UNIVERSITY OF LIVERPOOL

MATH102

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Two and a half hours

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

SECTION A

1. Write down the Taylor series about $x = 2$ for the function

$$f(x) = \ln x.$$

State whether this Taylor series converges to $f(x)$ for:

a) $x = 1$, b) $x = 4$. [5 marks]

2. Find the solutions of the following differential equations:

(i) $\dfrac{dy}{dx} + 2xy^2 = 0$ with $y(0) = 1$,

(ii) $x\dfrac{dy}{dx} - 2y = x^2$ with $y(1) = 1$.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 5$.

[5 marks]

4. Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^4 + x^2y^2}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x,y) = \cos(x^2 - y^2)$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 4(x^2 + y^2)f = 0.$$

[6 marks]

6. Suppose that $u = u(x,y)$ and $v = v(x,y)$ are functions of (x,y) such that

$$u(0,0) = 1, \quad v(0,0) = 2$$

and

$$\frac{\partial u}{\partial x}(0,0), = -2, \quad \frac{\partial v}{\partial x}(0,0) = -1.$$

Then work out

$$\frac{\partial F}{\partial x}(0,0)$$

where $F(x,y) = f(u,v)$, and

$$f(u,v) = u^3 + 3v^2u - v^2.$$

[5 marks]

7. Find the gradient vector $\nabla f(2,1,1)$, where

$$f(x,y,z) = y^3 - x^2z^2 + 2xyz.$$

Find also the tangent plane to the surface $f(x,y,z) = 1$ at the point $(2,1,1)$.

[4 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = 2x^3 + 9y + 6y^2 + y^3 - 3x^2y.$$

[10 marks]

9. Find the linear approximation near $(x, y) = (1, 1)$ to the function

$$f(x, y) = (2x^2 + y^2)^{1/2}$$

[4 marks]

10. By changing to polar coordinates, compute

$$\int\int_D \sin(\pi(x^2 + y^2))dxdy$$

where

$$D = \{(x, y) : x^2 + y^2 \leq 1\}.$$

[5 marks]

SECTION B

11.

(i) Write down the Maclaurin series of $f(x) = \sin x$, and the fourth Taylor polynomial $P_4(x, 0)$ at 0.

(ii) Give an expression for the remainder term $R_4(x, 0)$ of f at 0. Show that it satisfies

$$|R_4(x, 0)| \leq \frac{|x|^5}{5!}.$$

Hence show that if $x \geq 0$,

$$\int_0^1 R_4(t^2, 0)dt \leq \frac{1}{1320}.$$

(iii) Hence, or otherwise, compute

$$\int_0^1 \sin(x^2)dx$$

to 2 decimal places.

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' + 4y = 8x^2 - 4x$$

with $y(0) = 2$, $y'(0) = 1$.

(ii)

$$y'' + 4y = \cos x + \sin x$$

with $y(0) = 1$, $y'(0) = 3$.

[15 marks]

13. Find the minimum distance from the line

$$\underline{r}(t) = (t, 2t)$$

to the hyperbola

$$g(x, y) = x^2 - y^2 = 1.$$

Hint: You may assume that the square of the distance from $\underline{r}(t)$ to the point (x, y) is

$$f(x, y, t) = (x - t)^2 + (y - 2t)^2.$$

Try to show that the minimum must occur at a point (x, y) on the hyperbola satisfying either $2x - y = 0$ or $x - 2y = 0$, and discount one of these. [15 marks]

14.

a) Sketch the region R bounded by the two parabolas $y = 2x^2$ and $y = x^2 + 1$, indicating clearly where the parabolas cross.

b) Find the area of R.

c) Find the centroid $(\overline{x}, \overline{y})$ of R.

[15 marks]