Solutions to MATH102 Practice Exam

Section A

1. The Taylor series of $f(x) = e^{2x}$ is

$$1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

This series is convergent for all x
[4 marks]
2 a)

$$\int \frac{dy}{y} = \int \frac{(x+2)}{x^2} = \int \left(\frac{1}{x} + \frac{2}{x^2}\right) dx,$$

or

$$\ln|y| = \ln|x| - \frac{2}{x} + C.$$

Another way of writing this is

$$y = Axe^{-2/x}.$$

[4 marks]
b) The integrating factor is

$$\exp(-1dx) = e^{-x}.$$

So

$$\frac{d}{dx}(ye^{-x}) = e^{-2x}.$$

So

$$ye^{-x} = -\frac{1}{2}e^{-2x} + C,$$

or

$$y = -\frac{1}{2}e^{-x} + Ce^{x}.$$

[4 marks]
It is also possible to solve this by finding particular and complementary solutions.
3. Trying $y = e^{rx}$, we get

$$r^2 - 2r - 15 = 0 \Rightarrow (r+3)(r-5) = 0 \Rightarrow r = -3 \text{ or } r = 5.$$

[2 marks]
So the general solution is $y = Ae^{-3x} + Be^{5x}$, which gives $y' = -3e^{-3x} + 5Be^{5x}$. So $A + B = 0$ and $5B - 3A = 8B = 4$. So $B = \frac{1}{2}$ and $A = -\frac{1}{2}$. So the solution is

$$-\frac{1}{2}e^{-3x} + \frac{1}{2}e^{5x}.$$

[4 marks]
4. We have

$$\lim_{(x,y)\to 0, y=0} \frac{x^2+y^2}{x^2+2y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1,$$
$$\lim_{(x,y)\to 0, x=0} \frac{x^2+y^2}{x^2+2y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

So the limits along the axes are different and the overall limit does not exist.
[4 marks]
5.

$$\frac{\partial f}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial f}{\partial y} = -12x^2y + 4y^3,$$

[2 marks]

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 12y^2, \quad \frac{\partial^2 f}{\partial y \partial x} = -24xy,$$
$$\frac{\partial^2 f}{\partial y^2} = -12x^2 + 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -24xy.$$

[3 marks]
So we do indeed have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -24xy$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} =$$
$$12x^2 - 12y^2 - 12x^2 + 12y^2 = 0.$$

[1 mark]
6.

$$\frac{\partial z}{\partial x} = y + \frac{y}{x}, \quad \frac{\partial z}{\partial y} = x + \ln x.$$

We have $x(1,2) = 1$ and $y(1,2) = 1$. Then

$$\frac{\partial z}{\partial x}(1,1) = 1 + 1 = 2, \quad \frac{\partial z}{\partial y}(1,1) = 1 + \ln 1 = 1.$$

[2 marks]
We are given that

$$\frac{\partial x}{\partial u}(1,2) = 1, \quad \frac{\partial x}{\partial v}(1,2) = 2, \quad \frac{\partial y}{\partial u} = -1, \quad \frac{\partial y}{\partial v} = 0.$$

So

$$\frac{\partial z}{\partial u}(1,2) = \frac{\partial z}{\partial x}(1,1)\frac{\partial x}{\partial u}(1,2) + \frac{\partial z}{\partial y}(1,1)\frac{\partial y}{\partial u}(1,2)$$
$$= 2 \times 1 + 1 \times -1 = 1,$$

[2 marks]

$$\frac{\partial z}{\partial v}(1,2) = \frac{\partial z}{\partial x}(1,1)\frac{\partial x}{\partial v}(1,2) + \frac{\partial z}{\partial y}(1,1)\frac{\partial y}{\partial v}(1,2)$$
$$= 2 \times 2 + 1 \times 0 = 4.$$

[2 marks]
7.

$$\nabla f(x,y,z) = (2x - z)\mathbf{i} - 2y\mathbf{j} - x\mathbf{k}.$$

So

$$\nabla f(1,-1,2) = 0\mathbf{i} + 2\mathbf{j} - \mathbf{k} = 2\mathbf{j} - \mathbf{k}.$$

Now

$$\|2\mathbf{i} + \mathbf{j} + 2\mathbf{k}\| = \sqrt{2^2 + 1 + 2^2} = \sqrt{9} = 3.$$

So the derivative in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is

$$\frac{1}{3}\nabla \mathbf{f}(1,-1,2).(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

$$= \frac{1}{3}(2\mathbf{j} - \mathbf{k}).(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{1}{3} \times (2 - 2) = 0.$$

[4 marks]
8.

$$\frac{\partial f}{\partial x} = 5y^2 - 16x, \quad \frac{\partial f}{\partial y} = 10xy - 18y = 2y(5x - 9).$$

So for a stationary point we must have $y = 0$ or $x = 9/5$. If $y = 0$ then $\partial f/\partial x = 0$ gives $x = 0$. If $x = 9/5$ then $\partial f/\partial x = 0$ gives $y = \pm 12/5$. So the critical points are $(0,0)$ and $(9/5, \pm 12/5)$.
[4 points]
Now

$$\frac{\partial^2 f}{\partial x^2} = -16, \quad \frac{\partial^2 f}{\partial y \partial x} = 10y, \quad \frac{\partial^2 f}{\partial y^2} = 10x - 18.$$

So at $(x,y) = (0,0)$ we have $A = -16$, $B = 0$, $C = -18$. So $AC - B^2 > 0$ and $A < 0$ and $(0,0)$ is a maximum. At $(x,y) = (9/5, \pm 12/5)$ we have $C = 0$, $B = \pm 24$ and $AC - B^2 < 0$. So both these points are saddles.
[4 points]
9. We have $f(x,y) = (x^2 + y^2)^{1/2}$.

$$\frac{\partial f}{\partial x} = x(x^2 + y^2)^{-1/2}, \quad \frac{\partial f}{\partial y} = y(x^2 + y^2)^{-1/2}.$$

So

$$f(1,0) = 1, \ \frac{\partial f}{\partial x}(1,0) = 1, \ \frac{\partial f}{\partial y}(1,0) = 0.$$

So for (x,y) near $(1,0)$ we have

$$f(x,y) \approx 1 + (x - 1) = x.$$

[4 marks]
10. We are integrating over the region

$$\{(x,y) : 0 \le x \le 1,\ x \le y \le 1\}.$$

This can equally well be described as

$$\{(x,y) : 0 \le y \le 1,\ 0 \le x \le y\}.$$

[2 marks]
So

$$\int_0^1 \int_x^1 \frac{1}{y} \cos(\pi x/(2y)) dy dx = \int_0^1 \int_0^y \frac{1}{y} \cos(\pi x/(2y)) dx dy$$
$$= \int_0^1 \left[\frac{2}{\pi} \sin(\pi x/(2y)) \right]_0^y = \int_0^1 \frac{2}{\pi} (\sin(\pi/2) - 0) dx = \frac{2}{\pi}.$$

Section B

11a) We have $f'(y) = e^y = f(y)$. So $f^{(k)}(y) = e^y$ for all k and $f^k(0) = 1$ for all k. So

$$P_2(y,0) = 1 + y + \frac{y^2}{2},$$

$$R_2(y,0) = e^c \frac{y^3}{6}$$

for some c between 0 and y,

$$P_9(y,0) = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{4!} + \frac{y^5}{5!}$$
$$+ \frac{y^6}{6!} + \frac{y^7}{7!} + \frac{y^8}{8!} + \frac{y^9}{9!}$$

and

$$R_9(y,0) = e^c \frac{y^{10}}{10!}$$

for some c between 0 and y.
[6 marks]
If $y = -x$ for $x \ge 0$ and c is between 0 and $-x$ then $c \le 0$ and $0 < e^c \le 1$. So since

$$e^{-x} - P_2(-x,0) = R_2(-x,0),$$

$$|e^{-x} - P_2(-x,0)| \le \frac{x^3}{6}$$

and similarly

$$|e^{-x} - P_9(-x,0)| \le \frac{x^{10}}{10!}$$

If $0 \leq x \leq \frac{1}{2}$ we obtain

$$|e^{-x} - P_2(-x,0)| \leq \frac{(\frac{1}{2})^3}{6} = \frac{1}{48}$$

and if $0 \leq x \leq 2$ we obtain

$$|e^{-x} - P_9(-x,0)| \leq \frac{2^{10}}{10!} = \frac{2^{10}}{10!} = \frac{1024}{5040 \times 720} = \frac{1024}{3628800} < .0003$$

[4 marks]
b) The Taylor series for e^x is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \cdots$$

The Taylor series for e^{-x} is

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$$

The Taylor series for e^x is convergent for all x, and equal to e^x, respectively for all x. [3 marks] Since the Taylor series for e^{-x} is simply obtained by substituting $-x$ for x, the same is true for e^{-x}. So

$$\lim_{x\to 0} \frac{1 - x + \frac{x^2}{2} - e^{-x}}{1 + x + \frac{x^2}{2} - e^x} =$$

$$\lim_{x\to 0} \frac{1 - x + \frac{x^2}{2} - (1 - x + \frac{x^2}{2} - \frac{x^3}{3!} \cdots)}{1 + x + \frac{x^2}{2} - (1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \cdots)}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3!}}{-\frac{x^3}{3!}} = -1.$$

[2 marks]

It is also possible to prove this using l'Hopital's Rule.

12a) Making the substitution $y = vx$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

So

$$v + x\frac{dv}{dx} = \frac{xy}{x^2 - y^2} = \frac{v}{1 - v^2}.$$

So

$$x\frac{dv}{dx} = \frac{v - (v - v^3)}{1 - v^2} = \frac{v^3}{1 - v^2}.$$

So

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

and

$$-\frac{v^{-2}}{2} - \ln|v| = \ln|x| + C.$$

So

$$-\frac{x^2}{2y^2} - \ln|y| + \ln|x| = \ln|x| + C.$$

So

$$-\frac{x^2}{2y^2} - \ln|y| = C.$$

[6 marks]
12b) Trying $y = e^{rx}$ for the complementary solution, we have

$$(r^2 + 2r - 3 = 0 \Leftrightarrow (r+3)(r-1) = 0 \Leftrightarrow r = 1 \text{ or } r = -3.$$

So the complementary solution is $y = Ae^x + Be^{-3x}$.
[2 marks]

For the particular soution we try $y = C\cos x + D\sin x$. So $y' = -C\sin x + D\cos x$ and $y'' = -C\cos x - D\sin x$ and

$$y'' + 2y' - 3y = (-4C + 2D)\cos x + (-4D - 2C)\sin x = \cos x.$$

So equating coeffiients of $\sin x$, $C = -2D$ and equating coefficients of $\cos x$, $D = \frac{1}{10}$. So

$$y(x) = -\frac{1}{5}\cos x + \frac{1}{10}\sin x + Ae^x + Be^{-3x}.$$

So

$$y'(x) = \frac{1}{5}\sin x + \frac{1}{10}\cos x + Ae^x - 3Be^{-3x}.$$

[4 marks] So

$$1 = \frac{-1}{5} + A + B, \quad -1 = \frac{1}{10} + A - 3B.$$

Subtracting these,

$$2 = -\frac{3}{10} + 4B \Rightarrow B = \frac{23}{40}$$

So

$$A = \frac{6}{5} - \frac{23}{40} = \frac{25}{40} = \frac{5}{8}.$$

So

$$y = \frac{5}{8}e^x + \frac{23}{40}e^{-3x} - \frac{1}{5}\cos x + \frac{1}{10}\sin x.$$

[3 marks]
13.

a) The area of the triangle is

$$\frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(2x \times (1-y)) = x - xy = f(x,y).$$

[2 marks]

We want to maximise $f(x,y)$ subject to $g(x,y)=1$. We have

$$\nabla f(x,y) = (1-y)\mathbf{i} - x\mathbf{j},$$

$$\nabla g(x,y) = \frac{x}{2}\mathbf{i} + 2y\mathbf{j}.$$

So Lagrange equations become

$$1-y = \frac{\lambda}{2}x, \quad -x = 2\lambda y.$$

This gives

$$4y - 4y^2 + x^2 = 0.$$

Combining with $4g(x,y) = x^2+4y^2 = 4$, this gives

$$4+4y-8y^2 = 0 \Rightarrow (1-y)(1+2y) = 0 \Rightarrow y = 1 \text{ or } y = \frac{-1}{2}.$$

If $y=1$ we have $x=0$ and the area is 0. If $y=-\frac{1}{2}$ then $x=\pm\sqrt{3}$ and the area is $\frac{1}{2}3\sqrt{3}$.

[6 marks]

b) We have

$$\nabla f(x,y) = 2x\mathbf{i} - 4y\mathbf{j} = \underline{0} \Leftrightarrow x = y = 0.$$

Clearly $f(0,0)=0$.

[2 marks]

Let $g(x,y) = x^2+y^2$. The maximum and minimum of f in the region $g(x,y) \le 1$ either occurs at $(0,0)$ or on the boundary $g(x,y)=1$. We have

$$\nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]

A maximum or minum on the boundary must occur at a point (x,y) where

$$\nabla f(x,y) = \lambda\mathbf{nabla}g(x,y),$$

that is,

$$2x = 2\lambda x, \quad 2y = -4\lambda y.$$

Mutliplying the first equation by $2y$ and the second by x, we obtain $xy=0$. If $x=0$ we have $y=\pm1$ and if $y=0$ we have $x=\pm1$. Checking the values of f at $(\pm1,0)$, $(0,\pm1)$, $(0,0,$ we see that the maximum value 1 of f occurs at $(\pm1,0)$, the minimum value -2 occurs at $(0,\pm1)$. We can also compute that $(0,0)$ is a saddle point and so cannot be a maximum or minimum but we do not need to do this.

We do not have to use Lagrange' method. Alternatively we can say that because $y^2 = 1-x^2$ when $g(x,y)=1$, we have $f(x,y) = 3x^2-2$ when $g(x,y)=1$. On this set we have $-1 \le x \le 1$, so we need to find the maximum and minimum of $3x^2-2$ on the set of x where $-1 \leq x \leq 1$. The maximum and minimum can occur at $x=\pm 1$, of or the stationary point $x=0$ f $3x^2-2$. The points $x=\pm 1$ give the maximum value 1 and $x=\pm 1$ give the minimum value -2, the same answers as the other method.

[5 marks]

14.

The centre of mass is $(\overline{xy})$ where

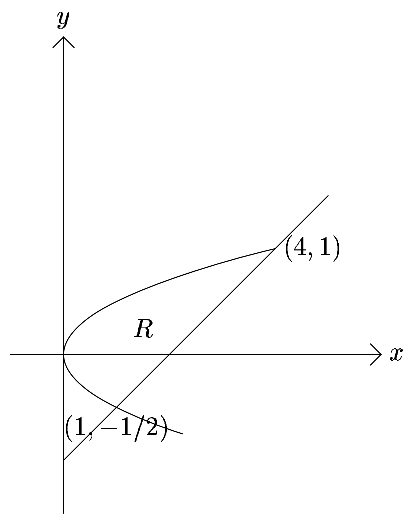
$$A = \int\int_R dxdy,$$

$$\overline{x} = \frac{1}{A}\int\int_R xdxdy, \quad \overline{y} = \frac{1}{A}\int\int_R ydxdy.$$

We have

$$4y^2 = 2y+2 \Leftrightarrow 2y^2-y-1=0 \Leftrightarrow (2y+1)(y-1)=0.$$

So a sketch of the region R is as shown.



[3 marks]

The area A is

$$\int_{-1/2}^{1}\int_{4y^2}^{2y+2} dxdy = \int_{-1/2}^{1}(2y+2-4y^2)dy$$

$$= \left[y^2+2y-\frac{4y^3}{3}\right]_{-1/2}^{1} = 1+2-\frac{4}{3}-\frac{1}{4}+1-\frac{1}{6}=\frac{9}{4}$$

[4 marks]

14b) Then

$$\overline{x} = \frac{1}{A}\int_{-1/2}^{1}\int_{4y^2}^{2y+2} xdxdy$$

$$= \frac{4}{9}\int_{-1/2}^{1}\left[\frac{x^2}{2}\right]_{4y^2}^{2y+2} dy = \frac{2}{9}\int_{-1/2}^{1}(4y^2+8y+4-16y^4)dy$$

$$= \frac{2}{9}\left[\frac{4y^3}{3}+4y^2+4y-\frac{16y^5}{5}\right]_{-1/2}^{1} = \frac{2}{9}\left(\frac{4}{3}+4+4-\frac{16}{5}+\frac{1}{6}-1+2-\frac{1}{10}\right)$$

$$= \frac{2}{9}.\frac{36}{5} = \frac{8}{5}$$

[3 marks]

$$\overline{y} = \frac{1}{A}\int_{-1/2}^{1}\int_{4y^2}^{2y+2} ydxdy$$

$$= \frac{4}{9}\int_{-1/2}^{1}(2y^2+2y-4y^3)dy = \frac{4}{9}\left[\frac{2y^3}{3}+y^2-y^4\right]_{-1/2}^{1}$$

$$= \frac{4}{9}\left(\frac{2}{3}+1-1+\frac{1}{12}-\frac{1}{4}+\frac{1}{16}\right) = \frac{4}{9}.\frac{9}{16} = \frac{1}{4}.$$

So

$$(\overline{xy}) = \left(\frac{8}{5},\frac{1}{4}\right).$$

[5 marks]

[3 + 4 + 3 + 5 = 15 marks.]