All questions are similar to homework problems. A question similar to 13 will be set for revision.

MATH102 Solutions May 2008
Section A

1. The Taylor series at 2 of

$$f(x) = \ln(x) = \ln(2 + (x-2)) = \ln 2 + \ln(1 + (x-2)/2)$$

is

$$\ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + \cdots = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{2^n n}.$$

This can also be worked out by computing all derivatives of f at $x = 2$.
[3 marks]
a) When $x = 1$ the series is convergent.
[1 mark]
b) When $x = 4$ the series is also convergent.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int \frac{dy}{y^2} = -\int 2x dx,$$

$$-\frac{1}{y} = -x^2 + C.$$

Putting $x = 0$ and $y = 1$ gives $C = -1$. So we obtain

$$y = \frac{1}{1+x^2}.$$

2(ii) Using the integrating factor method, the standard form is

$$\frac{dy}{dx} - \frac{2}{x} y = x.$$

the integrating factor is

$$\exp\left(\int (-2/x) dx\right) = e^{-2\ln x} = x^{-2}.$$

So the equation becomes

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = \frac{d}{dx}(yx^{-2}) = x^{-1}$$

Integrating gives

$$yx^{-2} = \int x^{-1} dx = \ln x + C.$$

So the general solution is

$$y = x^2 \ln x + Cx^2.$$

Putting $y(1) = 1$ gives $1 = C$ and

$$y = x^2 \ln x + x^2.$$

3 marks for (i) 4 marks for (ii).
[7 marks]

3. Try $y = e^{rx}$. Then

$$r^2 + 2r + 5 = 0 \Rightarrow r = -1 \pm 2i.$$

So the general solution is

$$y = e^{-x}(A \cos 2x + B \sin 2x).$$

[2 marks]

So $y' = e^{-x}(-A \cos 2x - B \sin 2x - 2A \sin 2x + 2B \cos 2x)$ and the initial conditions $y(0) = 1$, $y'(0) = 5$ give

$$A = 1, \quad 2B - A = 5 \Rightarrow A = 1, \ B = 3.$$

So

$$y = e^{-x}(\cos 2x + 3 \sin 2x).$$

[3 marks]
[$2 + 3 = 5$ marks]
4. We have, for example,

$$\lim_{(x,y)\to(0,0), x=0} \frac{xy^3}{x^4 + y^4 + x^2y^2} = \lim_{x\to 0} \frac{0}{y^4} = 0,$$

$$\lim_{(x,y)\to(0,0), y=x} \frac{xy^3}{x^4 + y^4 + x^2y^2} = \lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{3}.$$

So the limits along two different lines as $(x, y) \to (0, 0)$ are different, and the overall limit does not exist.
[4 marks]

5.

$$\frac{\partial f}{\partial x} = -2x \sin(x^2 - y^2)$$

$$\frac{\partial f}{\partial y} = 2y \sin(x^2 - y^2),$$

$$\frac{\partial^2 f}{\partial x^2} = -2\sin(x^2 - y^2) - 4x^2\cos(x^2 - y^2),$$
$$\frac{\partial^2 f}{\partial y \partial x} = 4xy\cos(x^2 - y^2),$$
$$\frac{\partial^2 f}{\partial x \partial y} = 4xy\cos(x^2 - y^2)$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = 2\sin(x^2 - y^2) - 4y^2\cos(x^2 - y^2)$$

So we also have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4(x^2 + y^2)\cos(x^2 - y^2) = -4(x^2 + y^2)f.$$

as required.[6 marks]

6. We have

$$\frac{\partial f}{\partial u} = 3u^2 + 3v^2,$$
$$\frac{\partial f}{\partial v} = 6vu - 2v.$$

[2 marks]

By the Chain Rule,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

So

$$\frac{\partial F}{\partial x}(0,0) = \frac{\partial f}{\partial u}(1,2) \times (-2) + \frac{\partial f}{\partial v}(1,2) \times (-1)$$
$$15 \times (-2) + 8 \times (-1) = -38.$$

[3 marks]
$[2 + 3 = 5$ marks]

7. For

$$f(x,y,z) = y^3 - x^2z^2 + 2xyz.$$

we have

$$\nabla f(x,y,z) = (-2xz^2 + 2yz)\mathbf{i} + (3y^2 + 2xz)\mathbf{j} + (2xy - 2x^2z)\mathbf{k}.$$

So

$$\nabla f(2,1,1) = -2\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$$

[2 marks]

The tangent plane at $(2,1,1)$ is

$$\nabla f(2,1,1).((x-2)\mathbf{i}+(y-1)\mathbf{j}+(z-1)\mathbf{k})=0,$$

that is

$$-2(x-2)+7(y-1)-4(z-1)=0$$

or

$$-2x+7y-4z+1=0.$$

[2 marks]
[$2+2=4$ marks.]

8. For

$$f(x,y)=2x^3+9y+6y^2+y^3-3x^2y,$$

we have

$$\frac{\partial f}{\partial x}=6x^2-6xy=6x(x-y),\quad \frac{\partial f}{\partial y}=9+12y+3y^2-3x^2.$$

[2 marks]

So at a stationary point, from the equation for $\frac{\partial f}{\partial x}$ we have $x=0$ or $x=y$ If $x=0$ then the equation for $\frac{\partial f}{\partial y}$ gives

$$3(3+y)(1+y)=0,$$

that is, $y=-3$ or $y=-1$. If $x=y$ then we obtain

$$3+4y=0$$

that is, $x=y=-\frac{3}{4}$. So the stationary points are

$$(0,-3),\ (0,-1),\ \left(-\frac{3}{4},-\frac{3}{4}\right).$$

[3 marks]

$$A=\frac{\partial^2 f}{\partial x^2}=12x-6y,\ B=\frac{\partial^2 f}{\partial y\partial x}=-6x,\ C=\frac{\partial^2 f}{\partial y^2}=12+6y.$$

For $(x,y)=(0,-3)$, $A=18$, $B=0$ and $C=-6$. So $AC-B^2<0$ and $(0,-3)$ is a saddle.

For $(x,y)=(0,-1)$, we have $A=6$, $B=0$, $C=6$. So $AC-B^2>0$, and $A>0$ and $(0,-1)$ is a local min.

For $(x,y)=(-\frac{3}{4},-\frac{3}{4})$, we have $A=-\frac{9}{2}$, $B=\frac{9}{2}$, $C=\frac{15}{2}$. So $AC-B^2<0$, and $(-\frac{3}{4},-\frac{3}{4})$ is a saddle.
[5 marks]
[$2+3+5=10$ marks]

9. For

$$f(x,y) = (2x^2 + y^2)^{1/2},$$

we have

$$\frac{\partial f}{\partial x} = 2x(2x^2 + y^2)^{-1/2}, \quad \frac{\partial f}{\partial y} = y(2x^2 + y^2)^{-1/2}.$$

So

$$f(1,1) = \sqrt{3}, \quad \frac{\partial f}{\partial x}(1,1) = \frac{2}{\sqrt{3}}, \quad \frac{\partial f}{\partial y}(1,1) = \frac{1}{\sqrt{3}}.$$

So the linear approximation is

$$\sqrt{3} + \frac{2}{\sqrt{3}}(x-1) + \frac{1}{\sqrt{3}}(y-1).$$

[It would be acceptable to realise that

$$f(x,y) = (3 + 4(x-1) + 2(x-1)^2 + 2(y-1) + (y-1)^2)^{1/2}$$

$$= \sqrt{3}\left(1 + \frac{4}{3}(x-1) + \frac{2}{3}(y-1) + \frac{2}{3}(x-1)^2 + \frac{1}{3}(y-1)^2\right)^{-1}$$

and to expand out.]
[4 marks]

10. Using polar coordinates, $rdrd\theta = dxdy$ and $x^2 + y^2 = r^2$ This integral can be written as

$$\int_0^{2\pi}\int_0^1 r\sin(\pi r^2)drd\theta = 2\pi\left[-\frac{1}{2\pi}\cos(\pi r^2)\right]_0^1$$

$$-\frac{2\pi}{2\pi}(-1-1) = 2.$$

[5 marks]

Section B

11. (i) The Maclaurin series of f is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

[2 marks]

The fourth Taylor polynomial is

$$x - \frac{x^3}{6}.$$

[1 mark]

(ii) We have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = f(x) = \sin x$ and $f^{(5)}(x) = f'(x) = \cos x$ So the remainder term

$$R_4(x, 0) = (\cos c)\frac{x^5}{5!}$$

for some c between 0 and x. Since $|\cos c| \leq 1$ for all c we obtain

$$|R_4(x, 0)| \leq \frac{|x|^5}{5!}.$$

[3 marks.] So

$$\left|\int_0^1 R_4(t^2, 0)dt\right| \leq \int_0^1 |R_4(t^2, 0)|dt$$
$$\leq \int_0^1 \frac{t^{10}}{120}dt = \left[\frac{t^{11}}{1320}\right]_0^1 = \frac{1}{1320}.$$

[3 marks]
(iii) We have

$$\sin(x^2) = P_4(x^2, 0) + R_4(x^2, 0),$$

and so

$$\int_0^1 \sin(x^2)dx = \int_0^1 P_4(x^2, 0)dx + \int_0^1 R_4(x^2, 0)dx.$$

Now

$$\int_0^1 P_4(x^2, 0) = \int_0^1 \left(x^2 - \frac{x^6}{6}\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{42}\right]_0^1$$
$$= \frac{1}{3} - \frac{1}{42} = \frac{13}{42} = 0.3095$$

to 4 decimal places. Since $\frac{1}{1320} = 0.001$ to 3 decimal places, we have

$$\int_0^1 \sin(x^2)dx = 0.31$$

to 2 decimal places.
[6 marks]
$2 + 1 + 3 + 3 + 6 = 15$ marks.

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 4 = (r + 2i)(r - 2i) = 0,$$

that is, $r = \pm 2i$. So the complementary solution is

$$A'e^{2ix} + B'e^{-2ix} = A\cos 2x + B\sin 2x$$

for suitable constants A and B [3 marks]

(i) We try $y_p = Cx^2 + Dx + E$. Then $y_p' = 2Cx + D$ and $y_p'' = 2C$. So $y_p'' + 4y_p = 2C + 4(Cx^2 + Dx + E)$. So

$$2C + 4E = 0, \quad 4D = -4, \quad 4C = 8.$$

So

$$D = -1, \quad C = 2, \quad E = -1.$$

So the general solution is

$$y = A\cos 2x + B\sin 2x + 2x^2 - x - 1.$$

[3 marks]

This gives

$$y' = -2A\sin 2x + 2B\cos 2x + 4x - 1.$$

So putting $x = 0$, the boundary conditions give

$$A - 1 = 2, \quad 2B - 1 = 1 \quad \Rightarrow \quad A = 3, \quad B = 1$$

So the solution is

$$y = 3\cos 2x + \sin 2x + 2x^2 - x - 1.$$

[3 marks]

(ii) We try $y_p = C\cos x + D\sin x$. Then $y_p'(x) = -C\sin x + D\cos x$ and $y_p'' = -C\cos x - D\sin x$. So

$$y_p'' + 4y_p = 3C\cos x + 3D\sin x.$$

Comparing coefficients, we obtain

$$C = \frac{1}{3}, \quad D = \frac{1}{3}.$$

So the general solution is

$$A\cos 2x + B\sin 2x + \frac{1}{3}(\cos x + \sin x).$$

[3 marks] This gives

$$y'(x) = -2A\sin 2x + 2B\cos 2x - \frac{1}{3}\sin x + \frac{1}{3}\cos x.$$

So putting $x = 0$, the boundary conditions give

$$A + \frac{1}{3} = 1, \quad 2B + \frac{1}{3} = 3 \Rightarrow A = \frac{2}{3}, \quad B = \frac{4}{3}.$$

So

$$y = \frac{2}{3}\cos 2x + \frac{4}{3}\sin 2x + \frac{1}{3}(\cos x + \sin x).$$

[3 marks]

[3 + 3 + 3 + 3 + 3 = 15 marks]

13. We have

$$f(x, y, t) = (x - t)^2 + (y - 2t)^2.$$

and

$$g(x, y) = x^2 - y^2$$

We want to minimise $\sqrt{f}$ subject to $g = 1$. This is the same as minimising f subject to $g = 1$.

We have

$$\nabla f = 2(x - t)\mathbf{i} + 2(y - 2t)\mathbf{j} - (2(x - t) + 4(y - 2t))\mathbf{k}$$
$$\nabla g = 2x\mathbf{i} - 2y\mathbf{j}.$$

[3 marks]

At a minimum of f subject to $g = 1$ we have

$$\nabla f = \lambda \nabla g,$$

[1 mark]

that is,

$$\begin{array}{lll} 2(x - t) & = & 2x\lambda \\ 2(y - 2t) & = & -2y\lambda \\ 2(x - t) + 4(y - 2t) & = & 0 \end{array}$$

From the third equation we obtain

$$5t = x + 2y.$$

Then the first two equations can be rewritten as

$$\begin{array}{lll} 5x - x - 2y & = 4x - 2y & = 5x\lambda \\ 5y - 2x - 4y & = y - 2x & = -5y\lambda \end{array}$$

So multiplying the first equation by y and the second by x and adding, we obtain

$$y(4x - 2y) + x(y - 2x) = -2y^2 + 5xy - 2x^2 = 0$$

[6 marks]

So

$$2x^2 - 5xy + 2y^2 = (2x - y)(x - 2y) = 0.$$

Combining with $g = x^2 - y^2 = 1$, if $y = 2x$ we obtain $-3x^2 = 1$, which is impossible. So we must have $x = 2y$ which yields $3y^2 = 1$ and $y = \pm 1/\sqrt{3}$. So we have

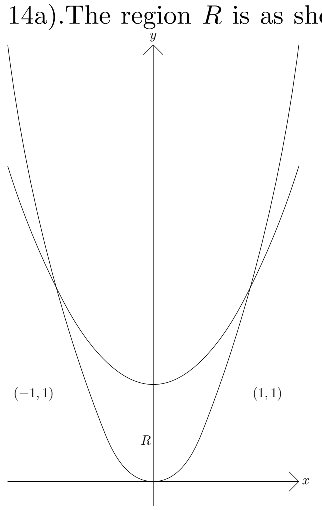
$$(x, y, t) = \pm \frac{1}{\sqrt{3}} \left(2, 1, \frac{4}{5} \right)$$

At both these points

$$f = \frac{1}{3}\left(\frac{36}{25} + \frac{9}{25}\right) = \frac{9}{15} = \frac{3}{5}.$$

So the minimum distance is $\sqrt{3/5}$.
[5 marks]
[$1 + 3 + 6 + 5 = 15$ marks]

14a).The region R is as shown. The two parabolas cross at the points $(\pm 1, 1)$



[3 marks]
14b) The area is

$$\int_{-1}^{1}\int_{2x^2}^{x^2+1} dydx = \int_{-1}^{1}(1 - x^2)dx$$
$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

[4 marks]
14c) By symmetry $\overline{x} = 0$. This answer will be accepted If this needs any confirmation

$$\overline{x} = \frac{3}{4}\int_R xdydx = \frac{3}{4}\int_{-1}^{1} x \int_{2x^2}^{x^2+1} dydx$$
$$= \frac{3}{4}\int_{-1}^{1}(x - x^3)dx$$

Since the integrand is odd, it is clear that the integral will be 0.
[2 marks]
For $\overline{y}$,

$$\overline{y} = \frac{3}{4}\int_R ydydx = \frac{3}{4}\int_{-1}^{1}\int_{2x^2}^{x^2+1} ydydx$$

$$= \frac{3}{4}\int_{-1}^{1}\left[\frac{y^2}{2}\right]_{2x^2}^{x^2+1} = \frac{3}{4}\int_{-1}^{1}\frac{1}{2}(x^4+2x^2+1-4x^4)dx$$

$$= \frac{3}{8}\left[-\frac{3}{5}x^5+\frac{2}{3}x^3+x\right]_{-1}^{1} = \frac{3}{4}\left(-\frac{3}{5}+\frac{5}{3}\right)$$

$$= \frac{4}{5}$$

[6 marks]
[$3+4+2+6=15$ marks.]