MATH 105(R)

Examiner: Prof. S.M. Rees, Extension 44063.

Time allowed: Two and a half hours.

Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations or inequalities), and determine whether each one is true.

a) $x^2 = 4 \Rightarrow x \in \mathbb{Z}$.

b) $x \in \mathbb{Q} \wedge x < 1 \Rightarrow \exists y \in \mathbb{Q}$ such that $x < y < 1$ [6 marks]

2. Negate each of the following statements, using logical symbols where possible.

a) $-1 < x < 0$

b) $x \in (0, \pi/2) \Rightarrow \tan x > x$.

[4 marks]

3. Write each of the sets below as either a single interval or a set containing just one element.

a) $((-3,-1) \cup (2,4]) \cap [0,3]$

b) $(0,2) \cup ((1,3) \cap [2,4))$.

c) $([-1,3] \cup [2,3) \cup (6,7]) \cap [3,5)$.

[6 marks]

4. In each of the following, find the set of all $x \in \mathbb{R}$ satisfying the inequalities.

a) $3x^2 > 2x + 1$

b) $\left|2 + \frac{3}{x}\right| \leq 1$.

[8 marks]

5. Prove by induction on n that $n^3 < 4^n$ for all $n \in \mathbb{N}$ with $n \geq 2$. Prove separately that this also holds for $n = 0$ and $n = 1$

[5 marks]

6. Let $a = 567$ and $b = 387$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of a and b;

(ii) integers r and s such that $a = dr$ and $b = ds$;

(iii) integers m and n such that $d = ma + nb$;

(iv) the l.c.m. of a and b.

[10 marks]

7. Write down what it means for a function $f : X \to Y$ to be injective. Define the image of a function $f : X \to Y$, and define what it means for $f : X \to Y$ to be a bijection. Find the images of the following functions, and also determine whether the functions are injective:

a) $f : [0, \infty) \to [0, \infty)$ given by $f(x) = \frac{1}{1+x^2}$;

b) $f : [-\pi/2, \pi/2] \to \mathbb{R}$ given by $f(x) = 2\cos x$.

[10 marks]

8. State the inclusion/exclusion principle for two sets A_1 and A_2.

15 travel companies offer package holidays in Florida or New York. 8 offer holidays in both, and 3 more offer holidays in New York than in Florida. Determine how many offer holidays in New York and how many offer holidays in Florida.

Hint: Write x for the number of companies offering holidays in Florida and write down an equation for x.

[6 marks]

SECTION B

9. An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

(i) Let $X = \mathbb{R}$ and define $x \sim y \Leftrightarrow x - y \geq 0$. Show that $\sim$ is not an equivalence class on X

(ii) Now let $X = (-\infty, 0) \cup (0, \infty)$ and define $x \sim y \Leftrightarrow x/y > 0$. Show that $\sim$ is an equivalence relation on X. Determine the number of equivalence classes and write down a representative of each equivalence class.

(iii) Now let $X = \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$ and define $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ such that $(x_2, y_2) = \lambda(x_1, y_1)$. Assuming that $\sim$ is an equivalence relation (which is true) show that each element of $\{(1, y) : y \in \mathbb{R}\}$ is in a different equivalence class. Show also that there is just one more equivalence class and give an element of it.

[15 marks]

10. Let x_n be defined inductively by $x_0 = 1$ and

$$x_{n+1} = \frac{1 + x_n + x_n^2}{4}$$

(i) Prove by induction that $\frac{1}{3} < x_n \leq 1$ for all $n \in \mathbb{N}$.

(ii) Show that

$$x_{n+1} - x_{n+2} = \frac{(1 + x_n + x_{n+1})(x_n - x_{n+1})}{4},$$

and hence, or otherwise, show by induction that x_n is a decreasing sequence.

(iii) Prove by induction that

$$|x_{n+1} - x_n| \leq \left(\frac{3}{4}\right)^n \frac{1}{4}.$$

[15 marks]

11. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut*. Define also what it means for A Dedekind cut to be *rational* and write down the Dedekind cut which represents the rational number q.

Determine which of the following is a Dedekind cut (if any).

a) $A = \{x \in \mathbb{Q} : -2 < x < \frac{1}{3}\}$;

b) $A = \{x \in \mathbb{Q} : \frac{1}{3} < x\}$;

c) $A = \{x \in \mathbb{Q} : x^2 + 2x + 3 < 0\}$.

Now let A be a Dedekind cut, and define

$$B = \{-x : x \in \mathbb{Q} \wedge x \notin A\}$$

Show from the definition of Dedekind cut for A, that B is bounded above and non-empty.

[15 marks]

12. Define what it means for a set A to be *finite*, *of cardinality* n, and what it means for A to be *countable*

State which of the following sets A, B, C and D are countable and which are uncountable. Complete proofs are not required.

a) $A = \{2n : n \in \mathbb{N}\}$;

b) $B = (0, \infty)$;

c) $C = \mathbb{Z}$;

d) $D = \mathbb{R}^2$.

Find a bijection $f : \mathbb{R} \to (0, \infty)$ and a bijection $g : \mathbb{N} \to \mathbb{Z}$. You should prove that g is a bijection, possibly by showing that it has an inverse function. *Hint*: One way to construct $g : \mathbb{N} \to \mathbb{Z}$ is to map the even natural numbers to the positive integers and the odd natural numbers to the strictly negative integers (for example).

[15 marks]