Some examples of Taylor Series

For all n,

$$f(x) = P_n(x,a) + R_n(x,a).$$

Suppose that

$$\lim_{n\to\infty} R_n(x,a) = 0. \tag{1}$$

Then

$$f(x) = \lim_{n\to\infty} P_n(x,a),$$

that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{2}$$

Now (1), and hence also (2), hold in the following cases.

. $f(x) = e^x$, or $f(x) = \sin x$, or $f(x) = \cos x$, for all x and all a, and so (with $a = 0$) for all x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

- $f(x) = (1+x)^{\alpha}$ for $a = 0$ and $|x| < 1$. So for all $|x| < 1$,

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

- $f(x) = \ln(1+x)$ for $a = 0$ and $-1 < x \leq 1$. So for all $-1 < x \leq 1$,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Properties of Taylor Series

If, for some $r > 0$, and all $|x - a| < r$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n,$$

then for all n

$$a_n = \frac{f^{(}n)(a)}{n!}$$

and

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is the Taylor series of f at a.

Integration

If $P_n(x, a)$ is the Taylor polynomial of f at a,

$$\int_a^x P_n(t, a) dt = \int_a^x \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (t-a)^k dt$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{(k+1)!} (x-a)^{k+1}.$$

So if

$$\lim_{n\to\infty} \int_a^x R_n(t,a)dt = 0,$$

then

$$\int_a^x f(t)dt = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{(k+1)!}(x-a)^{k+1}.$$

This is true, for example, if $|x-a| < R$ and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $|x-a| < R$.