Symmetric $X_9$ singularities
and the complex affine reflection groups

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to Vladimir Igorevich
on the occasion of his 70th birthday

Abstract

We establish a natural correspondence between the finite order automorphisms of the function singularities $X_9$ and the complex crystallographic groups. The complete list of the related objects is obtained.

Relations between singularities and Coxeter groups is a classical area of singularity theory, going back to the fundamental works by Arnold [1] and Brieskorn [6]. Recently it was observed that these relations can be extended to include symmetric simple functions singularities on one hand and certain Shephard-Todd groups on the other [11, 12, 13]. In this paper we are making a further natural step in this direction by relating symmetries of the function singularities $X_9$ to a number of Popov’s complex crystallographic groups [16]. Appearance of complex affine reflection groups in equivariant monodromy of parabolic function singularities with symmetry is the first appearance of such groups in any singularity context (see also [14]).

The structure of the paper is as follows.

Section 1 introduces the crystallographic groups to be related to the function singularities. In addition, in Subsection 1.2 we describe a way to construct a complex affine reflection group from a semi-definite hermitian form of corank 1.

Section 2 lists finite order automorphisms of the $X_9$ functions. It also shows how the rank 2 kernel of the $X_9$ hermitian intersection form is shared by various character subspaces $H_\chi$ of the symmetry action on the middle vanishing homology.

Section 3 is devoted to the proof of the main result of the paper that all the complex affine reflection groups arising from the equivariant monodromy of the symmetric $X_9$ singularities on the appropriate $H_\chi$ via the construction of Subsection 1.2 are actually crystallographic.
1 Affine reflection groups

1.1 The complex crystallographic groups

An affine reflection in $\mathbb{C}^n$ is an affine unitary transformation identical on a hyperplane. The hyperplane is called the mirror of the reflection. A group generated by such reflections and having a compact fundamental domain is called complex crystallographic. These groups were classified by V. L. Popov in [16].

For a complex crystallographic group $W$, we denote by $L \subseteq U_n$ its linear part, that is the image of $W$ under the natural map $W \rightarrow U_n$. The group $L$ is a Shephard-Todd group. Let $T$ be the maximal translation subgroup of $W$. Then $W$ is an extension of $L$ by $T$. Unlike the real case, $W$ may not be the semi-direct product of its linear and translation parts. However, all the groups we will need in our current singularity context are such products.

We shall now describe the five groups to be involved. Mirrors of $L$ will be identified by their normals which we shall call roots.

The linear parts of the groups we will need are the Shephard-Todd groups $L = G(4,1,2), G(6,2,2), G_3(6), G_8, G_{26}$ (see [17, 16]). Their Dynkin diagrams are given in Figure 1. The vertex set of a diagram there represents a set of generating reflections. Each vertex is a unit root and is marked with the order of the reflection, order 2 omitted. An edge $a \rightarrow b$ is equipped with the hermitian product $\langle a, b \rangle$. As usual, $\omega = e^{2\pi i/3}$. The edge orientation is omitted if the product is real, and there is no edge at all if the roots are orthogonal. All the diagrams were constructed using the roots from Table 2 of [16] (see also [9]). The rank of the group $G(6,2,2)$ is 2. The rank of any other group is equal to the number of vertices in its diagram.

![Dynkin diagrams of the Shephard-Todd groups. All roots are unit.](image)

In the notation of [16], the crystallographic groups $W$ with the above linear parts that will be related to function singularities in this paper are $[G(4,1,2)]_2$, $[G(6,2,2)]_2$, $[K_3(6)]$, $[K_8]$, $[K_{26}]$. The lattice $T$ is spanned by the $L$-orbit of any root of $L$ of order 2 in the first two cases, of any root in the next two, and of any root of order 3 in the last case.

All the crystallographic groups have the conjugate versions, with $i$ and $\omega$ replaced by their conjugates. However, the conjugations yield the same groups.
1.2 Affine groups defined by corank 1 hermitian forms

The relation between the crystallographic groups and function singularities we are going to establish is based on the following construction of a complex reflection group from a corank 1 hermitian form (cf. [5]).

Let \( \tilde{q} \) be a corank 1 semi-definite hermitian form on \( \tilde{V} = \mathbb{C}^{n+1} \). Choose a basis \( e_0, e_1, \ldots, e_n \) in \( \tilde{V} \) so that \( e_0 \) is in the kernel \( K \) of the form. The span of the \( e_{j>0} \) will be denoted \( \tilde{V} \), and \( v \) will stay for the \( V \)-component of \( \tilde{v} \in \tilde{V} \): \( \tilde{v} = v_0e_0 + v \). In all the matrix expressions below, with a minor abuse of the notation, elements \( v \in \tilde{V} \) will be treated as columns of their coordinates \( v_{j>0} \). For example, \( \tilde{q}(\tilde{v}, \tilde{w}) = v^T Q \tilde{w} \), where \( Q = (\tilde{q}(e_i, e_j))_{i,j>0} \) is the matrix of the restriction \( q = \tilde{q}|_{V} \).

We consider the space \( \tilde{V}^* \) dual to \( \tilde{V} \) as \( K^* \oplus V \). For coordinates on it we choose \( \alpha_0, \alpha_1, \ldots, \alpha_n \) so that a linear functional \( \tilde{\alpha} \) on \( \tilde{V} \) is written as

\[
\tilde{\alpha}(v) = v_0\alpha_0 + v^T Q \alpha = v_0\alpha_0 + q(v, \tilde{\alpha}).
\]

Take a pseudo-reflection on \( \tilde{V} \) (that is a transformation given by the same formula as a reflection had the form \( \tilde{q} \) been non-degenerate) with a root \( \tilde{u} \notin K \) and the eigenvalue \( \lambda \):

\[
A : \tilde{v} \mapsto \tilde{v} - (1 - \lambda)\tilde{q}(\tilde{v}, \tilde{u})u/\tilde{q}(\tilde{u}, \tilde{u}) = \left( v_0 + \gamma q(v, u)u \right) e_0 + \left( v + \gamma q(v, u)u \right),
\]

where \( \gamma = (\lambda - 1)/q(u, u) \). For the dual transformation \( A^* \), we have

\[
(A^* \tilde{\alpha})\bar{v} = \tilde{\alpha}(A^{-1}\bar{v}) = v_0\alpha_0 + v^T Q \alpha + \overline{\gamma} v^T Q(u_0\alpha_0 + u^T Q \alpha)\overline{\bar{u}} = v_0\alpha_0 + v^T Q(\alpha + \overline{\gamma} \bar{\alpha}(\bar{u})\overline{\bar{u}}).
\]

Therefore, the dual transformation sends each of the hyperplanes \( \alpha_0 = \text{const} \) into itself and on such a hyperplane it acts as

\[
\alpha \mapsto \alpha + \overline{\gamma} \bar{\alpha}(\bar{u})\overline{\bar{u}} = \alpha - (1 - \overline{\lambda})\frac{\alpha_0u_0 + \overline{q(\alpha, \bar{u})}}{\overline{\tilde{q}(\bar{u}, \overline{\bar{u}})}}\overline{\bar{u}},
\]

where \( \overline{\tilde{q}} \) is the hermitian form on \( V \) conjugate to \( q \): it has the matrix \( \overline{Q} = Q^T \) in the basis \( e_{j>0} \). If \( \alpha_0 \neq 0 \), then this is an affine reflection on the hyperplane \( \Gamma = \{ \alpha_0 = \text{const} \} \simeq V \), with the root \( \overline{\bar{u}} \), mirror \( \bar{\alpha}(\bar{u}) = \alpha_0u_0 + \overline{q(\alpha, \bar{u})} = 0 \) and eigenvalue \( \overline{\lambda} \). For \( u_0 = 0 \), the transformation is linear.

2 Smoothable symmetries of \( X_9 \)

Now we introduce the function singularities we will be dealing with.

Let \( f \) be a holomorphic function-germ on \( (\mathbb{C}^n, 0) \), with an isolated singularity at the origin. Consider a diffeomorphism-germ \( g \) of \( (\mathbb{C}^n, 0) \) sending the hypersurface \( f = 0 \) into itself. It multiplies \( f \) by a function \( c \) not vanishing at the origin. In what follows we assume \( g \) is of a finite order, so \( c \) is a constant, a root of unity.

Let \( \mathcal{O}(g, c) \) be the space of all holomorphic function-germs on \( (\mathbb{C}^n, 0) \) multiplied by \( c \) under the action of \( g \). The group \( \mathcal{R}_g \) of biholomorphism-germs of \( (\mathbb{C}^n, 0) \) commuting with
$g$ acts on $\mathcal{O}(g, c)$. The corresponding equivalence is a geometric equivalence in the sense of Damon [10]. Therefore, the base of an $\mathcal{R}_g$-miniversal deformation of $f$ in $\mathcal{O}(g, c)$ is smooth and such a deformation can be constructed in the standard way [10, 4].

**Definition 2.1** An automorphism $g$ of a hypersurface $f = 0$ is called **smoothable** if an $\mathcal{R}_g$-versal deformation of function $f$ contains members with smooth zero sets.

If $g$ is such an automorphism, then the zero level $M$ of a generic member of an $\mathcal{R}_g$-versal deformation is a $g$-invariant Milnor fibre of $f$. Hence, $g$ acts on the homology of $M$ and provides the splitting

$$H_{n-1}(M, \mathbb{C}) = \oplus_{\chi} H_{\chi}, \quad \chi^{\text{order}(g)} = 1,$$

of the middle homology, in which $g$ acts on an individual summand as a multiplication by the character $\chi$. The **equivariant monodromy group**, that is the monodromy within an $\mathcal{R}_g$-versal deformation of $f$, preserves the splitting. The monodromy action on the $H_{\chi}$ will be our source of complex crystallographic groups, upon an application of the construction of Section 1.2.

We restrict our attention to classification of smoothable automorphisms of curves of the $X_9$ family

$$x^4 + ax^2y^2 + y^4 = 0, \quad a^2 \neq 4.$$  

(3)

The classification is up to holomorphic changes of the coordinates. Our actual major aim is to obtain the homology splitting (2), therefore we will not distinguish between automorphisms generating the same cyclic groups. Moreover, we prefer to have a hermitian intersection form on the middle homology rather than skew-hermitian. Because of that, we stabilise equation (3) by adding $z^2$ to the left-hand side. Respectively, $g$ starts acting on $z$ by multiplication by one of two possible square roots of $c$. We call this action **stabilised**. The ambiguity in choosing a root affects only the character assignment in (2), not the direct summands themselves. Since only the summands are crucial for us, we give just one of the choices in our classification. In particular, we set $g$ act trivially on $z$ if the function is $g$-invariant.

**Theorem 2.1** The complete list of stabilised smoothable automorphisms of all $X_9$ curves is given in Table 1.

In the Table:

$\varepsilon_r = e^{2\pi i/r}$;

the **versal monomials** are those to add with arbitrary coefficients to $f$ to obtain an $\mathcal{R}_g$-miniversal deformation;

the **kernel** $\chi$ are the values of the character for which the restrictions of the hermitian intersection form from $H_2(M, \mathbb{C})$ to the $H_{\chi}$ are degenerate (see Proposition 2.1 below);

the **affine groups** are the complex crystallographic groups which will be constructed in Section 3 from the monodromy on the $H_{\chi}$ on which the intersection form has corank 1;

similar to [7, 11], if the discriminant of a symmetric function singularity coincides with that of a Weyl group, the group enters the **notation**, the superscripts indicating the orders of the Picard-Lefschetz operators (see Section 3);
the $K_{1,2}$ is the unimodular boundary function singularity of [2, 3];

in all the other cases, the notation shows the symmetry group of the singularity, with the
vertical line telling that the function is invariant under the action and the slash indicating
that it is equivariant (cf. [13, 12]).

\textit{About the proof of Theorem 2.1.} The classification process is based on the consideration
how the automorphism permutes the four branches of the curve (3). The smoothability is
heavily restricted by an obvious observation that, once a smoothable diffeomorphism of the
plane has been diagonalised, it multiplies function $f$ by the same factor by which it multiplies
one of the monomials $1, x, y$ (otherwise the zero level of any symmetric perturbation of
$f$ would have had a critical point at the origin). The rest of the classification is rather
straightforward. \hfill $\square$

For an application of the construction of Section 1.2, it is crucial to know how the rank
2 kernel of the $X_9$ hermitian intersection form is shared by the character subspaces.

\textbf{Proposition 2.1} The kernel values of the character $\chi$ for the symmetric $X_9$ singularities
are those given in Table 1.

\textit{Proof.} We distinguish between invariant and equivariant cases, that is when 1 respectively
is or is not among the versal monomials.

a) In the invariant cases, the kernel characters are the eigenvalues of the action of $g$ on
the residue forms $dx dy dz/df$ and $q_4(x, y) dx dy dz/df$, where $q_4(x, y)$ is a degree 4 monomial
defining a non-trivial element in the local algebra of $f$. The span of the two forms is dual to
the kernel of the intersection form on the homology.

b) We do the equivariant functions case-by-case, mainly using the fact that cycles in the
kernel of the intersection form are invariant under any monodromy.

$X_9/\mathbb{Z}_3$. The monodromy $\alpha = e^{2\pi i t}$, $0 \leq t \leq 1$, in the family $f(x, y, z) + \alpha x = 0$
coincides with the transformation $g$, hence all the kernel of the $X_9$ intersection form is in $H_{\chi=1}$.

$X_9/\mathbb{Z}_9$. The top-dimensional strata of the discriminant of $X_9/\mathbb{Z}_3$ are $3A_1$ only. Three
ordinary Morse 2-cycles $e, ge$ and $g^2e$ vanishing simultaneously provide an element

$$e + \chi^{-1} ge + \chi^{-2} g^2 e \in H_\chi, \quad \chi^3 = 1. \quad (4)$$

This implies that the ranks of all of the three $H_\chi$ are the same, 3. On the other hand,
the automorphism of $X_9/\mathbb{Z}_3$ is the cube of that of $X_9/\mathbb{Z}_9$. Hence the kernel characters of
$X_9/\mathbb{Z}_9$ are cubic roots of unity. Since the kernel character set must be sent into itself by
the complex conjugation, we see that for $X_9/\mathbb{Z}_9$ the kernel of the $X_9$ form is spanned by the
one-dimensional spaces $H_\chi$ and $H_\overline{\chi}$.

$X_9/\mathbb{Z}_{12}$. Take $M = \{x^3 + y^4 + z^2 - x = 0\}$ as a symmetric Milnor fibre. It retracts to
the $\mathbb{Z}_{12}$-orbit of the 2-cell $\sigma = \{(x, y, z) : 0 \leq x \leq 1, y \geq 0, z \in \mathbb{R}\} \subset M \cap \mathbb{R}^3$. The linear combination

$$\sum_{j=0}^{11} \chi^{-j} g^j \sigma, \quad \chi^{12} = 1, \quad \chi^3 \neq 1, \quad (5)$$
Table 1: Symmetric $X_9$ singularities

| $f$ | $g: x, y, z \mapsto |g|$ | versal monomials | kernel $\chi$ | affine group | notation |
|-----|-----------------|-----------------|-------------|-------------|---------|
| $x^4 + y^4 + z^2$ | $ix, -y, z$ | $4$ | $1, y^2, x^2y$ | $\pm i$ | $[G(4, 1, 2)]_2$ | $X_9\lbrack \mathbb{Z}_4\rbrack$ |
| $x^4 + iy, z$ | $i\omega x, i\omega y, \overline{\omega} z$ | $12$ | $x$ | $\pm i$ | $-$ | $X_9/\mathbb{Z}_{12}$ |
| $x^4 + xy^3 + z^2$ | $ix, i\omega y, z$ | $6$ | $1, x^2$ | $\omega, \overline{\omega}$ | $K_3(6)$ | $B_2^{(6,3)}$ |
| $x, \omega y, z$ | $12$ | $3$ | $1, x, x^2, x^3$ | $\omega, \overline{\omega}$ | $[K_{28}]_1$ | $C_4^{(2,3)}$ |
| $\omega x, \omega y, -\overline{\omega} z$ | $6$ | $6$ | $x, y^2, x^2y$ | $-\omega, -\overline{\omega}$ | $[G(6, 2, 2)]_2$ | $X_9/\mathbb{Z}_6$ |
| $\varepsilon y, \varepsilon_1^xy, \varepsilon_2^zy$ | $9$ | $9$ | $y$ | $\omega, \overline{\omega}$ | $-$ | $X_9/\mathbb{Z}_9$ |
| $x^4 + a x^2y^2 + y^4 + z^2$ | $\varepsilon_8 x, -\varepsilon_8 y, z$ | $8$ | $1$ | $\pm i$ | $-$ | $X_9/\mathbb{Z}_8$ |
| $x^4 + ax^2y^2 + y^4 + z^2$ | $ix, -iy, z$ | $4$ | $1, xy, x^2y^2$ | $1$ | $-$ | $(X_9\lbrack \mathbb{Z}_4\rbrack)^g$ |
| $-x, y, z$ | $2$ | $2$ | $1, y^2, x^2, x^2y, x^2y^2$ | $-1$ | $-$ | $(X_9/\mathbb{Z}_6)^g$ |
| $\omega x, -\omega y, \overline{\omega} z$ | $6$ | $6$ | $x, x^2y$ | $-1$ | $-$ | $(X_9/\mathbb{Z}_6)^g$ |
| $ix, iy, z$ | $4$ | $4$ | $1, x^2y^2$ | $-1$ | $-$ | $(X_9/\mathbb{Z}_4)^g$ |
| $-x, -y, z$ | $2$ | $2$ | $1, x^2, xy, y^2, x^2y^2$ | $1$ | $-$ | $(X_9/\mathbb{Z}_4)^g$ |
| $\omega x, \omega y, \overline{\omega} z$ | $3$ | $3$ | $x, y, x^2y^2$ | $1$ | $-$ | $X_9/\mathbb{Z}_3$ |
spans $H_{\chi}$. On the other hand, the quasi-homogeneous monodromy in the family $x^4 + y^4 + z^2 - e^{2\pi i t} x = 0$, $0 \leq t \leq 1$, is $g^4$. Hence the kernel characters satisfy $\chi^4 = 1$. With $\chi = 1$ prohibited, this gives $\chi = \pm i$.

$(X_9/\mathbb{Z}_6)'$. The square of the $X_9/\mathbb{Z}_{12}$ automorphism is the inverse of that of $(X_9/\mathbb{Z}_6)'$. So, the above implies that the kernel of the $X_9$ form is now the rank 2 space $H_{\chi=-1}$.

$X_9/\mathbb{Z}_6$. The deformation $f(x, y, z) + \alpha x^2 y$ gives an adjacency of $X_9/\mathbb{Z}_6$ to the singularity $D_6/\mathbb{Z}_6$ of [13, 12], all of whose $H_{\chi}, \chi^3 = -1$, are of rank 2. The multiplicity of the $X_9/\mathbb{Z}_6$ discriminant is 4, one higher than that of $D_6/\mathbb{Z}_6$, the increase due to the $3A_1$ stratum. This implies that the dimension of each of the three character spaces of $X_9/\mathbb{Z}_6$ is 3. Since the ranks of the intersection forms on them are at least 2, the characters $-\omega$ and $-\bar{\omega}$ are kernel.

\[ \square \]

**Questions 2.1** a) A bit more careful calculations show that, for all symmetric $X_9$ singularities, the rank of a character subspace with a degenerate intersection form is equal to the dimension of the base of an equivariant universal deformation, that is to the number of the versal monomials. The same is true for the $J_{1,5}$ symmetries of [14]. Why is this so?

b) It would be also good to understand why the kernel of the intersection form does not split exactly when the symmetric singularity has a module.

### 3 Relating symmetric $X_9$ singularities and crystallographic groups

We call a symmetric $X_9$ singularity interesting if the monodromy group on one of its character subspaces gives rise to an affine complex reflection group (not necessarily crystallographic) via the construction of Section 1.2. Necessary conditions for this are:

- the rank 2 kernel of the $X_9$ hermitian intersection form splits between two character subspaces;
- each of the two subspaces must be of rank at least 2;
- the multiplicity of the discriminant of a symmetric singularity must be at least 2, since an affine reflection group has at least two generators which must be coming from the Picard-Lefschetz operators.

According to Table 1, the first condition eliminates all moduli cases. The last condition eliminates four further singularities with one-dimensional bases of miniversal deformations. This leaves exactly 5 interesting symmetries, those to which the table assigns affine groups.

In Figure 2 the discriminants of three interesting $X_9$ singularities are shown. The degeneration types to which the top strata correspond are indicated. The $X_9/\mathbb{Z}_6$ discriminant is that of $B_3$ with an additional smooth component. The order $\alpha, \beta, \gamma$ of the deformation parameters is by the increase of their quasi-homogeneous weight in the deformations using the versal monomials of Table 1. The equation of the $X_9|\mathbb{Z}_4$ discriminant is

$$\gamma(\beta^2 - 4\gamma)((\beta - \alpha^2/4)^2 - 4\gamma) = 0.$$
Two discriminants missing from Figure 2 are those of singularities $A_3^{(4)}$ and $C_4^{(2,3)}$. The first of them is the standard $A_3$ swallowtail, with the top stratum $A_3$. The second is the standard $C_4$ discriminant with the smooth and singular components $3A_1$ and $A_2$ respectively.

The main result of this paper is

**Theorem 3.1** Consider an interesting symmetric $X_9$ singularity. Let $\chi$ be one of its kernel characters, and $\Gamma$ the hyperplane in $H^*_\chi$ formed by all linear functionals taking a fixed non-zero value on a fixed element of the kernel of the hermitian intersection form on $H^*_\chi$. Then the equivariant monodromy group of the singularity acting on $\Gamma$ is the complex crystallographic group given in Table 1.

**Proof.** By the methods developed in [11, 12, 13], it is possible to construct, for each of the $H^*_\chi$ of the Theorem, distinguished sets of vanishing $\chi$-cycles whose Dynkin diagrams are those of Figure 3. The sets are bases of the $H^*_\chi$, except for the $X_9/Z_6$ case that has one relation.

We use the following conventions in the diagrams. The vertices are elements of a distinguished set of $\chi$-cycles. A $\chi$-cycle vanishing at a $kA_\nu$ stratum has the self-intersection number $-k(\nu + 1)$ which is written by the vertex. The order of the corresponding Picard-Lefschetz operator is $\nu + 1$ (written inside the vertex, order 2 omitted). Simple, double and triple edges indicate that the relations between the pairs of the operators are $aba = bab$, $(ab)^2 = (ba)^2$ and $(ab)^3 = (ba)^3$ respectively. The marking and orientation of the edges are similar to those in Figure 1.

The idea behind the cycle construction is as follows. Consider the quotient set $M' = M/Z_m$ of a symmetric Milnor fibre by the group generated by the automorphism $g$. This set is stratified according to the stationary subgroups of the points. Let $M'' \subset M'$ be the union of all strata whose dimension is less than $\dim M'$. When the deformation parameter approaches its discriminant value, it is easy to define geometrically a relative vanishing
cycle in \((M', M'')\). Let \(\sigma \subset M'\) be this cycle and \(\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\) its inverse images in \(M\) ordered so that \(g(\sigma_j) = \sigma_{(j+1) \mod m}\). Then \(\sum_{j=0}^{m-1} \chi^{-j} \sigma_j \in H_{\chi}\) is the \(\chi\)-cycle we are for. The cycles (4) and (5) are examples of the construction.

Each tree diagram of Figure 3 serves both kernel values of the character since vanishing \(\chi\)-cycles are defined up to multiplication by powers of \(\chi\) and orientation change.

A vanishing \(\chi\)-cycle defines the Picard-Lefschetz operator on \(H_{\chi}\). This is a pseudo-reflection with the cycle as its root. Thus we are ready to apply the construction of Section 1.2. To introduce the notations used in it, we denote \(e'_0, e_1, e_2, \ldots\) the vertices in each tree diagram going from left to right, and in the \(X_9/\mathbb{Z}_6\) diagram starting from the top left and going clockwise (in this case \(e_3 = -\chi e_1 + \chi e_2\)). For all the singularities, a generator of the kernel of the hermitian intersection form can be taken in the form \(e_0 = e'_0 + \mathbf{a}\) where \(\mathbf{a}\) is a linear combination of the \(e_{j>0}\). The vector \(\mathbf{a}\) will be called the truncated kernel vector. It is an analog of the negative of the maximal root of a Weyl group.

Now drop the vertex \(e'_0\) from each tree diagram of Figure 3, change the sign of the intersection form and divide all the roots by appropriate positive numbers to make all of them unit. The result will be exactly the diagrams of Figure 1 of the linear parts \(L\) of the affine groups assigned to the singularities by Table 1 (for \(\chi = -\varpi\) in the \(X_9/\mathbb{Z}_6\) case, the additional complex conjugation is required). Therefore, the Picard-Lefschetz operators corresponding to the \(\chi\)-cycles \(e_{j>0}\) define the Shephard-Todd group \(L\) on the hyperplane \(\Gamma \subset H_{\chi}^*\).

The translation vector of the transformation (1) is proportional to its root, as it should be in an affine reflection. Thus, Theorem 3.1 will be proven if it turns out that, in all the cases, the truncated kernel vector \(\mathbf{a}\) is a root of a reflection from \(L\) of the order specified at the end of Section 1.1. And, indeed, we have:

\[
X_9/\mathbb{Z}_4: \quad \mathbf{a} \sim (2,1) \quad = \quad A_1^2 e_2 \\
A_3^{(4)}: \quad \mathbf{a} = (-1, i, i) \quad = \quad -A_1 A_2^{-1} e_1 \\
B_2^{(6,3)}: \quad \mathbf{a} = e_1 \\
C_4^{(2,3)}: \quad \mathbf{a} \sim (\omega - 1, 2, 1) \quad = \quad -A_1^{-1} A_2 A_3 A_2 e_1 \\
X_9/\mathbb{Z}_6: \quad \mathbf{a} = -2 e_1 - e_2 \quad = \quad \left\{ \begin{array}{ll} A_1 e_3, & \chi = -\omega \\ A_1^{-1} e_3, & \chi = -\varpi \end{array} \right. 
\]

Figure 3: Dynkin diagrams of the symmetric \(X_9\) singularities in 3 variables.
where the vector $a$ or its multiple are written in the basis $e_{j>0}$ and the $A_j$ are the linear reflections defined by the roots $e_j$ and having the eigenvalues $-1, \omega, i$. This yields the result required.

**Remarks 3.1**

a) The eigenvalue of the Picard-Lefschetz operator corresponding to a multiple Morse degeneration is $-1$. The eigenvalues of all the other operators in the $X_9/\mathbb{Z}_4$ and $X_9/\mathbb{Z}_6$ singularities are $-\chi$. They are $\chi$ in the $A_3^{(4)}$ and $C_4^{(2,3)}$ cases. And finally, for the $B_2^{(6,3)}$ singularity, the operators of orders 3 and 6 have the eigenvalues $\chi$ and $-\chi$ respectively. This follows from easy quasi-homogeneous considerations similar to those in [11, 12, 13].

b) The standard order of vanishing cycles in the distinguished set used to construct the $X_9/\mathbb{Z}_6$ diagram is $e_2, e_0, e_1, e_3$. As usual, for the tree diagram the order may be done arbitrary.

c) The three crystallographic groups corresponding to the three symmetric $X_9$ singularities with the Weyl groups in the notations are representations of the corresponding generalised braid groups.

We should also notice that the fact that the equivariant monodromies of Theorem 3.1 are at most factor-groups of the crystallographic groups in question already follows from the description of the discriminants of our singularities and the information about the orders of the Picard-Lefschetz operators. Indeed, consider first the four string diagrams of Figure 3 omitting their egle orientations and all the labellings. Applying Zariski’s method to calculate the fundamental groups of the complements to our discriminants, we see that the reduced diagrams are exactly the diagrams of relations between the generators of these groups. If we now restore the orders of the vertices then we come to the diagrammatic presentations of the corresponding crystallographic groups obtained in [15]. To obtain similar coincidence with [15] for $X_9/\mathbb{Z}_6$, we use the interpretation of the triple intersection of the discriminant: the lower right triangle of the Dynkin diagram corresponds to the circular relations $abc = bca = cba$ in the fundamental group (see [8, 15]). Finally, the additional relations in [15] are the orders of the classical monodromy in our cases.

**Question 3.1** A relation between the discriminant of an interesting symmetric parabolic function and the orbit space of the related crystallographic group should be investigated. In particular, it would be interesting to find out why function singularities with non-isomorphic discriminants may give rise to the same crystallographic groups. At the moment, there are two examples of such a duplication: symmetric $J_{10}$ singularities with the discriminants $G_2$ and $C_3$ (see [14]) correspond to the same affine groups, $[K_3(6)]$ and $[K_8]$, as respectively the singularities $B_2^{(6,3)}$ and $A_3^{(4)}$ of this paper.

The skew-hermitian versions of the five affine reflection groups are given by the Dynkin diagrams of the two-variable symmetric $X_9$ singularities of Figure 4. The diagrams are drawn for $\chi = i$ and $\chi = \omega$ for the 2-variable automorphisms of Table 1 of orders respectively 4 and 3 or 6. For $\chi = -i, \omega$, all the numbers must be conjugated. Inside the vertices are the eigenvalues of the Picard-Lefschetz operators. The empty vertices correspond to the $kA_1$ degenerations, hence all the eigenvalues for them are 1 and the Picard-Lefschetz operators.
are $a \rightarrow a - (a, e)e/k$. The three cycles forming the lower right triangle of the $X_9/Z_6$ diagram are linearly dependent.

References


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