SECTION A

1. Write down the Taylor series about $x = 1$ for the function

$$f(x) = \ln x.$$

State whether this Taylor series is equal to $f(x)$ for:

a) $x = 3$, b) $x = 1.5$. [5 marks]

2. Find the general solutions of the following differential equations:

(i) $x\dfrac{dy}{dx} + 2y = 0$,

(ii) $x\dfrac{dy}{dx} + 2y = x$.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 2$.

[6 marks]

4. Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$$

does not exist.

[4 marks]

5. For

$$f(x, y) = e^{x^2 - y^2} \sin(2xy),$$

verify that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[9 marks]

6. Find the tangent plane and normal line at the point $(1,1,1)$ to the surface

$$2x^2y + y^2z - xz^2 = 2.$$

[6 marks]

7. Locate and classify all stationary points of the function

$$f(x,y) = 2y^3 - 6xy + x^2.$$

[8 marks]

8. Find the linear approximation near $(x,y) = (1,0)$ to the function

$$f(x,y) = \ln(x^2 + y^2).$$

[4 marks]

9. Let T be the triangle in the plane bounded by the lines

$$y = 0, \quad x = 0, \quad x + y = 1.$$

Work out the double integral

$$\int\int_T (x - y)dxdy.$$

You may change the order of integration if you prefer.

[6 marks]

SECTION B

10. Let $f(y) = (1+y)^{-1/2}$. Find the Taylor polynomial $P_2(y)$ near $y = 0$ and the associated error term $R_2(y)$. Hence, or otherwise, show that, for any x,

$$\left|(1+x^2)^{-1/2} - P_2(x^2)\right| \leq \frac{5x^6}{16}.$$

Now work out

$$\int_0^{1/3} P_2(x^2)dx$$

and compare the answer with $\ln((1+\sqrt{10})/3)$, worked out on your calculator. Explain why you expect the two answers to be close, by working out

$$\frac{d}{dx}(\ln(x+\sqrt{1+x^2}))$$

or otherwise.

[15 marks]

11. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' + y = x$$

with $y(0) = 1$, $y'(0) = -1$,

(ii)

$$y'' + y = e^x$$

with $y(0) = 1$, $y'(0) = -1$.

[15 marks]

12. Find the minimum distance from the origin $(0,0,0)$ to the surface

$$g(x,y,z) = 4xy - 3y^2 + 2z^2 = 1$$

[*Hint*: consider the function $f(x,y,z) = x^2 + y^2 + z^2$.]

[15 marks]

13. Find the centre of mass of the plane triangle bounded by the lines

$$x = 0, \quad y = 0, \quad 2x + y = 1,$$

where mass is uniformly distributed.

[15 marks]