of attraction is the method of Liapunov functions, which will be developed in Chapter 4. In this section, we will establish some of the basic topological properties of basin of attractions. Henceforth, all of our maps are assumed to be continuous. We begin our exposition by defining the important notion of invariance.

DEFINITION 2.2 *A set M is said to be invariant under a map f if $f(M) \subset M$. In other words, for every $x \in M$, $O(x) \subset M$.*

Clearly every orbit is invariant.
Next, we show that a basin of attraction is invariant and open.

LEMMA 2.3
Let x^* be an attracting fixed point of a map f. Then $W^s(x^*)$ is an invariant open interval.

PROOF

1. Let $x \in W^s(x^*)$. Then, $f^n(x) \to x^*$ as $n \to \infty$. Now, $f^n(f(x)) = f[f^n(x)]$. Since f is continuous, $f[f^n(x)] \to f(x^*) = x^*$. Thus, $f(x) \in W^s(x^*)$ and, consequently, $W^s(x^*)$ is invariant.

2. Let $a \in W^s(x^*)$ be a left end point. There exists an open interval $(x^* - 2\varepsilon, x^* + 2\varepsilon) \subset W^s(x^*)$. For some $k \in \mathbb{Z}^+$, $|f^k(a) - x^*| < \varepsilon$. Since f^k is continuous, there exists $\delta > 0$ such that $z \in (a - \delta, a + \delta)$ implies $|f^k(z) - f^k(a)| < \varepsilon$. Now, by the triangle inequality, it follows that $|f^k(z) - x^*| \le |f^k(z) - f^k(a)| + |f^k(a) - x^*| < 2\varepsilon$. Thus, $f^k(z) \in W^s(x^*)$ and consequently $z \in W^s(x^*)$, a contradiction to the maximality of $W^s(x^*)$. ∎

2.4 The Schwarzian Derivative

We are still plagued with many unresolved issues concerning periodic attractors of one-dimensional maps. The main question that we are going to address here is, how many periodic attractors can a differentiable map have? In 1978, David Singer [64] more or less answered the above question. The main tool

used in Singer's theorem is the Schwarzian derivative[2] introduced in Chapter 1. In this section all maps are assumed to be at least C^3, i.e., their third derivatives exist and are continuous. Recall from Definition 1.3 that the Schwarzian derivative $Sf(x)$ of a map f at x is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 . \tag{2.1}$$

Example 2.2
For $x \in R$, find the Schwarzian derivative for the sine map $G_\beta(x) = \beta \sin \pi x$. ▯

SOLUTION Now, $G'_\beta(x) = \beta\pi \cos \pi x$, $G''_\beta(x) = -\beta\pi^2 \sin \pi x$, $G'''_\beta(x) = -\beta\pi^3 \cos \pi x$.
Thus,

$$SG_\beta(x) = -\pi^2 - \frac{3}{2}\left(\pi^2 \tan^2 \pi x\right) < 0 \text{ for all } x \in R .$$

Recall from calculus that a point $\tilde{x}$ is a **critical point** of a differential map f if $f'(\tilde{x}) = 0$. ∎

We are now ready to state the main result of this section.

THEOREM 2.4
(Singer's Theorem). *Let $f : I \to I$ be defined on the closed interval I such that $Sf(x) < 0$ [$Sf(x) = -\infty$ is allowed] for all $x \in I$. If f has n critical points in I, then for every $k \in \mathbb{Z}^+$, the map f has at most $(n+2)$ attracting k-cycles.*

The proof of the above theorem depends on the following two lemmas.

LEMMA 2.4
Let a_1, a_2, a_3 be fixed points of a continuously differentiable map g with $a_1 < a_2 < a_3$ and such that $Sg < 0$ on the open interval (a_1, a_3). If $g'(a_2) \le 1$, then g has a critical point in (a_1, a_3).

[2]The Schwarzian derivative is named after its creator, Hermann Schwarz, who introduced it in 1869.

PROOF Since $g(a_1) = a_1, g(a_2) = a_2$, it follows by the mean value theorem that there exists a number $b_1 \in (a_1, a_2)$ such that $g'(b_1) = 1$. Similarly, there exists a number $b_2 \in (a_2, a_3)$ such that $g'(b_2) = 1$ (see Fig. 2.6).

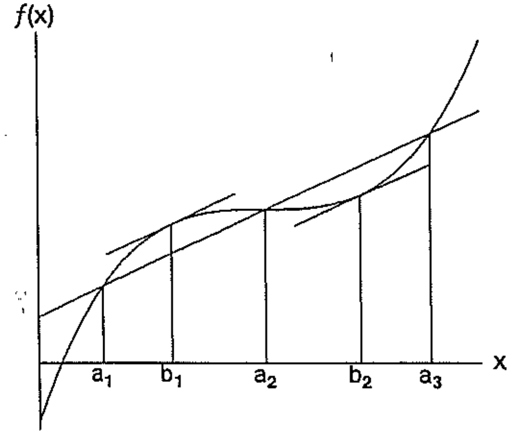

FIGURE 2.6
There exist a number $b_1 \in (a_1, a_2)$ with $g'(b_1) = 1$ and a number $b_2 \in (a_2, a_3)$ with $g'(b_2) = 1$.

Furthermore, since g' is continuous on $[b_1, b_2]$, g' attains its minimum value at some point $c \in [b_1, b_2]$. If c is either b_1 or b_2, then $g'(a_2)$ is a local minimum of g'. Thus, without loss of generality, we may assume that $c \in (b_1, b_2)$ and $g'(c)$ is a local minimum of g'. Hence, $g''(c) = 0$ and $g'''(c) > 0$. Because $Sg(c) = \frac{g'''(c)}{g'(c)} < 0$, it follows that $g'(c) < 0$. By the intermediate value theorem, there is $d \in (b_1, b_2)$ such that $g'(d) = 0$ [since $g'(c) < 0$ and $g'(b_2) > 0$]. The point d is a critical point of g. ∎

LEMMA 2.5
If $Sf < 0$ *and* $Sg < 0$, *then* $S(fog) < 0$.

PROOF Using the chain rule, one may show that

$$(fog)'(x) = (f'(g(x)))g'(x),$$
$$(fog)''(x) = (f''(g(x))) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x) \qquad (2.2)$$

and

$$(fog)'''(x) = (f'''(g(x))) \cdot g'(x)^3 + 3f''(g(x)) \cdot g''(x) \cdot g'(x)$$

$$+ f'(g(x)) \cdot g'''(x) . \qquad (2.3)$$

After some computations (Problem 11), it follows that

$$S(fog)(x) = Sf(g(x)) \cdot [g'(x)]^2 + Sg(x)) . \qquad (2.4)$$

Since $Sf < 0$ and $Sg < 0$, Formula (2.4) implies that $S(fog) < 0$. ∎

COROLLARY 2.1
If $Sg < 0$, *then* $Sg^k < 0$ *for all* $k \in \mathbb{Z}^+$.

PROOF This follows from Lemma 2.5 (Problem 13). ∎

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4 Let $I = [a, b]$, where a may take the value of $-\infty$ and b may be ∞, and let p be an attracting k-periodic point of f. Then, p is an attracting fixed point of $g = f^k$. Let $W^s(p)$ be the basin of attraction of p. If p is in (a, b), then $W^s(p)$ is of the form (c, d), $[a, c)$, or $(d, b]$. Then, for each $x \in W^s(p)$, $g^n(x) = f^{nk}(x) \to p$ as $n \to \infty$. Furthermore, from Lemma 2.3, $g(W^s(p)) \subset W^s(p)$. Assume first that $W^s(p) = (c, d)$. Since g is continuous and $W^s(p)$ is the maximum interval of attraction to p, it follows that g must map (c, d) into itself. However, g will not map the points c or d into (c, d) since both points are not in $W^s(p)$. Hence, we have three cases to consider:

1. $g(c) = c$, and $g(d) = d$
2. $g(c) = d$, and $g(d) = c$
3. $g(c) = g(d)$, $(= c$ or $d)$

Case 1: $g(c) = c$ and $g(d) = d$.
Since p is an attracting fixed point of g, it follows from Definition 1.2 that $g'(p) \leq 1$. By Corollary 2.1, $Sg = Sf^k < 0$. Now, using Lemma 2.4, we conclude that g has a critical point $\tilde{x}$ in the interval (c, d). It remains to show that this implies that f itself has a critical point in (c, d). This may be seen using the chain rule as follows:

$$0 = g'(\tilde{x}) = f'(f^{k-1}(\tilde{x}))f'(f^{k-2}(\tilde{x})) \ldots f'(\tilde{x}) .$$

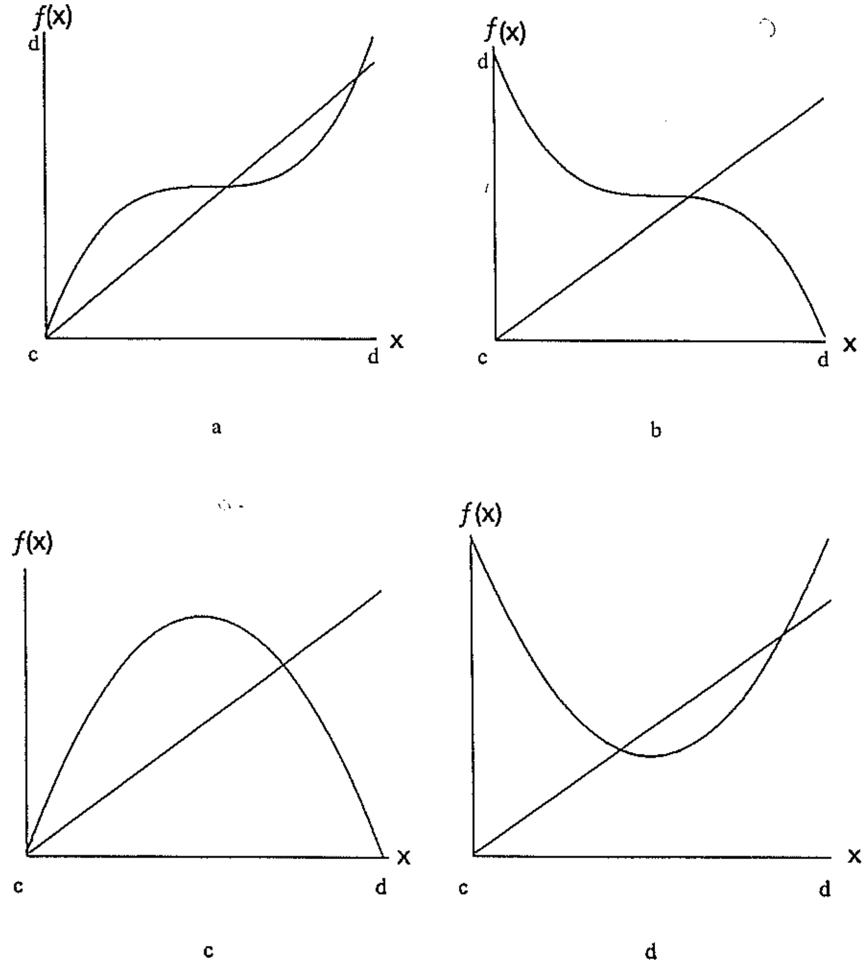

FIGURE 2.7
(**a**) $g(c) = c$, $g(d) = d$, (**b**) $g(c) = d$, $g(d) = c$ (**c**) $g(c) = g(d) = c$, (**d**) $g(c) = g(d) = d$

Thus, either $f'(\tilde{x}) = 0$ or $f'(f(\tilde{x})) = 0$ ... or $f'(f^{k-1}(\tilde{x})) = 0$. But since $f(J) \subset J, \tilde{x}, f(\tilde{x}), f^2(\tilde{x}), \ldots, f^{k-1}(\tilde{x}) \in J$. This implies that indeed $f'(y) = 0$, where y is one of the points $\tilde{x}, f(\tilde{x}), \ldots, f^{k-1}(\tilde{x})$.

Case 2: $g(c) = d$ and $g(d) = c$.

Note that in this case $g^2(c) = c, g^2(d) = d$. Hence, this case reduces to Case 1 (Problem 12).

Case 3: $g(c) = g(d)$.

By the mean value theorem, $g'(\tilde{x}) = 0$ for some $\tilde{x} \in (c, d)$. Then, as in the proof of Case 1, one may show that f has a critical point in (a, b).

In all cases, we have shown that if there are n critical points for f, then f has at most n attracting k-cycles that are associated with finite intervals like (c, d).

Let us now consider the case when $W^s(p) = [a, c)$, then p must attract a. Similarly, if $W^s(p) = (d, b]$, then p must attract b. The last two cases add a maximum of two more possible attracting cycles. Hence, f has at most $(n+2)$ attracting k-cycles. ∎

Remarks about Singer's Theorem

1. The above proof shows that if the basin of attraction $J = (c, d)$ of a periodic point p is bounded, then p must attract a critical point. This remark leads to the solution of the following two examples.

Example 2.3

The map $f(x) = 1 - 2x^2$ on $[-1, 1]$ has no attracting periodic points. To show this, note that 0 is the only critical point. But the orbit of 0 is $0, 1, -1, -1, -1, \ldots$. Thus, -1 is a possible attracting fixed point of f. But, this is impossible because $f'(-1) = 4$ (Theorem 1.3). □

Example 2.4

The logistic map $F_\mu(x) = \mu x(1 - x), 0 < \mu \le 4, x \in [0, 1]$, has at most one attracting cycle. As we saw in Chapter 1, for $0 < \mu \le 1$, 0 is the only attracting fixed point where the region of attraction is [0,1]. For $1 < \mu < 4$, F_μ has only one critical point $\frac{1}{2}$. By Theorem 2.4, there are at most three attracting cycles associated with intervals of the form $[0, c), (c, d)$, and $(d, 1]$ with $0 < c < d < 1$. Since $F'_\mu(0) = \mu > 1$, the fixed point 0 is unstable (Theorem 1.3); therefore, $[0, c)$ cannot be a basin of attraction. Furthermore, $F_\mu(1) = 0$ and hence $(d, 1]$ is not a basin of attraction either. We conclude that there is at most one attracting cycle (c, d) in [0,1] for a given $\mu \in (1, 4]$. Figure 2.8 is part of the bifurcation diagram of F_μ which shows a window with six horizontal curves. We may wonder whether this represents two attracting 3-cycles, one attracting 6-cycle, or another combination. The above example gives us the definite answer, namely, that this is an attracting 6-cycle. Moreover, since there is only one attracting 3-cycle, only one 3-cycle will appear in the bifurcation diagram precisely at window 3 (see Fig. 2.9). □

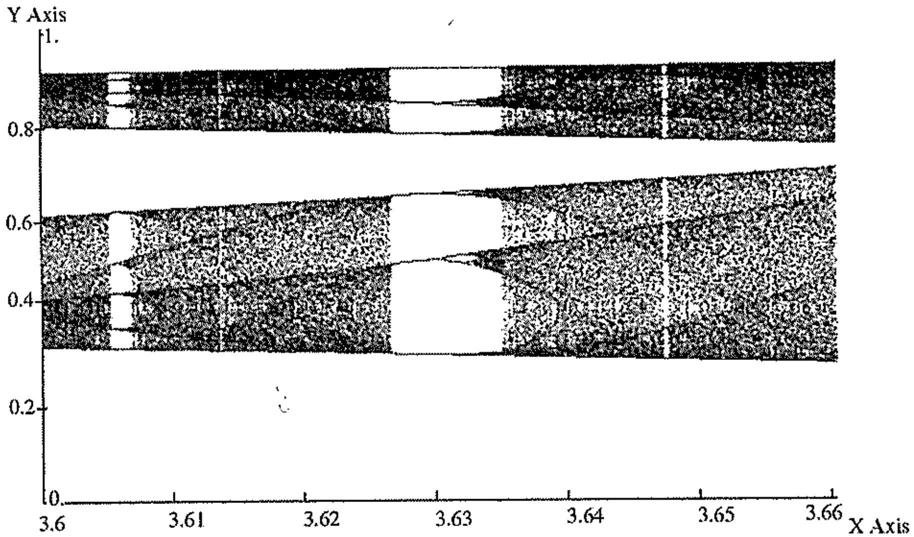

FIGURE 2.8
The appearance of a 6-cycle. (This figure is a second generation computer image.)

2. If the basin of attraction for an attracting cycle is unbounded or contains an end point, then this cycle may not attract a critical point, as the following example shows.

Example 2.5

Let $G_\mu(x) = \mu\tan^{-1}(x)$, $\mu \neq 0$. Then, $G'_\mu(x) = \frac{\mu}{1+x^2}$. Clearly $G_\mu(x)$ has no critical points. Now, if $|\mu| < 1$, then $x^* = 0$ is an asymptotically stable fixed point where the basin of attraction is $(-\infty, \infty)$. While if $\mu > 1$, then G_μ has two attracting fixed points x_1^* and x_2^* with basins of attraction of the form $(-\infty, c)$ and (d, ∞), respectively (see Fig. 2.10). Finally, if $\mu < -1$, then G_μ has an attracting 2-cycle $\{\overline{x}_1, \overline{x}_2\}$ with a basin of attraction of the form $(-\infty, c)(d, \infty)$ (see Fig. 2.11). □

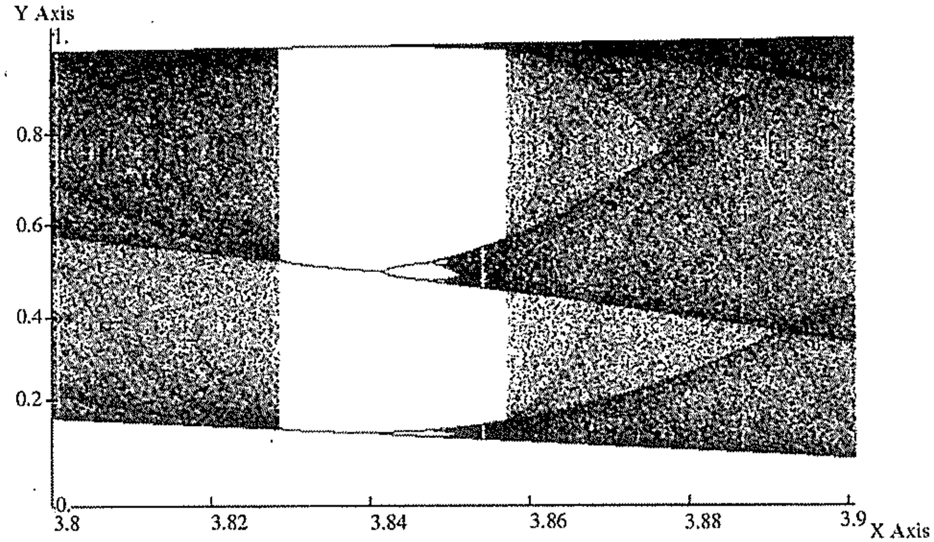

FIGURE 2.9
The appearance of a 3-cycle. (This figure is a second generation computer image.)

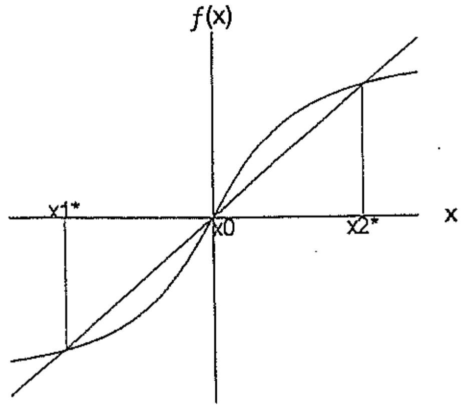

FIGURE 2.10
Basin of attraction of $x_1^* = (-\infty, c)$, basin of attraction of $x_2^* = (d, \infty)$.

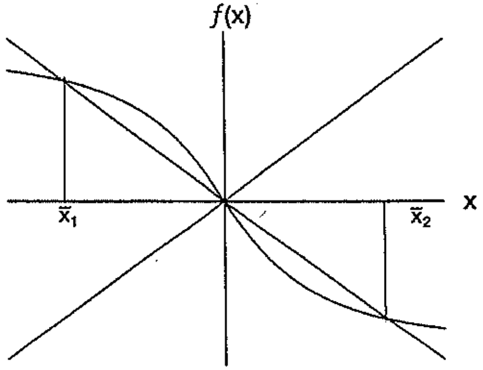

FIGURE 2.11
An attracting 2-cycle $\{\overline{x_1}, \overline{x_2}\}$ **with basin of attraction** $(-\infty, c) \cup (d, \infty)$.

Exercises - (2.3 and 2.4)

1. Give an example of a polynomial that does not have a negative Schwarzian derivative.

2. Give an example of a continuous function f that has an attracting fixed point but $Sf > 0$.

3. Sketch a graph of a continuous function that has two asymptotically stable fixed points.

4. Let f be a continuous map on $\mathbb{R}$. Prove that

   (a) if $f(b) = b$ and $x < f(x) < b$ for all $x \in [a, b)$, then $a \in W(b)$.

   (b) if $f(b) = b$ and $b < f(x) < x$ for all $x \in (b, c]$, then $c \in W(b)$.

5. Let $P(x)$ be a polynomial of degree 4 such that all the roots of $P'(x)$ are real and distinct. Show that $SP < 0$.

6. Show that for any polynomial $P(x)$ of degree at least 2, such that all the roots of $P'(x)$ are real and distinct, $SP < 0$.

7. Show that $Sf < 0$ for $f(x) = x\exp(r(1 - \frac{x}{k}))$, $r, k \in \mathbb{R}$.

8. Show that the logistic map F_μ has no attracting periodic points if $\mu > 2 + \sqrt{5}$.

9. Show that the logistic map $F_4(x) = 4x(1-x)$ has no attracting periodic points.

10. Prove Formula (2.4).

11. Let $G_\mu(x) = \mu \sin x$ for $0 \le x \le \pi$. Determine the maximum possible number of attracting cycles of G_μ for $0 < \mu < \pi$.

12. Let f be a C^3 map. Show that $Sf = 0$ if and only if $f(x) = (ax + b)/(cx + d)$ for some real numbers a, b, c, d.

13. Prove Corollary 2.1.

14. In the proof of Theorem 2.4, give a detailed proof of the existence of a critical point of f in Case 2.4 where $g(c) = d$ and $g(d) = c$.

15. Suppose that $Sf < 0$ for a C^3 map of f. Prove that f' cannot have a positive local minimum or a negative local maximum.

2.5 Bifurcation

In this section, we resume our investigation of the bifurcation phenomena discussed in Sec. 1.8. But before embarking on such a task, we need to explain what bifurcation really means. Roughly speaking, the term bifurcation refers to the phenomenon of a system exhibiting new dynamical behavior as the parameter is varied. As we have seen in Chapter 1 (see Fig. 1.25), the logistic map $F_\mu(x) = \mu x(1-x)$ undergoes a period doubling at an infinite sequence of values of the parameter $\mu : \mu_1, \mu_2, \mu_3, \ldots$, where $\mu_1 = 3$, $\mu_2 = 1 + \sqrt{6}, \ldots$. Note that for the fixed point $x^* = \frac{\mu - 1}{\mu}$, $F'_{\mu_1} = -1$. Similarly, if $\{\overline{x}_1, \overline{x}_2\}$ is the 2-cycle of F_{μ_2} (or the fixed points of $F^2_{\mu_2}$), then $[F^2_{\mu_2}(\overline{x}_i)]' = -1$. This is a trademark of a period-doubling bifurcation that is always associated with the appearance of a slope of -1.

The logistic map F_μ undergoes another important type of bifurcation, commonly called **saddle node** or **tangent** bifurcation. This bifurcation is associated with the appearance of a slope of 1. Now, it can be shown that period 3 appears at $\tilde{\mu} = 1 + \sqrt{8} \approx 3.8284$ (see Saha and Strogatz [59] for an elementary derivation). Figure 2.12a depicts the graph of F^3_μ for $\mu < \tilde{\mu}(\mu = 3.75)$. Here, F^3_μ has only two fixed points that are fake 3-cycles; they are fixed points of F_μ. Figure 2.12b shows how period 3 appears when $\mu = \tilde{\mu}$ at which $F^3_{\tilde{\mu}}$