The dictionary between holomorphic maps and Kleinian groups

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Möbius transformations

Möbius transformations are simply the degree one rational maps of \( \mathbb{C} \):

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\sigma_A : Z \mapsto \frac{az + b}{cz + d} : \mathbb{C} \rightarrow \mathbb{C}
\]

where

\[
ad - bc \neq 0
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and

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A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
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Möbius transformations are simply the degree one rational maps of \( \overline{\mathbb{C}} \):

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Then

\[A \mapsto \sigma_A : \text{GL}(2\mathbb{C}) \to \{ \text{Möbius transformations} \}\]

is a homomorphism whose kernel is

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The homomorphism is an isomorphism restricted to \( SL(2, \mathbb{C}) \), the subgroup of matrices of determinant 1.
We have an action of $GL(2, \mathbb{C})$ on $\overline{\mathbb{C}}$ by

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The action of \( SL(2, \mathbb{R}) \) preserves the upper half-plane

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\{ z \in \mathbb{C} : \text{Im}(z) > 0 \}
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and also \( \mathbb{R} \cup \{ \infty \} \) and the lower half-plane. The action of the subgroup

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SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}
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preserves the open unit disc, the closed unit disc, and its exterior.
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All of these actions are transitive that is, for all $z$ and $w$ in the domain there is $A$ in the group with $A.z = w$. 


Definition 1 A *Kleinian group* is a subgroup $\Gamma$ of $PSL(2, \mathbb{C})$ which is discrete, that is, there is an open neighbourhood $U \subset PSL(2, \mathbb{C})$ of the identity element $I$ such that

$$U \cap \Gamma = \{I\}.$$
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**Definition 2** A *Fuchsian group* is a discrete subgroup of $PSL(2, \mathbb{R})$. 

Equivalently (as usual with topological groups) there is an open neighbourhood $V$ of $I$ such that $\gamma V \cap \gamma' V = \emptyset$ for all $\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'$. To get this, choose $V$ with $V = V - 1$ and $V \subset U$. 

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Limit set

**Definition 3** The *domain of discontinuity* $\Omega_\Gamma$ of $\Gamma$ is the set of all $z \in \mathbb{C}$ such that, for some open neighbourhood $U$ of $z$,\n\[
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Every Möbius transformation which is not the identity has 1 or 2 fixed points in $\mathbb{C}$.

**Definition 5** A *hyperbolic* element of $\Gamma$, is an element which has two fixed points in $\overline{\mathbb{C}}$ with multipliers at both points off the unit circle. An *elliptic* element has two fixed points with multiplier on the unit circle. A *parabolic* element has just one fixed point.
Properties in brief

- \( \Gamma \) is always nonempty, closed and invariant under \( \Gamma \).
- It is infinite except when \( \Gamma \) is elementary, that is, abelian-by-finite. If it is infinite elementary, it can consist of one or two points, depending on whether the infinite order generator is parabolic or hyperbolic.
- If \( \Gamma \) is nonelementary, \( L_{\Gamma} \) is the closure of the set of fixed points of hyperbolic elements of \( \Gamma \).
- The domain of discontinuity is open, invariant under \( \Gamma \) and possibly empty.
- \( \Gamma \) acts minimally on \( L_{\Gamma} \), that is, for every \( z \in L_{\Gamma} \), the set \( \{ \gamma \cdot z : \gamma \in \Gamma \} \) is dense in \( L_{\Gamma} \).
- \( \Gamma \) acts transitively on \( L_{\Gamma} \) that is, for any open sets \( U \) and \( V \) intersecting \( L_{\Gamma} \), there is \( \gamma \in \Gamma \) such that \( \gamma \cdot U \cap V \neq \emptyset \).
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- $\Gamma$ acts transitively on $L_\Gamma$ that is, for any open sets $U$ and $V$ intersecting $L_\Gamma$, there is $\gamma \in \Gamma$ such that $\gamma.U \cap V \neq \emptyset$. 
Extension of the $SL(2, \mathbb{C})$ action

There is an extension of the $SL(2, \mathbb{C})$ action to upper half space which mimics the action of $SL(2, \mathbb{R})$ on the upper half plane. One neat way of describing the action is to regard upper half space as a subset of the quaternions and to use multiplication and division in the quaternions. So write

$$H^3 = \{ x + yi + tj : t > 0, x, y \in \mathbb{R} \} = \{ z + tj : t > 0, z \in \mathbb{C} \}.$$

Then $SL(2, \mathbb{C})$ acts on $H^3$ by

$$A \cdot w = (aw + b)(cw + d)^{-1}$$

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\[ A.w = (aw + b)(cw + d)^{-1} \text{ if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
Why does it work?

Note that $w - 1 = \frac{w}{|w|}$. Then $A \cdot w = \frac{a}{c}|w|^2 + \frac{b}{d} + (\frac{ad}{c} - \frac{bc}{d})w$.
Why does it work?

Note that

\[ w^{-1} = \frac{\overline{w}}{|w|^2} \]

where

\[ x + iy + tj + uk = x - yi - tj - uk, \]

\[ |w|^2 = \overline{w}w. \]

Then

\[ A.w = \frac{ac|w|^2 + bd + (ad - bc)w}{|cw + d|^2} \]
Preservation of the hyperbolic metric.

The action of $SL(2, \mathbb{C})$ preserves the metric on $H^3$ given in classical form by

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With this metric, \( H^3 \) is hyperbolic space. The action also preserves the set of hemispheres with centres on the place \( \{ t = 0 \} \) and vertical half-planes — all of which surfaces are totally geodesic — and the horizontal planes and spheres in \( H^3 \) which are tangent to \( \{ t = 0 \} \).
The action of a Kleinian group on $H^3$. 

The stabiliser of $j$ under the $SL_2(\mathbb{C})$ is the compact group $SU_2(\mathbb{C}) = \{ (a \ b; -b \ a) : |a|^2 + |b|^2 = 1 \}$. 

It follows that if $\Gamma$ is Kleinian then there is an open neighbourhood $U$ of $j$ in $H^3$ such that $\{ \gamma \in \Gamma : \gamma U \cap U \neq \emptyset \} = \{ \gamma \in \Gamma : \gamma \cdot j = j \}$ and this set is finite and consists of finite order elements.

If $\Gamma$ has no finite order elements apart from the identity then this set is simply the identity element.

Since $SL_2(\mathbb{C})$ acts isometrically, it follows that the action of a Kleinian group on $H^3$ is discrete.

If $\Gamma$ has no finite order elements apart from the identity then $H^3/\Gamma$ is a hyperbolic manifold with covering group $\Gamma$.

If $\Gamma$ does have finite order elements then $H^3/\Gamma$ is a hyperbolic orbifold.
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Is there an analogue?

Analogues of the extension from $C$ to $H_3$ have been sought in holomorphic dynamics, in particular for rational maps, for example in work of Lyubich and co-workers. But there is no easy analogue. But we continue the elementary part of the dictionary, promoted by Sullivan in the 1980's.
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- Analogues of the extension from \( \overline{\mathbb{C}} \) to \( H^3 \) have been sought in holomorphic dynamics, in particular for rational maps, for example in work of Lyubich and co-workers.
- But there is no easy analogue.
- But we continue the elementary part of the dictionary, promoted by Sullivan in the 1980’s.
We fix a Kleinian group $\Gamma$ with domain of discontinuity $\Omega_\Gamma$.

$\Gamma$ preserves $\Omega_\Gamma$. 

Since $\Gamma_1$ acts discretely on $U$, the quotient $U / \Gamma_1$ is a Riemann surface, and if $\Gamma_1$ has no finite order elements (apart from the identity) then $\Gamma_1$ is a quotient group of the covering group.

If $U$ is simply connected then $\Gamma_1$ is the covering group of $U / \Gamma_1$. 

**Action on the domain of discontinuity**
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- If $U$ is simply connected then $\Gamma_1$ is the covering group of $U/\Gamma_1$. 
Here is an analogue of Sullivan’s theorem on the nonexistence of wandering domains in the Fatou set for rational functions. The theorem was first proved by Ahlfors (Tulane Symposium on quasiconformal mappings, 1967), but Sullivan gave a proof based on a variant of his wandering domains proof (Ann of Math 122, 1985).
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**Ahlfors’ finiteness theorem**  Let $\Gamma$ be finitely generated. Then for any component $U$ of $\Omega_\Gamma$ with stabiliser $\Gamma_1$, $U/\Gamma_1$ is always an analytically finite surface, that is, a compact surface minus finitely many punctures. There are only finitely many orbits of the $\Gamma$-action in $\Omega_\Gamma$. 
Analogue of hyperbolicity

We say that $\Gamma$ acts hyperbolically or is convex cocompact if one of the two following equivalent properties holds.

▶ There is a covering of $L_\Gamma$ by finitely many open balls $U_i$ ($1 \leq i \leq n$) such that, for each $\varepsilon > 0$, there is a covering of $L_\Gamma$ by sets of the form $\gamma \cdot U_i$ with $\gamma \in \Gamma$ and of radius $< \varepsilon$ in the spherical metric.

▶ $(H_3 \cup \Omega_\Gamma)/\Gamma$ is compact.

Necessarily, if $\Gamma$ acts hyperbolically then $\Gamma$ is finitely generated and every element is either hyperbolic or elliptic.
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Necessarily, if $\Gamma$ acts hyperbolically then $\Gamma$ is finitely generated and every element is either hyperbolic or elliptic.
A maximal parabolic subgroup in a Kleinian group $\Gamma$ is the stabiliser $\Gamma_z$ of an element $z \in \mathbb{C}$, if this group contains at least one parabolic element. If one element in $\Gamma_z$ is parabolic then all elements are either parabolic or elliptic. The group $\Gamma_z$ preserves any ball or sphere in $\mathbb{H}^3$ tangent at $z$. We shall call $z$ a parabolic point. Such balls and spheres are called horoballs and horospheres at $z$. There is at least one horoball $B$ at $z$ such that $\gamma \cdot B \cap B \neq \emptyset$ $\iff$ $\gamma \in \Gamma_z$, in which case, of course, $\gamma \cdot B = B$. The quotient space $B/\Gamma_z$ in $\mathbb{H}^3/\Gamma$ is called a cusp neighbourhood. The following theorem was proved by Sullivan (Acta Math 147 1981, 289-299). Sullivan's finite cusps theorem Let $\Gamma$ be finitely generated. There are only finitely many conjugacy classes of maximal parabolic subgroups in $\Gamma$. 
Cusps

A maximal parabolic subgroup in a Kleinian group $\Gamma$ is the stabiliser $\Gamma_z$ of an element $z \in \overline{\mathbb{C}}$, if this group contains at least one parabolic element. If one element in $\Gamma_z$ is parabolic then all elements are either parabolic or elliptic. The group $\Gamma_z$ preserves any ball or sphere in $H^3$ tangent at $z$. We shall call $z$ a parabolic point.
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A maximal parabolic subgroup in a Kleinian group $\Gamma$ is the stabiliser $\Gamma_z$ of an element $z \in \overline{\mathbb{C}}$, if this group contains at least one parabolic element. If one element in $\Gamma_z$ is parabolic then all elements are either parabolic or elliptic. The group $\Gamma_z$ preserves any ball or sphere in $H^3$ tangent at $z$. We shall call $z$ a parabolic point.

Such balls and spheres are called horoballs and horospheres at $z$. There is at least one horoball $B$ at $z$ such that $\gamma \cdot B \cap B \neq \emptyset$ $\iff \gamma \in \Gamma_z$, in which case, of course, $\gamma \cdot B = B$.

The quotient space $B/\Gamma_z$ in $H^3/\Gamma$ is called a cusp neighbourhood. The following theorem was proved by Sullivan (Acta Math 147 1981, 289-299).

Sullivan’s finite cusps theorem  Let $\Gamma$ be finitely generated. There are only finitely many conjugacy classes of maximal parabolic subgroups in $\Gamma$. 
Geometrically finite groups

A finitely generated Kleinian group is called geometrically finite if for representatives $z_i, 1 \leq i \leq n$ of the parabolic point orbits, $(H^3 \cup \Omega \Gamma \cup \Gamma \cdot \{z_i : 1 \leq i \leq n\})/\Gamma$ is compact. Geometrically finite groups have some nice properties that are reasonably easy to prove.

- Either the limit set is $C$ or it has zero measure.
- If the limit set is connected then it is locally connected.

The first property is now known to hold for all finitely generated Kleinian groups and not to hold for rational maps, nor even for polynomials (proved by Buff and Cheritat in 2005). The second property has been claimed at least for a large class of groups, by Mitra (also known as Brahmachaitanya).
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Structural Stability

A convex cocompact group $\Gamma$ is fairly easily proved to be structurally stable, that is, if the generators $\gamma_i$ of $\Gamma$ are moved sufficiently little then the resulting group $\Gamma'$ with generators $\gamma'_i$ is also Kleinian and quasiconformally conjugate to $\Gamma'$, that is there is a q-c map $\phi: \mathbb{C} \to \mathbb{C}$ such that $\phi(\gamma_i(z)) = \gamma'_i \phi(z)$ for all generators $\gamma_i$ and all $z \in \mathbb{C}$. 
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$$\varphi(\gamma_i \cdot z) = \gamma'_i \cdot \varphi(z)$$

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Structural Stability and no invariant line fields


Sullivan's no invariant line field theorem

If $\Gamma$ is finitely generated Kleinian and structurally stable, then every conjugacy to a sufficiently nearby group is quasiconformal.

The quasi-conformal deformation space of any finitely generated Kleinian group is naturally isomorphic to the Teichmüller space of $\Omega_{\Gamma}/\Gamma$.

There are no invariant line fields on the limit set.
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The dictionary on density and structural stability

Sullivan was able to prove that all structurally stable Kleinian groups are "good" (convex cocompact) but was unable to prove that structurally stable groups are dense.

He proved with Mane and Sad that structurally stable rational maps are dense but was unable to prove that structurally stable rational maps are hyperbolic.

It is now known that geometrically finite groups are dense. (First main results due to Brock and Bromberg.)

**Theorem**

Let $M = \mathbb{H}^3 / \Gamma$ be any hyperbolic 3-manifold such that $\pi_1(M)$ is finitely generated and a representation $\rho: \pi_1(M) \to \Gamma$ is fixed. Then there is a sequence $\rho_n: \pi_1(M) \to \Gamma_n$ such that $\rho_n \to \rho$ and $\mathbb{H}^3 / \Gamma_n \to \mathbb{H}^3 / \Gamma$ and $\Gamma_n$ is geometrically finite.

Density of hyperbolicity in any reasonable family of rational maps is conjectured but still unknown.
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