SECTION A

1. Write down the Taylor series of $f(x) = e^{2x}$ about $x = 0$. State for which x this is convergent.

[4 marks]

2. Find the general solutions of the differential equations

a) $x^2 \dfrac{dy}{dx} = (x+2)y$,

b) $\dfrac{dy}{dx} - y = e^{-x}$.

[8 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0$$

subject to $y(0) = 0$, $y'(0) = 4$.

[6 marks]

4. Show, using two different paths through the origin, that

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+2y^2}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x,y) = x^4 - 6x^2y^2 + y^4.$$

verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[6 marks]

6. Find $\dfrac{\partial z}{\partial u}(1,2)$ and $\dfrac{\partial z}{\partial v}(1,2)$ where $z = xy + y\ln x$ and

$$x(1,2) = 1, \quad y(1,2) = 1,$$

$$\frac{\partial x}{\partial u}(1,2) = 1 \ \frac{\partial y}{\partial u}(1,2) = -1, \quad \frac{\partial x}{\partial v}(1,2) = 2, \frac{\partial y}{\partial v}(1,2) = 0.$$

[6 marks]

7. Given that $f(x,y,z) = x^2 - y^2 - zx$, find the directional derivative of f at $(1,-1,2)$ in the direction $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

[4 marks]

8. Locate and classify all stationary points of

$$f(x,y) = 5xy^2 - 8x^2 - 9y^2.$$

[8 marks]

9. Find a linear approximation near $(1,0)$ to

$$f(x,y) = \sqrt{x^2 + y^2}.$$

[4 marks]

10. By changing to polar coordinates, compute

$$\int\int_D \frac{1}{1+x^2+y^2},$$

where D is the unit disc:

$$D = \{(x,y) : x^2+y^2 \leq 1\}.$$

[6 marks]

SECTION B

11. a) Write down the degree two Taylor polynomial $P_2(y,0)$ of e^y at 0, and the remainder term $R_2(y,0)$. Also write down $P_9(y,0)$ and the remainder term $R_9(y,0)$.

Now if $y=-x$ and $x>0$, show that

$$|P_2(-x,0) - e^{-x}| \leq \frac{x^3}{3!}$$

and

$$|P_9(-x,0) - e^{-x}| \leq \frac{x^{10}}{10!}.$$

Hence show that if $0 \leq x \leq \frac{1}{2}$ then

$$|P_2(-x) - e^{-x}| \leq \frac{1}{48}$$

and if $0 \leq x \leq 2$ then

$$|P_9(-x) - e^{-x}| < 0.0003.$$

b) Write down the first four terms of the Taylor series of e^x and e^{-x}. State for which values of x the Taylor series for e^x is convergent and equal to e^x. Hence, or otherwise, show that

$$\lim_{x\to 0} \frac{1-x+\frac{x^2}{2}-e^{-x}}{1+x+\frac{x^2}{2}-e^x} = -1.$$

[15 marks]

12.

a) Find the general solution to

$$(x^2 - y^2)\frac{dy}{dx} = xy.$$

b) Solve the equation

$$y'' + 2y' - 3y = \cos x$$

subject to

$$y(0) = 1, \quad y'(0) = -1.$$

[15 marks]

13. Find the minimum distance between the line $x + y = 2$ and the ellipse $x^2 + 2y^2 = 1$. You may assume (as is true) that this line and ellipse do not intersect.

Hint: A general point on the line is given by $(x, y) = (2 - t, t)$ for $t \in (-\infty, \infty)$. If suffices to find the minimum of the *square* of the distance. So it is necessary to find the minimum of

$$f(x, y, t) = (x - 2 + t)^2 + (y - t)^2$$

subject to

$$g(x, y) = x^2 + 2y^2 = 1.$$

Show that at a minimum, either $x + y = 2$ (which is impossible) or $x = 2y$.

[15 marks]

14.

a) Find the area of the region R bounded by the parabola $x = 4y^2$ and the line $2y + 2 = x$.

b) Find the centre of mass $(\overline{x}, \overline{y})$ of R, assuming uniform density.

[15 marks]