Ph.D. Projects.

As an example of what can occur the reader might like to consider the following family

$$f_a(x) = 1 - \frac{a+1}{x} + \frac{a}{x^2}.$$

We restrict to real parameter a and real variable x, so considering f_a as a function on the real line $\mathbb{R}$. corresponding family of complex functions wth complex parameter is very interesting but this will suffice for the moment.

The function f_a has a vertical asymptote at $x = 0$, a horizontal asymptote at $y = 1$, and a critical (or stationary) point at $x = \frac{2a}{a+1}$. Regarded as a holomorphic map of the Riemann sphere, 0 is another critical point of f_a, but just considering f_a as a real-valued function, we simply note that

$$\lim_{x \to 0} f_a(x) = \infty,$$

which we write as $f_a(0) = \infty$, and

$$\lim_{x \to \infty} f_a(x) = 1$$

which we write as $f_a(\infty) = 1$. We also have $f_a(1) = 0$.

It is a fact that, in the theory of complex dynamics, the dynamics is often completely determined by the dynamics of the critical pionts. Therefore, we are particularly interested in the sequence

$$f_a^n\left(\frac{2a}{a+1}\right)$$

for different values of a, where f_a^n denotes the n-fold iterate. There are some special values of a where this sequence is particularly simple:

$$a = 1: \ \frac{2a}{a+1} = 1, \text{ so } f_1^3\left(\frac{2a}{a+1}\right) = f_1^3(1) = 1,$$

$$a = -1: \ \frac{2a}{a+1} = -\infty, \text{ so } f_{-1}^3\left(\frac{2a}{a+1}\right) = f_{-1}^3(\infty) = \infty.$$

Also we note that for all a

$$f_a\left(\frac{2a}{a+1}\right) = -(a-1)^2/4a.$$

The equation

$$\frac{2a}{a+1} == -(a-1)^2/4a$$

has just one real root a_1, which is < 0. Note that

$$f'_{a_1}\left(\frac{2a_1}{a_1+1}\right) = 0.$$

The sequence $f_a^n(\frac{2a}{a+1})$ is then also particularly simple in somewhat larger sets:

$$a < a_1 + \delta_1, \quad -1 - \delta_1 < a < 0, \quad 0 < a < 1 + \delta_3,$$

for some numbers $\delta_j > 0$, $j = 1, 2, 3$. For $-1 - \delta_1 < a < 0$,

$$\lim_{n\to\infty} f_a^{3n}(\frac{2a}{a+1}) = \infty,$$

for $0 < a < 1 + \delta_3$

$$\lim_{n\to\infty} f_a^{3n}(\frac{2a}{a+1}) = 1,$$

and for $a < a_1 + \delta_1$, $f_a^n(\frac{2a}{a+1})$ converges to an attractive fixed point.

The sets

$$a_1 + \delta_1 \leq a \leq -1 - \delta_1, \quad 1 + \delta_3 \leq a$$

are more interesting. Write

$$X_1 = [-\infty, 0], \ X_2 = [0, 1], \ X_3 = [1, \infty].$$

Then for $a \leq -1$,

$$f_a(X_1) = X_1 \cup X_2, \quad f_a(X_2) = X_1, \quad f_a(X_3) \subset X_2 \cup X_3.$$

For $a \geq 1$,

$$f_a(X_1) = X_3, \quad f_a(X_2) = X_2 \cup X_3, \quad f_a(X_3) \subset X_1 \cup X_2.$$

One can then ask what are the possible *symbolic dynamics* of the sequence $f_a^n(\frac{2a}{a+1})$, that is, what are the possibilities for sequences X_{i_n} $(n \geq 0)$ where $f_a^n(\frac{2a}{a+1}) \in X_{i_n}$. For $a < -1$, the the sequences which occur are fairly easily characterised as follows: $X_{i_j} = X_3$ for $j \leq k$, some $k \geq 0$, and $X_{i_{k+1}} = X_2$, $X_{i_j} = X_1$ or X_2, such that if $X_{i_j} = X_2$ then $X_{i_{j+1}} = X_1$. For $a \geq 1$ it is similar but rather more tricky: the start of the sequence imposes a condition on the rest of the sequence. For $a \leq -1$ the nature of the sequence means that it is impossible for $\frac{2a}{a+1}$ to be periodic for $a < -1$, although it is possible for $\frac{2a}{a+1}$ to be eventually periodic, for example to have $f_a^n(\frac{2a}{a+1}) = 0$ for some $n > 0$. For $a > 1$ it is possible for $\frac{2a}{a+1}$ to be periodic.

These calculations in themselves are just that - nontrivial but a few hours' or days' work, not weeks or months on end. Looking at the real case in isolation is not the best strategy. Also, the coding used above is not the only possible one. There are other codings which recognise hidden Julia sets – more than one – and the relationship between the different codings could be clarified. Lookin at real parameter certaunly gives some information, but there is a much better description if one moves to the complex parameter, and uses more sophisticated methods. These methods have not yet been used on other families. So there is a lot of development possible.

There is also an analytical question which can probably be tackled now.

Problem: Is the set of parameter values for which $f_a^n(\frac{2a}{a+1})$ *converges to a periodic orbit dense in* $a < -1$ *(or in* $a > 1$*)?*

It is quite likely that the methods used to prove the corresponding result for real quadratic polynomials will work here. The real quadratic polynomial result was proved by Grazcyk and Swiatek, and by Lyubich, in the 1990's, and there have been a number of important extensions in the last few years by van Strien and Weixao Shen and Lyubich and collaborators, for example. But this would need checking.