The dictionary between holomorphic maps and Kleinian groups

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$$\sigma_A : z \mapsto \frac{az+b}{cz+d} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

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The homomorphism is an isomorphism restricted to $SL(2, \mathbb{C})$, the subgroup of matrices of determinant 1.

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The action of $SL(2,\mathbb{R})$ preserves the upper half-plane

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and also $\mathbb{R} \cup \{\infty\}$ and the lower half-plane. The action of the subgroup

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

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All of these actions are *transitive* that is, for all z and w in the domain there is A in the group with $A.z = w$.

Kleinian groups

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Equivalently (as usual with topological groups) there is an open neighbourhood V of I such that

$$\gamma V \cap \gamma' V = \emptyset$$

for all γ, $\gamma' \in \Gamma$, $\gamma \neq \gamma'$. To get this, choose V with $V = V^{-1}$ and $V.V \subset U$.

Limit set

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Definition 5 A *hyperbolic* element of Γ, is an element which has two fixed points in $\overline{\mathbb{C}}$ with multipliers at both points off the unit circle. An *elliptic* element has two fixed points with multiplier on the unit circle. A *parabolic* element has just one fixed point.

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- Γ acts transitively on L_Γ that is, for any open sets U and V intersecting L_Γ, there is $\gamma \in \Gamma$ such that $\gamma.U \cap V \neq \emptyset$.

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There is an an extension of the $SL(2, \mathbb{C})$ action to upper half space which mimics the action of $SL(2, \mathbb{R})$ on the upper half plane. One neat way of describing the action is to regard upper half space as a subset of the quaternions and to use mutliplication and division in the quaternions. So write

$$H^3 = \{x + yi + tj : t > 0, x, y \in \mathbb{R}\} = \{z + tj : t > 0, z \in \mathbb{C}\}.$$

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Then $SL(2, \mathbb{C})$ acts on H^3 by

$$A.w = (aw + b)(cw + d)^{-1} \text{ if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Why does it work?

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Note that

$$w^{-1} = \frac{\overline{w}}{|w|^2}$$

where

$$\overline{x + iy + tj + uk} = x - yi - tj - uk,$$

$$|w|^2 = w\overline{w}.$$

Then

$$A.w = \frac{a\overline{c}|w|^2 + b\overline{d} + (ad - bc)w}{|cw + d|^2}$$

Preservation of the hyperbolic metric.

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With this metric, H^3 is *hyperbolic space*.The action also preserves the set of hemispheres with centres on the place $\{t = 0\}$ and vertical half-planes — all of which surfaces are *totally geodesic* — and the horizontal planes and spheres in H^3 which are tangent to $\{t = 0\}$.

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  and this set is finite and consists of finite order elements .

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- If Γ does have finite order elements then H^3/Γ is a *hyperbolic orbifold*

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- If U is simply connected then Γ_1 is the covering group of U/Γ_1.

Here is an analogue of Sullivan's theorem on the nonexistence of wandering domains in the Fatou set for rational functions. the theorem was first proved by Ahlfors (Tulane Symposium on quasiconformal mappings, 1967), but Sullivan gave a proof based on a variaant of his wandering domains proof (Ann of Mathh 122, 1985).

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Ahlfors' finiteness theorem *Let Γ be finitely generated. Then for any component U of Ω_Γ with stabiliser Γ_1, U/Γ_1 is always an analytically finite surface, that is, a compact surface minus finitely many punctures. There are only finitely many orbits of the Γ-action in Ω_Γ.*

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- There is a covering of L_Γ by finitely many open balls U_i ($1 \leq i \leq n$) such that, for each $\varepsilon > 0$, there is a covering of L_Γ by sets of the form $\gamma.U_i$ with $\gamma \in \Gamma$ and of radius $< \varepsilon$ in the spherical metric.

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Necessarily, if Γ acts hyperbolically then Γ is finitely generated and every element is either hyperbolic or elliptic.

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A *maximal parabolic subgroup* in a Kleinian group Γ is the stabiliser Γ_z. of an element $z \in \overline{\mathbb{C}}$, if this group contains at least one parabolic element. If one element in Γ_z is parabolic then all elements are either parabolic or elliptic. The group Γ_z preserves any ball or sphere in H^3 tangent at z. We shall call z a *parabolic point*.

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Such balls and spheres are called *horoballs* and *horospheres at z*. There is at least one horoball B at z such that $\gamma.B \cap B \neq \emptyset$ $\Leftrightarrow \gamma \in \Gamma_z$, in which case, of course, $\gamma.B = B$.

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Sullivan's finite cusps theorem *Let Γ be finitely generated. There are only finitely many conjugacy classes of maximal parabolic subgroups in Γ.*

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A finitely generated Kleinian group is called *geometrically finite* if for representatives z_i, $1 \leq i \leq n$ of the parabolic point orbits,

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A convex cocompact group Γ is fairly easily proved to be *structurally stable*, that is, if the generators γ_i of Γ are moved sufficiently little then the resulting group Γ' with generators γ_i' is also Kleinian and quasiconformally conjugate to Γ', that is there is a q-c map $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\varphi(\gamma_i.z) = \gamma_i'.\varphi(z)$$

for all generators γ_i and all $z \in \overline{\mathbb{C}}$.

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- *The quasi-conformal deformation space of any finitely generated Kleinian group is naturally isomorphic to the Teichmuller space of Ω_Γ/Γ.*
- *There are no invariant line fields on the limit set.*

The dictionary on density and structural stablity

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- Sullivan was able to prove that all structurally stable Kleinian groups are “good” (convex cocompact) but was unable to prove that structurally stable groups are dense.

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- Density of hyperbolicity in any reasonable family of rational maps is conjectured but still unknown.