{\ln|w|}(\cos\operatorname{Arg} w + i\sin\operatorname{Arg} w)\\ &= |w|(\cos\operatorname{Arg} w + i\sin\operatorname{Arg} w) = w.\end{aligned}$$

Therefore the set of all roots of the equation $e^z = w$ $(w \neq 0)$ is given by the formula

$$z = \ln|w| + i\operatorname{Arg} w = \ln|w| + i(\arg w + 2k\pi), \tag{9.25}$$

where $k = 0, \pm 1, \pm 2, \ldots$. These points all lie on the same straight line parallel to the imaginary axis, and the distance between any two consecutive points along the line is 2π. Thus the function $w = e^z$ maps the finite z-plane onto the domain obtained from the finite w-plane by deleting the single point $w = 0$, but the mapping is not one-to-one, since every point $w \neq 0$ has an infinite number of inverse images (9.25). On the other hand, the mapping is conformal at every point of the finite z-plane, since the derivative

$$(e^z)' = \frac{\partial(e^x \cos y)}{\partial x} + i\frac{\partial(e^x \sin y)}{\partial x} = e^x(\cos y + i\sin y) = e^z$$

does not vanish for any value of z.

Now suppose z traces out a straight line parallel to one of the coordinate axes (see Figure 9.3). For example, consider the line

$$z = b + it, \tag{9.26}$$

parallel to the imaginary axis. Then the image of (9.26) under the mapping $w = e^z$ is the curve

$$w = e^b(\cos t + i\sin t), \tag{9.27}$$

i.e., w traces out a circle of radius e^b with its center at the origin. Moreover, as z describes the line (9.26) once in such a way that t, the ordinate of z, increases continuously from $-\infty$ to $+\infty$, w describes the circle (9.27) an infinite number of times in the positive (counterclockwise) direction.

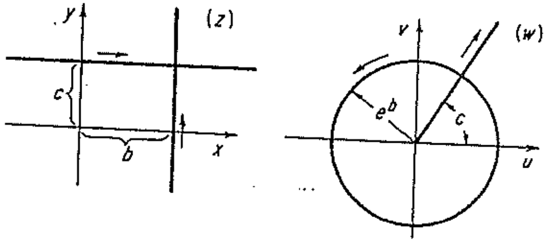

FIGURE 9.3

Next consider the line

$$z = t + ic, \tag{9.28}$$

parallel to the real axis. Then the image of (9.28) under the mapping $w = e^z$ is the curve

$$w = e^t(\cos c + i \sin c), \tag{9.29}$$

i.e., w traces out a ray of slope $\tan c$ emanating from the origin. Moreover, as z describes the line (9.28) once in such a way that t, the abscissa of z, increases continuously from $-\infty$ to $+\infty$, w describes the ray (9.29) once in such a way that the distance of w from the origin increases continuously from 0 to ∞ (of course, the limits 0 and ∞ are excluded, since $|w| = e^t$). Thus, *under the mapping $w = e^z$, a family of lines parallel to the imaginary axis is transformed into a family of concentric circles with the origin as center, and a family of lines parallel to the real axis is transformed into a family of rays emanating from the origin.*

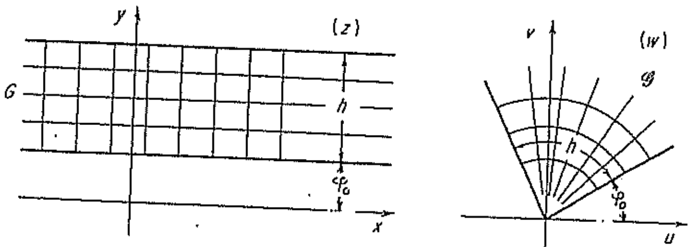

FIGURE 9.4

Now consider the domain G consisting of all points z such that

$$\varphi_0 < \operatorname{Im} z < \varphi_1,$$

where $\varphi_1 - \varphi_0 = h$; such a domain will be called an *(open) strip* of width h. Suppose $0 < h < 2\pi$, and let $\mathscr{G}$ be the image of G under the mapping $w = e^z$. It follows from the considerations just given that $\mathscr{G}$ is the interior of the angle of h radians with vertex at the origin, formed by the rays

$$\operatorname{Arg} w = \varphi_0 + 2k\pi, \quad \operatorname{Arg} w = \varphi_1 + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

(see Figure 9.4). Moreover, the correspondence between the domains G and $\mathscr{G}$ under the mapping $w = e^z$ is *one-to-one*. To see this, we recall that the inverse images of a point $w \in \mathscr{G}$ are all of the form (9.25), and hence differ only in the values of their imaginary parts. In fact, any two points (9.25) lie on a line parallel to the imaginary axis, and the distance between them is an integral multiple of 2π. However, by assumption, the width h of our strip does not exceed 2π, and G can contain only one inverse image of the point w, i.e., not only is $w = f(z)$ a single-valued function on G, but its inverse $z = f^{-1}(w)$ is a single-valued function on $\mathscr{G} = f(G)$. Thus, *the exponential function $w = e^z$ is a one-to-one conformal mapping of an open strip of width $h \leqslant 2\pi$ with sides parallel to the real axis onto the interior of an angle of h radians with vertex at the origin.*

Next consider a straight line with equation

$$z = (1 + i\alpha)t + ib \qquad (-\infty < t < \infty), \tag{9.30}$$

which is not parallel to one of the coordinate axes. Here $\alpha \neq 0$ is the slope of the line (9.30), and b is its y-intercept. The image of (9.30) under the mapping $w = e^z$ is the curve

$$w = \exp\,[t + i(\alpha t + b)] = e^t[\cos(\alpha t + b) + i \sin(\alpha t + b)].$$

Therefore

$$|w| = r = e^t, \qquad \varphi = \operatorname{Arg} w = \alpha t + b + 2k\pi,$$

and eliminating the parameter t, we obtain

$$r = \exp\,[(\varphi - b - 2k\pi)/\alpha]. \tag{9.31}$$

If we set $\theta = \varphi - 2k\pi$, (9.31) becomes

$$r = ce^{\theta/\alpha}, \tag{9.32}$$

where $c = e^{-b/\alpha}$. This is the equation (in polar form) of a *logarithmic spiral*. Since the mapping $w = e^z$ is conformal, and since (9.32) is the image of the line (9.30) intersecting all lines parallel to the real axis at the same angle $\arctan \alpha$, it follows that the logarithmic spiral intersects the images of all these lines, i.e., all rays emanating from the origin, at the same angle $\arctan \alpha$, a property which characterizes the logarithmic spiral (see Figure 9.5).

Remark. As is well known,[8]

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \tag{9.33}$$

for real x. Using (9.33), we can easily show that

$$e^z = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n \tag{9.34}$$

for complex z. In fact, writing

$$z_n = \left(1 + \frac{z}{n}\right)^n,$$

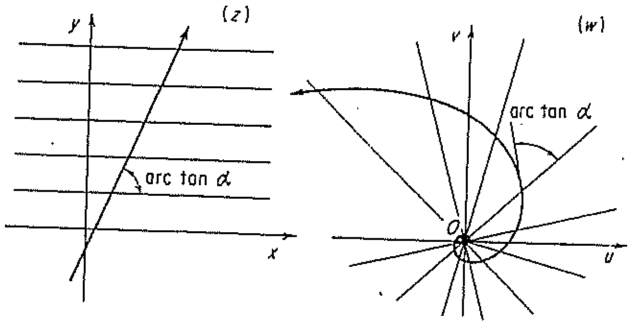

FIGURE 9.5

we have

$$|z_n| = \left|1 + \frac{x}{n} + i\frac{y}{n}\right|^n = \left[\left(1 + \frac{x}{n}\right)^2 + \frac{y^2}{n^2}\right]^{n/2},$$

$$\arg z_n = n \arc\tan \frac{y/n}{1 + (x/n)}.$$

Therefore

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty}\left(1 + \frac{2x}{n}\right)^{n/2} = e^x,$$

where we drop $(x^2 + y^2)/n^2$ in comparison to $2x/n$ and use (9.33). Moreover, replacing small angles by their tangents, we see that

$$\lim_{n\to\infty} \arg z_n = \lim_{n\to\infty} \frac{n(y/n)}{1 + (x/n)} = y.$$

[8] See e.g., R. Courant, *Differential and Integral Calculus, Vol. I*, second edition (translated by E. J. McShane), Interscience Publishers, Inc., New York (1959), p. 175.

But then, according to p. 34,

$$\begin{aligned}\lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n &= \lim_{n\to\infty} z_n \\ &= \lim_{n\to\infty} |z_n| \,[\cos(\lim_{n\to\infty} \arg z_n) + i \sin(\lim_{n\to\infty} \arg z_n)] \\ &= e^x(\cos y + i \sin y),\end{aligned}$$

and comparison of this result with (9.22) proves (9.34).

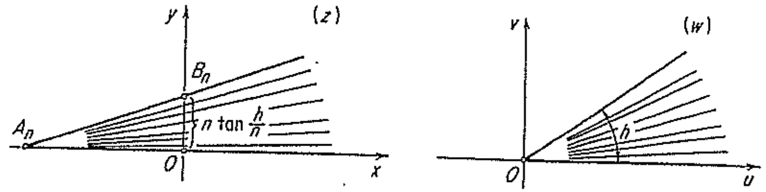

FIGURE 9.6

Formula (9.34) shows the connection between the mapping $w = e^z$ and the mapping $w = (z - a)^n$ studied in Sec. 37. We note that e^z is the limit as $n \to \infty$ of the mapping

$$w = \left(1 + \frac{z}{n}\right)^n = \frac{1}{n^n}[z - (-n)]^n, \tag{9.35}$$

which, as we know, maps the interior of an angle of h/n radians $(0 < h \leqslant 2\pi)$ with vertex at the point $A_n = (-n, 0)$ and sides consisting of the rays

$$x \geqslant -n, \qquad y = 0$$

and

$$\operatorname{Arg}(z + n) = \frac{h}{n} + 2k\pi,$$

onto the interior of an angle of h radians with vertex at the origin and sides consisting of the rays

$$\operatorname{Arg} w = 0, \qquad \operatorname{Arg} w = h + 2m\pi.$$

As $n \to \infty$, the vertex A_n approaches infinity along the negative real axis and the length of the segment $\overline{OB_n}$ (see Figure 9.6) approaches

$$\lim_{n\to\infty} n \tan\frac{h}{n} = h,$$

so that the limiting position of the ray $\overline{A_nB_n}$ is the line $y = h$, which together with the real axis forms the boundary of a strip of width h. Moreover, as $n \to \infty$, the rays emanating from the vertex A_n approach lines parallel to the real axis and the arcs of circles with A_n as center approach perpendiculars to the real axis lying inside the strip. In other words, in the limit as $n \to \infty$, the effect of the mapping (9.35) is exactly the same as that of the mapping $w = e^z$.

40. Some Functions Related to the Exponential

According to the formulas

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x, \tag{9.36}$$

we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{9.37}$$

for arbitrary real x. If z is an arbitrary finite complex number, we define two (entire) *trigonometric functions* $\cos z$ and $\sin z$, called the *cosine* and *sine*,[9] by simply changing x to z everywhere in (9.37):

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{9.38}$$

This seems quite natural, since the functions $\cos z$ and $\sin z$ are obviously analytic for all z, and reduce to the familiar functions $\cos x$ and $\sin x$ when $z = x$ is real. It follows from the definitions (9.38) that $\cos z$ is even and $\sin z$ is odd, i.e., that

$$\cos(-z) = \cos z, \qquad \sin(-z) = -\sin z.$$

Moreover, (9.38) implies the formulas

$$e^{iz} = \cos z + i \sin z, \qquad e^{-iz} = \cos z - i \sin z, \tag{9.39}$$

which generalize (9.36).

The functions $\cos z$ and $\sin z$ are both periodic with period 2π, since changing z to $z + 2\pi$ in (9.38) amounts to multiplying the exponentials by $e^{\pm 2\pi i} = 1$. Actually, 2π is the fundamental period of $\cos z$ and $\sin z$, i.e., any other period is an integral multiple of 2π, as we now verify for $\cos z$. If ω is any period of $\cos z$, then

$$\cos(z + \omega) = \cos z,$$

and hence, setting $z = \pi/2$, we obtain

$$\cos\left(\omega + \frac{\pi}{2}\right) = 0.$$

But this implies

$$\exp\left[i\left(\omega + \frac{\pi}{2}\right)\right] + \exp\left[-i\left(\omega + \frac{\pi}{2}\right)\right] = 0,$$

or

$$\exp[i(2\omega + \pi)] = -1.$$

[9] Trigonometric functions of a more general nature are discussed in Sec. 52.

Therefore, according to formula (9.25),

$$i(2\omega + \pi) = \ln|-1| + i \operatorname{Arg}(-1) = i(\pi + 2k\pi),$$

so that

$$\omega = 2k\pi,$$

as asserted. Similarly, it can easily be verified that 2π is the fundamental period of $\sin z$.

Next we derive *addition theorems* for the functions $\cos z$ and $\sin z$, i.e., formulas relating the quantities $\cos(z_1 + z_2)$ and $\sin(z_1 + z_2)$ to the quantities $\cos z_1$, $\sin z_1$, $\cos z_2$ and $\sin z_2$, where z_1 and z_2 are arbitrary complex numbers. As might be expected, the required relations are immediate consequences of the addition theorem

$$\exp(z_1 + z_2) = \exp z_1 \exp z_2$$

for the exponential. In fact, replacing z by $z_1 + z_2$ in the formulas (9.39), we find that

$$\begin{aligned} \cos(z_1 + z_2) + i \sin(z_1 + z_2) &= \exp[i(z_1 + z_2)] = \exp(iz_1)\exp(iz_2) \\ &= (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2) \\ &= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \end{aligned} \tag{9.40}$$

and

$$\begin{aligned} \cos(z_1 + z_2) - i \sin(z_1 + z_2) &= \exp[-i(z_1 + z_2)] = \exp(-iz_1)\exp(-iz_2) \\ &= (\cos z_1 - i \sin z_1)(\cos z_2 - i \sin z_2) \\ &= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2). \end{aligned} \tag{9.41}$$

First adding (9.41) to (9.40), and then subtracting (9.41) from (9.40), we obtain the addition theorems

$$\begin{aligned} \cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \\ \sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \end{aligned} \tag{9.42}$$

which are basic in the theory of trigonometric functions. In particular, the so-called *reduction formulas* are implicit in (9.42). For example, setting $z_1 = z$, $z_2 = \pi/2$ in (9.42) gives

$$\cos\left(z + \frac{\pi}{2}\right) = \cos z \cos\frac{\pi}{2} - \sin z \sin\frac{\pi}{2} = -\sin z,$$
$$\sin\left(z + \frac{\pi}{2}\right) = \sin z \cos\frac{\pi}{2} + \cos z \sin\frac{\pi}{2} = \cos z,$$

setting $z_1 = z$, $z_2 = \pi$, gives

$$\cos(z + \pi) = -\cos z,$$
$$\sin(z + \pi) = -\sin z,$$

and so on. Moreover, substituting $z_1 = z$, $z_2 = -z$ into the first of the formulas (9.42), we obtain the following basic relation between $\cos z$ and $\sin z$:

$$\cos^2 z + \sin^2 z = 1. \tag{9.43}$$