Unitary reflection groups associated with singularities of functions with cyclic symmetry

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Abstract

Finite groups generated by Euclidean reflections became a very common object in various problems of singularity theory since their importance in classification of critical points of functions was demonstrated by Arnold [1, 2]. We show that a number of finite groups generated by *unitary* reflections are also naturally related to function singularities, namely to those invariant under a unitary reflection of finite order. To establish this one has to consider function-germs on a manifold with boundary and lift them to a cyclic covering of the manifold ramified over the boundary. The construction provides a new notion of roots for the groups under consideration and skew-Hermitian versions of these groups.

The present paper can be considered as an initial step in solving a problem of finding singularity theory interpretations of finite groups generated by unitary reflections which was posed by Arnold some twenty years ago.

In 1972 Arnold discovered a natural one-to-one correspondence between simple function singularities and Weyl groups whose Dynkin diagrams have no multiple edges, that is, the groups A_k, D_k and E_k [1]. It was observed that for a simple function there exists a distinguished basis in the homology of its Milnor fibre for which the intersection graph is just the relevant Dynkin diagram. The monodromy group of a simple singularity is exactly the corresponding Weyl group. The set of irregular orbits of the complexification of the Weyl group is isomorphic to the discriminant of the function.

In 1978 Arnold extended his results to the relation between functions on a manifold with boundary and the groups with double edges in their Dynkin diagrams (B_k, C_k and F_4) [2]. The crucial idea there was to pass from the relative homology of the pair of Milnor fibres of a function and of its restriction to the boundary, to a space with a well-defined intersection from. This was provided by the introduction of the double covering of the manifold ramified over the boundary.

The last Weyl group, G_2, appeared as an S_3-symmetric function singularity [2].

Arnold's discoveries were followed by a successful hunt for a singularity theory interpretation of the reflection groups H_3, H_4 and $I_2(p)$ [13, 17, 12].

Analysing Arnold's approach to the definition of the intersection form of a boundary singularity, one arrives at a natural question: What happens if the double covering is replaced by a cyclic of an arbitrary order? The answer to this question brings us to singularities related to some of finite groups generated by unitary reflections. The relation generalises that above by Arnold. The groups are almost singled out from the complete classification list of Shephard and Todd [18] by the two requirements:

- the groups are generated by n reflections in $\mathbf{C}^n$,
- the degrees of the basic invariants must be symmetric with respect to their arithmetical mean.

'Almost' here means that we omit H_3 and H_4 and some two-dimensional groups.

In fact it would be very interesting to understand which intrinsic group property picks up the reflection groups appearing in our context from the entire list of Shephard-Todd groups.

We have to remark that five of the unitary reflection groups we are considering went unnoticed in Givental's paper [11]. Those are groups $A_k^{(m)}$ (as we denote them) and they correspond to particular values of the parameter in the Burau representation of braid groups. The latter was approached in [11] from a point of view which is rather close to our constructions.

Also we must say that classification of functions with a cyclic group symmetry is not a new topic in singularity theory. For example, it was considered by Wassermann [21] and Tibar [20]. In particular, paper [20] contains the singularities which we call $B_k^{(m)}$. But no relation to unitary reflection groups has been spotted.

The paper is organised as follows.

In Sections 1 and 2 we recall basic facts about simple functions on a smooth manifold and on a manifold with boundary emphasizing their relations with the Weyl groups.

Section 3 discusses the unitary reflection groups under consideration.

In Section 4 we consider finite cyclic coverings of a manifold with boundary ramified over the boundary. Lifting a function to the covering space we construct a distinguished basis in the homology of the lifted (cyclically symmetric) Milnor fibre which respects the action of the group of the covering.

In Section 5 we study functions on a complex linear space which are symmetric with respect to the action of a finite cyclic group $\mathbf{Z}_m$ fixing a hyperplane. We classify elliptic cyclically symmetric functions, that is, those with finite monodromy. Consideration of elliptic singularities turns out to be more natural than a traditional study of simple objects. We obtain a realisation of the unitary reflection groups as monodromy groups acting on the character subspaces of $\mathbf{Z}_m$ in the homology of symmetric Milnor fibres of the elliptic singularities.

In Section 5.5 we observe that a geometric basis in a character subspace has a lattice property very similar to that of the root system of a Weyl group. Apparently this has not been noticed in the group theory where the main approach to the cases under consideration was to choose an *ortho*basis in which the Hermitian matrix of the products would be real (cf. [7, 9]). In our approach the matrix is not real (cf. [16]) and the roots naturally have different length.

Every elliptic function with cyclic symmetry is simple as a non-symmetric singularity. This inscribes the related unitary group into a Weyl group. In Section 5.6 we give an interpretation of this in the language of Dynkin diagrams which generalises the folding operation producing, for example, the canonical diagram of B_k from that of A_{2k-1}.

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1 Isolated singularities of functions

We start with a necessary singularity theory background.

1.1 Milnor fibre of a function

Consider a holomorphic function-germ $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ with an isolated critical point at the origin (we call a point *critical* if the gradient of the function at this point vanishes). In what follows we identify germ f with its representative.

Consider a sufficiently small closed ball $B \subset \mathbf{C}^{n+1}$ centred at 0. Choose a non-zero complex number ε with its modulus much smaller than the radius of B.

Definition 1.1 A *Milnor fibre* of f is the variety $V = \{f = \varepsilon\} \cap B$.

A Milnor fibre is a real $2n$-dimensional smooth manifold with boundary. The following is a classical result by Milnor [14]:

Theorem 1.2 *A Milnor fibre of a function f with an isolated singularity is homotopy equivalent to a wedge of a finite number of n-dimensional spheres.*

Definition 1.3 The finite number $\mu = \mu(f)$ of the spheres is called the *Milnor number* of function f.

1.2 Vanishing cycles

A basis for the nth reduced homology $\overline{H}_n(V, \mathbf{Z})$ of a Milnor fibre can be obtained in the following way.

Recall that a critical point p of function g is called *Morse* if the matrix of second derivatives of g at p is non-degenerate. In a neighbourhood of such a point one can choose coordinates $x_0, \ldots, x_n$, with the origin at p, so that

$$g = x_0^2 + \ldots + x_n^2 + g(p) \ .$$

Let $\widetilde{f}$ be a *morsification* of f, that is, a small perturbation of f which has only Morse critical points, all situated on different levels. We assume $\widetilde{f}$ to be sufficiently close to f which, in particular, guarantees that the number of critical points (and thus critical values) of $\widetilde{f}$ is μ, all of them in B.

Take a non-critical value $* \in \mathbf{C}$ of $\tilde{f}$. The level $\{\tilde{f} = *\} \cap B$ is diffeomorphic to a Milnor fibre of f. We denote it by the same letter V.

Consider the homotopy of V defined by a non-self-intersecting path γ in $\mathbf{C}$ which joins point $*$ with a critical value c of $\tilde{f}$ and does not pass through other critical values (Figure 1). Approaching c, we get the local changes on a non-critical level of $\tilde{f}$ exactly the same as those on the variety $x_0^2 + \ldots + x_n^2 = \delta$ in $\mathbf{C}^{n+1}$ when positive number δ tends to zero. In the latter case the n-dimensional sphere defined by the equation in the real space $\mathbf{R}^{n+1} = \{\operatorname{Im} x_i = 0\}$ contracts to a point. This gives an n-sphere in V called a *vanishing cycle.*

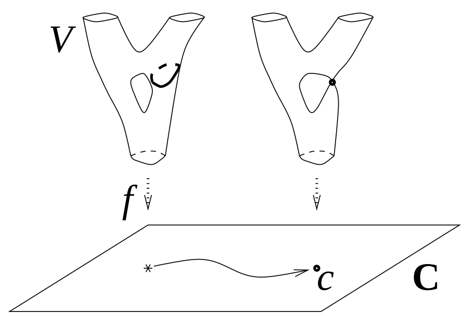

Figure 1: *A cycle vanishing at a critical point.*

Now join point $*$ in $\mathbf{C}$ with all μ critical values of $\tilde{f}$ by a star-like system of μ paths with no mutual and self-intersections. The collection of μ vanishing cycles in V defined by the system is called a *distinguished set* of vanishing cycles.

Theorem 1.4 [4] *A distinguished set of vanishing cycles forms a basis in* $\overline{H}_n(V, \mathbf{Z})$.

Example 1.5 Consider function $f = z^3$ on the complex line. Take $\tilde{f} = z^3 - 3z$ for its morsification and the zero-level of $\tilde{f}$ for the Milnor fibre V. Then $\overline{H}_0(V) = \{\alpha_1[-\sqrt{3}]+\alpha_2[0]+\alpha_3[\sqrt{3}], \sum \alpha_i = 0\} = \mathbf{Z}^2$. The morsification has critical values 2 and -2. For the system of paths from $* = 0$ to the critical values take the two straight intervals. The two vanishing cycles are respectively $[0] - [-\sqrt{3}]$ and $[\sqrt{3}] - [0]$. The intersection matrix of this distinguished set of vanishing cycles is the Cartan matrix of A_2.

In general, the self-intersection index of a vanishing cycle e is

$$\langle e, e\rangle = \begin{cases} 0, & n \text{ odd} \\ 2, & n \equiv 0 \bmod 4 \\ -2, & n \equiv 2 \bmod 4 \end{cases} \qquad (1)$$

1.3 Monodromy group of a singularity

A loop in $\mathbf{C}$ based at $*$ and missing critical values $c_1, \ldots, c_\mu$ of $\widetilde{f}$ defines an automorphism of the Milnor fibre V and, therefore, of $\overline{H}_n(V)$. The image of the representation on $\overline{H}_n(V)$ of the fundamental group $\pi_1(\mathbf{C} \setminus \{c_i\}_{i=1}^{\mu}, *)$ is called the *monodromy group* of the isolated function singularity f. It does not depend on the various choices done in the construction.

The star-like system of paths of the previous section defines a system of generators of the monodromy group in the following way. A non-self-intersecting path γ in $\mathbf{C}$, from a regular value $*$ to a critical value c of the morsification $\widetilde{f}$, defines a loop in $\mathbf{C}$ based at $*$ which leaves $*$ along γ, nearly enters c, goes around c once in the positive direction and comes back to $*$ along γ^{-1}. The described loop is called a *simple loop*. It defines a *Picard-Lefschetz automorphism* h_γ of $\overline{H}_n(V, \mathbf{Z})$:

$$h_\gamma : a \mapsto a + (-1)^{(n+1)(n+2)/2} \langle a, e\rangle e \ , \qquad (2)$$

where e is the cycle vanishing along γ, and $\langle \cdot, \cdot \rangle$ the intersection form. For even n, h_γ is an involution which is identical on the hyperplane $\langle a, e\rangle = 0$.

Example 1.6 The system of paths of Example 1.5 defines two monodromy operators on $\overline{H}_0(V)$, interchanging $[-\sqrt{3}]$ with $[0]$ and $[0]$ with $[\sqrt{3}]$ respectively. Hence the monodromy group of $f = z^3$ is the Weyl group A_2.

1.4 Classification of simple function singularities

Definition 1.7 Two function-germs $f, g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are said to be *equivalent* ($f \sim g$) if there exists a biholomorphism-germ b of $(\mathbf{C}^{n+1}, 0)$ such that $f = g \circ b$.

Definition 1.8 Two function-germs, f and g, defined on two different spaces $(\mathbf{C}^r, 0)$ and $(\mathbf{C}^s, 0)$, are said to be *stably equivalent* if they become equivalent after addition of squares of appropriate number of new variables:

$$f(x_1, \ldots, x_r) + x_{r+1}^2 + \ldots + x_m^2 \sim g(y_1, \ldots, y_s) + y_{s+1}^2 + \ldots + y_m^2 \ .$$

Definition 1.9 An equivalence class X of function-germs is *adjacent* to an equivalence class Y, $X \to Y$, if a representative of X can be deformed into a representative of Y by an arbitrary small perturbation.

Definition 1.10 An equivalence class of function-germs is *simple* if it is adjacent to only finitely many equivalence classes.

Theorem 1.11 (Arnold [1]) *Up to the stable equivalence, the list of simple function-germs is*

$A_k, k \geq 0$	$D_k, k \geq 4$	E_6	E_7	E_8
x_1^{k+1}	$x_1^2 x_2 + x_2^{k-1}$	$x_1^3 + x_2^4$	$x_1^3 + x_1 x_2^3$	$x_1^3 + x_2^5$

If the number of the variables is odd, the monodromy group of a simple function is the corresponding Weyl group. If the number of the variables is $1 \bmod 4$, *there exists a distinguished basis in the reduced homology of a Milnor fibre of a simple function in which the intersection matrix is the Cartan matrix of the Weyl group.*

Notice that the index in the notation of a simple singularity is its Milnor number.

Stabilisation of a function by addition of squares of two new variables provides the double suspension of each of the elements of a distinguished basis which canonically changes the sign of their intersection [10]. Therefore if the number of the variables is $3 \bmod 4$, a distinguished basis for a simple singularity may be chosen with the intersection matrix being negative of the Cartan matrix.

All possible adjacencies of the listed singularities are compositions of

$$A_k \to A_{k-1} \qquad D_k \to D_{k-1}, A_{k-1} \qquad E_k \to E_{k-1}, D_{k-1}, A_{k-1}$$

Remark 1.12 We see that simple functions of odd number of variables are *elliptic*: their intersection forms are non-degenerate and sign-definite. This implies finiteness of the monodromy group. None of non-simple singularities shares these properties.

Yet another appearance of the Weyl groups in the context is as follows.

Let $\mathcal{O}_{n+1}$ be the space of all holomorphic function-germs on $(\mathbf{C}^{n+1}, 0)$. A *versal deformation* of a function-germ $f \in \mathcal{O}_{n+1}$ is that which in a sense contains all possible deformations of f. It is *miniversal* if its base has minimal possible dimension. The latter, referred to as the *codimension* of f, coincides with the Milnor number. For a miniversal deformation one may take

$$f + \lambda_1 e_1 + \ldots + \lambda_\mu e_\mu \ ,$$

where the $e_i \in \mathcal{O}_{n+1}$ represent a linear basis of $\mathcal{O}_{n+1}/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}})$, and the $\lambda_i \in \mathbf{C}$ are the parameters of the deformation [3].

The *discriminant* of f, $\Delta(f) \in \mathbf{C}^\mu$, is the set of those values of the parameters of its miniversal deformation for which the corresponding functions on $\mathbf{C}^{n+1}$ have critical value 0.

For a simple function X_μ, the discriminant $\Delta \subset \mathbf{C}^\mu$ is biholomorphic to the set of irregular orbits in the orbit set $\mathbf{C}^\mu$ of the complexification of the Weyl group X_μ. Therefore, the complement $\mathbf{C}^\mu \setminus \Delta$ is an Eilenberg-MacLane $k(\pi, 1)$-space for Brieskorn's group $B(X_\mu)$ of generalised braids [5] (the group of ordinary braids on $k+1$ threads is $B(A_k)$). The monodromy group of singularity X_μ is a representation of $B(X_\mu)$.

Example 1.13 A miniversal deformation of A_2-singularity z^3 can be taken in the form $z^3 + \lambda_1 z + \lambda_2$. The discriminant of this family is the set of values $(\lambda_1, \lambda_2) \in \mathbf{C}^2$ corresponding to the polynomials with multiple roots.

The complex version of Weyl group A_2 is the symmetric group on 3 elements acting on the plane $\mathbf{C}^2 = \{z_1 + z_2 + z_3 = 0\} \subset \mathbf{C}^3$ by permutations of the coordinates. The orbit space $\mathbf{C}^2/A_2 \simeq \mathbf{C}^2$ is the same space of polynomials $z^3 + \lambda_1 z + \lambda_2$. The set of irregular orbits is that coming from the mirrors $z_i = z_j$, that is, the set of the polynomials with multiple roots again.

2 Boundary singularities

2.1 Vanishing cycles and semi-cycles

Assume now that in the domain $(\mathbf{C}^{n+1}, 0)$ of a function-germ f we have a distinguished hyperplane $(\mathbf{C}^n, 0)$. This hyperplane will be called a *boundary*. The restriction of f to the boundary will be denoted f_0.

Taking an appropriate ball $B \subset \mathbf{C}^{n+1}$ and small ε, we obtain a pair of Milnor fibres:

$$V = \{f = \varepsilon\} \cap B \ \supset \ V_0 = \{f_0 = \varepsilon\} \cap (B \cap \mathbf{C}^n) \ .$$

We have an obvious

Proposition 2.1

$$H_n(V, V_0; \mathbf{Z}) = \mathbf{Z}^{\mu+\mu_0} \ ,$$

where μ and μ_0 are the Milnor numbers of f and f_0.

Similar to the absolute case one can define a distinguished basis of the relative homology. For this we take a morsification $\widetilde{f}$ of f whose restriction $\widetilde{f}_0$ to the boundary is a Morse function as well. Also we assume that the critical values of $\widetilde{f}$ and $\widetilde{f}_0$ are all distinct.

Now choose a number $* \in \mathbf{C}$ which is a regular value for both $\widetilde{f}$ and $\widetilde{f}_0$. As earlier, a non-self-intersecting path in $\mathbf{C}$ from $*$ to a critical value of $\widetilde{f}$ defines a vanishing cycle on V and thus an element of $H_n(V, V_0)$. Similar path going to a critical value of $\widetilde{f}_0$ defines an element of $H_n(V, V_0)$ called a *vanishing semi-cycle*. Its local model is the set

$$\{x_0 + x_1^2 + \ldots + x_n^2 = \delta > 0\}$$

in the half-space $\{x_0 \geq 0\}$ of the real x-space $\mathbf{R}^{n+1}$ (here $\{x_0 = 0\}$ is the boundary). With δ tending to zero the set contracts to a point.

As in the absolute case, a system of $\mu + \mu_0$ paths on $\mathbf{C}$ from $*$ to critical values of $\widetilde{f}$ and $\widetilde{f}_0$ defines a distinguished set of vanishing cycles and semi-cycles which turns out to be a basis of $H_n(V, V_0; \mathbf{Z})$. Of course, the paths are assumed to have no self-intersection and common points, except for point $*$.

2.2 The double cover

To get a good definition of intersection form and monodromy for a boundary function singularity, Arnold introduced its double covering ramified over the boundary [2]. For this one just sets $x_0 = z^2$. The setting lifts f to a function $\widehat{f}(z, x_1, \ldots, x_n) = f(z^2, x_1, \ldots, x_n)$ even in z. Its Milnor number is $\widehat{\mu} = 2\mu + \mu_0$.

The Milnor fibre $\widehat{V}$ of $\widehat{f}$ is a double cover of the Milnor fibre V of f ramified over the Milnor fibre V_0 of f_0. The involution $z \mapsto -z$ acts on $H_n(\widehat{V})$. The anti-invariant part $H_{-1} \subset H_n(\widehat{V})$ has rank $\mu + \mu_0$, the same as that of $H_n(V, V_0)$, and substitutes the latter space in all the considerations involving intersection forms.

A natural basis for H_{-1} comes from a distinguished basis of $H_n(V, V_0; \mathbf{Z})$. It is formed by *long* and *short cycles* which are respectively the complete inverse images (appropriately oriented) of vanishing cycles and semi-cycles on (V, V_0). The self-intersection number of a short cycle is the same as that of an ordinary vanishing cycle (1). The self-intersection of a long cycle is twice as large, that is, 0, 4 or -4 depending on the dimension.

Picard-Lefschetz operators which generate the monodromy group acting on H_{-1} are given by formula (2) if the corresponding vanishing cycle is short and by

$$h_\gamma : a \mapsto a + (-1)^{(n+1)(n+2)/2} \langle a, e \rangle e/2 \ , \tag{3}$$

if e is long.

2.3 Simple boundary singularities

We consider two function-germs, f and g, on $(\mathbf{C}^{n+1}, 0)$ with boundary $(\mathbf{C}^n, 0)$ to be *equivalent* if there exists a biholomorphism-germ b of the pair $(\mathbf{C}^{n+1}, \mathbf{C}^n)$ such that $f = g \circ b$.

Theorem 2.2 (Arnold [2]) *Up to the stable equivalence, the list of simple function-germs on* $(\mathbf{C}^{n+1}, 0) = \{(x_0, \dots, x_n)\}$ *with boundary* $x_0 = 0$ *is*

A_k, D_k, E_k	$B_k, k \geq 2$	$C_k, k \geq 3$	F_4
$x_0 + f_0(x_1, \dots, x_n)$, $f_0 \in A_k, D_k, E_k$	x_0^k	$x_0 x_1 + x_1^k$	$x_0^2 + x_1^3$

If the number of the variables is odd, the monodromy group of a simple function acting on the anti-invariant homology H_{-1} *is the Weyl group of the same name. If the number of the variables is* $1 \bmod 4$*, for a simple function there exists a distinguished basis of* H_{-1} *in which the intersection matrix is the Cartan matrix of the corresponding Weyl group.*

The ellipticity property generalising in the obvious way that of Remark 1.12 holds.

In terms of Dynkin diagrams of the Weyl groups, the double cover of the previous section is the quotient under the involution of the canonical diagram (Figure 2). In terms of the groups themselves, this is passing to the subgroup generated by the products of the commuting reflections in the roots glued together by the folding.

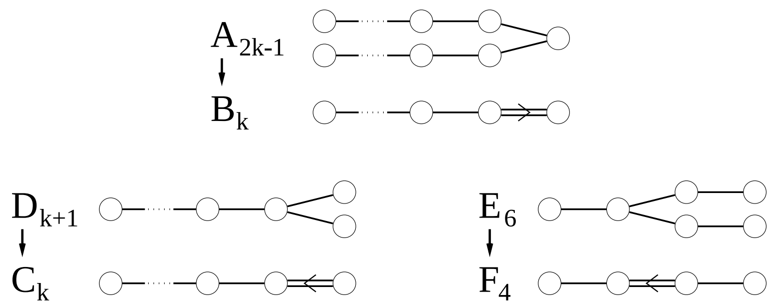

Figure 2: *Folding Dynkin diagrams*

All the adjacencies of singularities of Theorem 2.2 are those induced by the inclusions of the corresponding canonical Dynkin diagrams.

What was said by the end of Section 1.4 about the relations between the discriminant of a function and the set of irregular orbits of the associated Weyl group remains valid with minor adjustments. Namely, a *miniversal deformation* (with the base of dimension $\mu + \mu_0$) of a boundary function singularity $f \in \mathcal{O}_{n+1}$ is now

$$f + \lambda_1 e_1 + \ldots + \lambda_{\mu+\mu_0} e_{\mu+\mu_0} \ ,$$

where the e_i form a basis of

$$\mathcal{O}_{n+1}/(x_0 \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \ .$$

Also the *discriminant* of f now consists of two parts which correspond to either a function or its restriction to the boundary having critical value 0.

3 The finite unitary reflection groups

The list of unitary groups we will be dealing with is given in Figure 3. It contains one two-index series and seven exceptional groups.

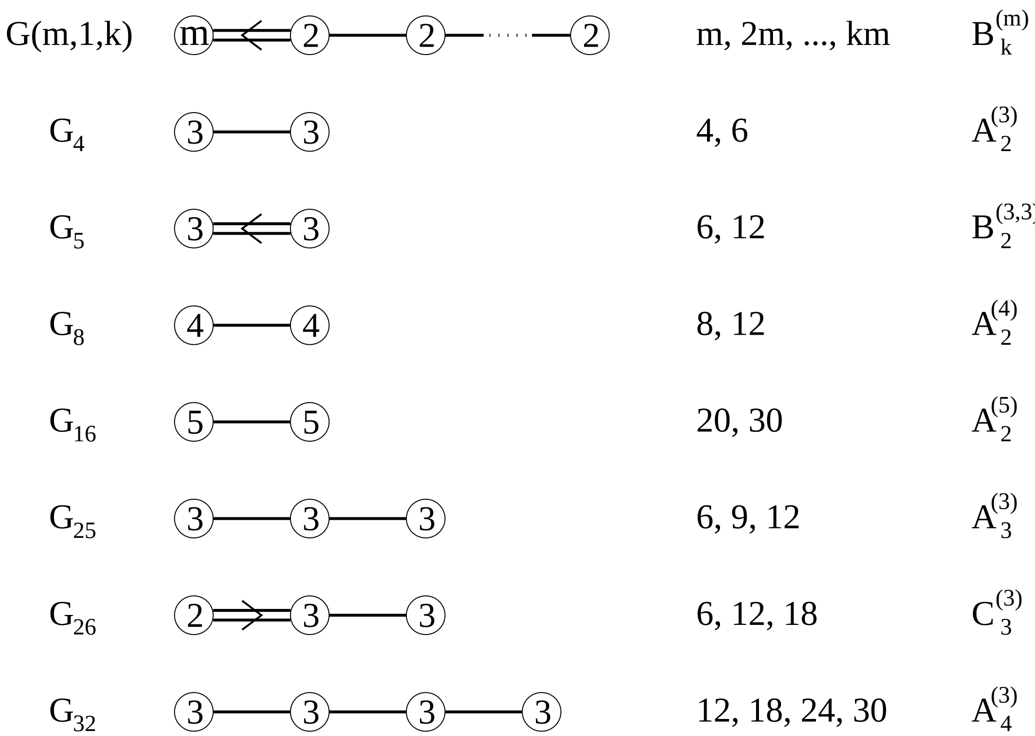

Figure 3: *Relations defining the finite unitary reflection groups and the degrees of the basic invariants.*

In the first column of the table we give the notation going back to the original paper by Shephard and Todd [18]. For example, the index in the notation of an exceptional group is its number in the classification list of [18]. We assume in the series $k \geq 1$ (the number of vertices of the graph) and $m \geq 3$ (group $G(2,1,k)$ would be just the Weyl group B_k).

The last column contains the notation of the same groups which is more illustrative from the singularity theory point of view (we shall see the reason) and is partially borrowed from [6]. The lower index is the dimension of the standard linear representation.

The third column gives the degrees of the basic invariants of the standard representation. The order of a group is the product of all the degrees as the corresponding orbit space is smooth.

The second column of the table contains diagrams encoding abstract representations of the groups. Each vertex is a generator whose order is written inside the circle. The multiplicity of the edge joining generators u and v is 2 less than the length of the *braiding* relation:

$uv = vu$, if there is no edge between the vertices;

$uvu = vuv$, if the edge is ordinary;

$uvuv = vuvu$, for the double edge.

Therefore, each of our unitary groups is a quotient of the generalised braid group corresponding to the Weyl group which enters the notation in the fourth column. The unitary reflection group is obtained by assigning the finite orders to the generators. Assigning order 2 to all the generators would give just the Weyl group itself.

The exact meaning of an inequality sign on a double edge of a diagram will be explained later, in Section 5.2. There a representation diagram will be interpreted as a Dynkin diagram of a unitary analog of a root system of a Weyl group. As in the Euclidean case, the sign indicates the difference in the root length.

We should remark that the series $G(m, 1, k)$ contains, for $k = 1$, all finite cyclic groups which make the third line in the original list of [18].

The standard linear representation of a group can be read from its graph as follows [9, 8]. Consider the vector space $\mathbf{C}^n$ formally spanned by the vertices $v_1, \ldots, v_n$ of the graph. The graph gives a positive-definite Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathbf{C}^n$:

$\langle v_i, v_i \rangle = \sin \frac{\pi}{p_i}$, where p_i is the number written at the vertex (the order of the generator);

$\langle v_i, v_j \rangle = 0$, if there is no edge joining the vertices;

$\langle v_i, v_j \rangle = -\cos \frac{\pi}{3} = -\frac{1}{2}$, for an ordinary edge;

$\langle v_i, v_j\rangle = -(\frac{1}{2}\sin\frac{\pi}{p_j})^{1/2}$, for a double edge with $p_i = 2$,

$\langle v_1, v_2\rangle = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, for G_5.

Now the generator (complex reflection) corresponding to the vertex is the rotation of order p_j around the hyperplane which is Hermitian orthogonal to v_j:

$$v \mapsto v + (e^{2\pi i/p_j} - 1)\frac{\langle v, v_j\rangle}{\langle v_j, v_j\rangle} v_j \ . \tag{4}$$

4 Cyclic coverings of functions on a manifold with boundary

We return to boundary function singularities of Section 2.1 keeping all the notations used there. Generalising the covering construction of Section 2.2 we set

$$x_0 = z^m \ ,$$

where $m > 1$ is an arbitrary integer. This gives an m-fold covering $c : \widehat{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ ramified along the boundary.

As earlier, a function f on the manifold $\mathbf{C}^{n+1}$ with boundary $\mathbf{C}^n$ lifts to a function $\widehat{f} = f \circ c$ on $\widehat{\mathbf{C}}^{n+1}$. This time

$$\widehat{\mu} = m\mu + (m-1)\mu_0 \ .$$

For a Milnor fibre of $\widehat{f}$ we take the inverse image $\widehat{V} = c^{-1}(V)$ of the Milnor fibre V of f. The group $\mathbf{Z}_m$ of the covering c acts on $\widehat{V}$ and on its homology. We get the splitting

$$\overline{H}_n(\widehat{V}, \mathbf{C}) = \oplus_{\chi^m=1} H_\chi$$

into subspaces on each of which $\mathbf{Z}_m$ acts by a character χ:

$$\dim H_{\chi=1} = \mu \ , \qquad \dim H_{\chi\neq 1} = \mu + \mu_0 \ .$$

A geometrical basis for each of the H_χ can be obtained from a distinguished basis of vanishing cycles and semi-cycles of the boundary Milnor pair (V, V_0) in the following way (see Figure 4).

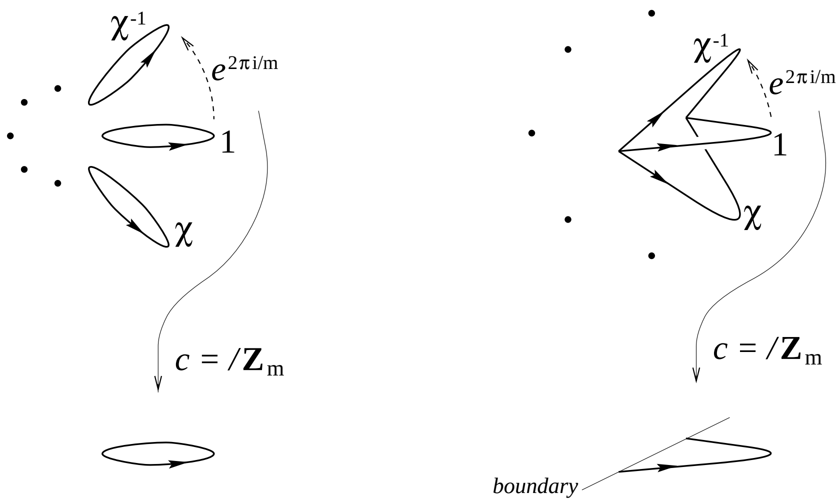

Figure 4: *Lifting a cycle and a semi-cycle of a boundary singularity f to long and short χ-cycles in a character subspace H_χ of the $\mathbf{Z}_m$-symmetric function $\widehat{f} = f \circ c$.*

The full inverse image under c of a vanishing cycle σ of the pair consists of m cycles $\sigma_0, \ldots, \sigma_{m-1}$ in $\widehat{V}$. We take them with the orientation inherited from σ. We assume that the ordering of the σ_j is chosen so that the rotation $z \mapsto e^{2\pi i/m} z$ provides the cyclic permutation $\sigma_j \mapsto \sigma_{(j+1) \bmod m}$. The linear combination

$$\sigma_\chi = \sum_{j=0}^{m-1} \chi^{-j} \sigma_j$$

defines a non-zero element in H_χ. We call it a *long χ-cycle*.

For a vanishing semi-cycle on (V, V_0), we can arrange a similar linear combination of its m preimages in $\widehat{V}$. This linear combination is a cycle if $\chi \neq 1$ (since $1 + \chi + \ldots + \chi^{m-1} = 0$) and defines a non-zero element in H_χ. We call this element a *short χ-cycle*.

Both long and short χ-cycles are defined up to multiplication by powers of χ and by -1.

Consider now a (skew-)Hermitian form on $\overline{H}_n(\widehat{V}, \mathbf{C})$ defined by the intersection index $\langle \cdot, \cdot \rangle$.

Proposition 4.1 *The self-intersection index of a long χ-cycle is 0 if n is odd and $(-1)^{n/2}2m$ if n is even. The self-intersection index of a short χ-cycle is is $(-1)^{(n-1)/2}\frac{1+\chi}{1-\chi}m$ and $(-1)^{n/2}m$ respectively.*

The fact is obvious for long cycles. For short ones it is based on

Lemma 4.2 *Consider a generic local intersection of two semi-cycles on V at a point of V_0. One can choose corresponding short χ-cycles on $\widehat{V}$ so that their local intersection index is $m/(1-\chi)$.*

The local lifting of the semi-cycles is given in Figure 5. Figure 6 proves that the local intersection is that stated in the lemma.

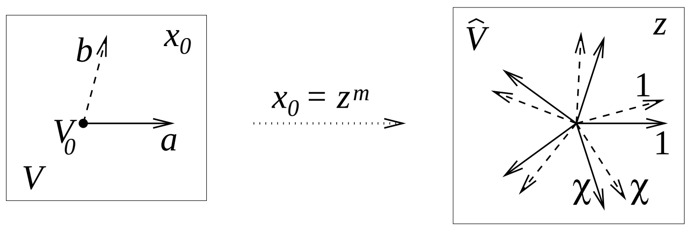

Figure 5: *Choice of the local lifting of intersection of semi-cycles to H_χ.*

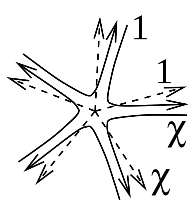

Figure 6: *Demonstration that for the lifted cycles of the previous Figure $\langle(1-\chi)\,\widehat{a},\widehat{b}\rangle = m$.*

The construction of the monodromy group of a boundary singularity provides the Picard-Lefschetz operators $h_\gamma : \overline{H}_n(\widehat{V},\mathbf{C}) \to \overline{H}_n(\widehat{V},\mathbf{C})$. Each of them splits into a direct sum $h_\gamma = \oplus_{\chi^m=1} h_{\gamma,\chi}$ of operators acting on individual character subspaces H_χ.

Proposition 4.3 *The action of the Picard-Lefschetz operator $h_{\gamma,\chi} : H_\chi \to H_\chi$ corresponding to a short vanishing χ-cycle e is given by the formula*

$$h_{\gamma,\chi} : a \mapsto a + (-1)^{(n+1)(n+2)/2}(1-\chi)\langle a, e\rangle e/m \ .$$

Similar operator corresponding to a long vanishing χ-cycle is

$$h_{\gamma,\chi} : a \mapsto a + (-1)^{(n+1)(n+2)/2}\langle a, e\rangle e/m \ .$$

Thus, in the symmetric case (n even) the operators are rotations around the hyperplanes Hermitian-orthogonal to the cycles. Their orders are equal to the order of χ and 2 respectively.

Example 4.4 Consider the covering $x_0 = z^m$ of a deformation $f(x) = x^2 - 4x + 3$ of the one-dimensional boundary singularity B_2. Take $\widehat{f} = 0$ as the covering Milnor fibre $\widehat{V}$ (see Figure 7). Join $0 \in \mathbf{C}$ by the straight paths with the critical values $f(0) = 3$ and $f(2) = -1$ of f. Take the linear combination of the points on the inner circle in Figure 7 for a short χ-cycle $e_1 \in H_\chi \subset \overline{H}_0(\widehat{V})$ vanishing on the level $\widehat{f} = 3$, and the difference between the linear combinations on the outer and inner circles for a long χ-cycle e_2 vanishing on $\widehat{f} = -1$. The intersection matrix $(\langle e_i, e_j\rangle)$ is $\begin{pmatrix} m & -m \\ -m & 2m \end{pmatrix}$.

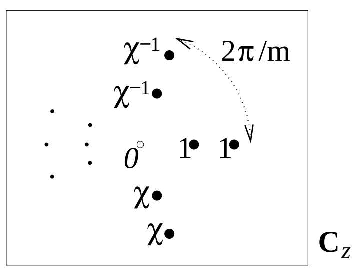

Figure 7: *Vanishing χ-cycles for the m-covering of B_2 singularity.*

The Picard-Lefschetz operator h_1 rotates the points on the inner circle anti-clockwise by $2\pi/m$ which means

$$h_1(e_1) = \chi e_1 \, ,$$

$$h_1(e_2) = e_2 + (1-\chi)e_1 = e_2 - (1-\chi)\langle e_2, e_1\rangle e_1/m = e_2 + (\chi - 1)\frac{\langle e_2, e_1\rangle}{\langle e_1, e_1\rangle}e_1 \,.$$

The operator h_2 swaps points on the same ray from the origin:

$$h_2(e_1) = e_1 + e_2 = e_1 - \langle e_1, e_2\rangle e_2/m = e_1 + (e^{2\pi i/2} - 1)\frac{\langle e_1, e_2\rangle}{\langle e_2, e_2\rangle}e_2 \,,$$

$$h_2(e_2) = -e_2 \,.$$

Both operators give a good illustration to Proposition 4.3 ($n = 0$). Comparing the above formulae with (4) and the settings of Section 3, we see that h_1 and h_2 generate in $U(2)$ a subgroup isomorphic to $B_2^{(p)}$, where p is the order of χ. Indeed, the orders of the generators coincide with those of the canonical generating reflections of $B_2^{(p)}$ and the intersection matrix in the rescaled basis $\{e_1/\sqrt{m \sin \frac{\pi}{p}}, e_2/\sqrt{2m \sin \frac{\pi}{p}}\}$ is exactly that encoded by the graph $B_2^{(p)}$. Notice also that once the braiding relation holds for generators of a reflection group, it holds for their powers.

We generalise these observations to all of our unitary reflection groups in the next section.

5 Functions with cyclic symmetry

5.1 Elliptic singularities

The group $\mathbf{Z}_m$ of the covering $c : \widehat{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ of the previous section acts on $\widehat{\mathbf{C}}^{n+1}$ by unitary reflections which fix the ramification locus $\mathbf{C}^n$. Forgetting for a moment about the covering itself, let us consider holomorphic function-germs on $(\widehat{\mathbf{C}}^{n+1}, 0)$ invariant under this action. We call them *functions with cyclic symmetry* or *cyclic singularities* for short.

The natural equivalence relation for such functions is that up to $\mathbf{Z}_m$-equivariant biholomorphisms of $(\widehat{\mathbf{C}}^{n+1}, 0)$. Of course, this is just the boundary equivalence of the functions lowered to the quotient space $(\widehat{\mathbf{C}}^{n+1}, 0)/\mathbf{Z}_m$ which is the base of the covering c. Stable equivalence adds to a function-germ squares of new variables on which the symmetry group acts trivially.

A (mini)versal deformation of a cyclic singularity within the space of functions with the same cyclic symmetry is induced by the covering c from a (mini)versal deformation of the related boundary function singularity. The dimension of the base of a miniversal deformation of a cyclic singularity (that is, its *codimension*) is $\mu + \mu_0$.

For the *monodromy group of a cyclic singularity* we take that generated by the Picard-Lefschetz operators $h_\gamma = \oplus_\chi h_{\gamma,\chi}$ of the previous section.

Definition 5.1 A function with cyclic symmetry is called *elliptic* if it is stably equivalent to a cyclic singularity in an odd number of variables with finite monodromy group.

Classification of elliptic cyclic singularities has more sense than that of simple cyclic singularities. The latter produces a larger and more straightforward list which consists of arbitrary cyclic liftings of simple boundary functions.

Up to the stable equivalence, elliptic cyclic singularities with $\mathbf{Z}_2$-symmetry are the double covers of the simple boundary singularities. For the higher symmetry we have

Theorem 5.2 *Up to the stable equivalence, the list of elliptic functions with cyclic symmetry* $\mathbf{Z}_m$, $m > 2$, *is as follows:*

notation	corresponding boundary singularity	the order of the covering $x_0 = z^m$	covering absolute singularity
$B_k^{(m)}$	$B_k: \ x_0^k, \ k \geq 1$	m	A_{km-1}
$A_2^{(3)}$	$A_2: \ x_0 + x_1^3$	3	D_4
$A_2^{(4)}$	$A_2: \ x_0 + x_1^3$	4	E_6
$A_2^{(5)}$	$A_2: \ x_0 + x_1^3$	5	E_8
$A_3^{(3)}$	$A_3: \ x_0 + x_1^4$	3	E_6
$C_3^{(3)}$	$C_3: \ x_0 x_1 + x_1^3$	3	E_7
$A_4^{(3)}$	$A_4: \ x_0 + x_1^5$	3	E_8

For an odd number of the variables, the monodromy group acting on H_χ, where χ is a primitive m*th root of unity, is the unitary reflection group whose notation coincides with the notation of the cyclic singularity.*

The dimension of the character subspace $H_{\chi\neq 1}$ is the codimension of the related boundary singularity. For convenience we set $B_1 = A_1$.

The adjacencies of the singularities are obvious (cf. Sections 1.4 and 2.3).

We prove Theorem 5.2 in the next two subsections.

Remarks 5.3 One can make a number of elementary but useful observations concerning the table:

(a) The complete set of the degrees of basic invariants of the unitary reflection group related to an elliptic cyclic singularity is a subset of that of the Weyl group in the last column. The maximal degrees (that is, Coxeter numbers) coincide. This shows how the orbit spaces (serving also as the bases of the corresponding miniversal deformations) are embedded one into the other.

(b) The complete set of the degrees of basic invariants of a unitary group $X^{(m)}$ is a multiple of that of the Weyl group X. According to [15], the two pairs {orbit space, set of irregular orbits} are isomorphic (as before, we are considering the complexification of the canonical representation of X).

(c) This is exactly the isomorphism (via c) between the bases of miniversal deformations of the cyclic and boundary singularities. Within such interpretation, the sets of irregular orbits are the discriminants of the singularities. The *discriminant of a cyclic singularity* is the set of those values of the deformation parameters for which the corresponding function has critical value 0.

(d) As we have already mentioned, the set of regular orbits of a Weyl group X is a $k(\pi, 1)$-space for the group $B(X)$ of generalised braids. Its generators and braiding relations between them are read from the (canonical) Dynkin diagram of group X in the same way as it was done in Section 3, without setting the orders of the generators to be finite. The monodromy group of a boundary singularity X is the representation of $B(X)$, which provides just X itself (by setting to 1 the square of each of the generators) in the case of an even-dimensional Milnor fibre. Similarly, the monodromy assertion of the theorem tells that the representation of $B(X)$ on the character subspace H_χ of the cyclic singularity $X^{(m)}$ is the unitary group $X^{(m)}$: it sets to 1 exactly the required powers of the generators and does not introduce any other relations.

5.2 Intersection diagrams

We start a proof of the theorem by demonstrating that its cyclic singularities do have the monodromy groups promised.

Let us show a way one can understand the diagrams of Figure 3 as Dynkin diagrams of distinguished bases of long and short χ-cycles in H_χ whose Picard-Lefschetz operators generate the unitary reflection groups required.

Forgetting the numbers at the vertices of these diagrams one obtains standard Dynkin diagrams of root systems B_k, A_k, C_3, within the traditional conventions: a vertex is a basic root, there are long roots of length 2 and short of length $\sqrt{2}$, no edge between the vertices means that the roots are orthogonal, a simple edge joining two vertices means that the scalar product of the roots is -1 if both are short and -2 if they are long, a double edge from a long root to a short one indicates that their scalar product is -2, edges joining roots of different length are equipped with the inequality sign open in the direction of the long root.

Each simple boundary function singularity X in $\mathbf{C}^{4r+1}$ has a distinguished basis of cycles and semi-cycles of $H_{4r}(V, V_0)$ that lifts to the double cover as a basis of long and short cycles in $H_{\chi=-1}$ which has the standard Dynkin diagram of the root system X as its intersection diagram (that is, the cycles are the roots, and the scalar product is defined by the intersection in the homology). Moreover, such a basis in H_{-1} has a geometrical realisation for which the number of points in the intersection of two cycles on $\widehat{V}$ is equal to the absolute value of their intersection number. Picard-Lefschetz operators on H_{-1} corresponding to the elements of this basis are orthogonal reflections in the roots and generate the Weyl group X.

The stable equivalence suspends the described relative basis to a basis of a simple boundary singularity in arbitrary dimension. We call the obtained basis *standard*.

Now take the standard basis of a boundary singularity and lift it to the m-cover to a distinguished basis of $H_{\chi\neq 1}$. What was said about the intersections in this subspace by the end of Section 4 implies (as a generalisation of the above discussion) that each diagram of Figure 3 (excluding $G_5 = B_2^{(3,3)}$)

can be understood as an intersection diagram of the latter basis within the following conventions:

- each vertex represents an element of the basis;
- a vertex corresponding to a long χ-cycle is assigned number 2, a vertex corresponding to a short χ-cycle is assigned order of χ;
- the squares of the basic χ-cycles are given by Proposition 4.1;
- no edge between the vertices means the cycles do not intersect;
- a simple edge joining two vertices means that the intersection index of the cycles is $m/(\chi - 1)$ if both are short and $-m$ if they are long (if n is odd, we must formally fix the ordering in the intersection, but this is not important at all, since all our graphs are trees);
- a double edge between cycles of different length indicates that their intersection index is $-m$;
- such an edge is equipped with the inequality sign open to the long cycle.

Now assume χ to be a primitive mth root of unity. It is easy to verify (cf. Example 4.4) that for even n one can multiply the elements of the standard distinguished basis of H_χ of a cyclic singularity of the table of Theorem 5.2 by appropriate complex numbers so that they would satisfy the normalisation conditions listed by the end of Section 3. Therefore, the Picard-Lefschetz operators $h_{\gamma,\chi}$ of Section 4 do generate the desired finite unitary group on H_χ.

For $B_k^{(m)}$ and $A_2^{(4)}$, character χ may happen to be non-primitive. In these cases one gets monodromy groups $B_k^{(\operatorname{ord}\chi)}$ and A_2.

5.3 End of the proof of Theorem 5.2

Now we show that the list of elliptic cyclic singularities given in the Theorem is complete.

Consider, for example, function-germs with the $\mathbf{Z}_3$-symmetry. Every such function not contained in the table is adjacent to at least one of the functions

$A_5^{(3)}$, $D_4^{(3)}$, $C_4^{(3)}$ and $F_4^{(3)}$ (in the obvious notations). If the number of the variables is odd, the monodromy group of each of these four singularities acting on $H_{\chi\neq 1}$ is that generated by the complex reflections satisfying the relations defined by the corresponding modified canonical Dynkin diagram (cf. Figure 3) with the vertices of orders 2 (if the root of the underlying Weyl group is long) or 3 (if it is short). None of such groups is finite according to the Shephard-Todd classification.

For a higher order symmetry one has to similarly consider functions $A_3^{(4)}$, $A_3^{(5)}$, $C_3^{(>3)}$ and $A_2^{(>5)}$.

5.4 Singularity $B_2^{(3,3)}$

The correspondence of Theorem 5.2 extends to include the last unitary reflection group $G_5 = B_2^{(3,3)}$ listed in Figure 3. For this one has to consider corner singularities of functions on $\mathbf{C}^{n+1}$, that is, those in presence of two transversal smooth hypersurfaces [19], and cyclically cover $\mathbf{C}^{n+1}$ twice with the branching along each of the hypersurfaces. Now take the $\mathbf{Z}_3 \times \mathbf{Z}_2$-covering of the only codimension 2 corner singularity

$$x_0 + x_1^2 + x_2^2 + \ldots + x_n^2\,, \qquad x_0 = z^3\,, \quad x_1 = w^2$$

which "embeds" it into E_6.

The rank 6 vanishing homology space of the symmetric E_6 splits into the direct sum $\oplus_{\chi_0^3=1,\chi_1^2=1} H_{\chi_0,\chi_1}$ of the character subspaces of $\mathbf{Z}_3 \times \mathbf{Z}_2$ (in fact, $H_{1,\pm 1} = 0$). A geometrical basis for each of these subspaces can be constructed from a geometrical basis of H_{χ_0} of $A_3^{(3)}$ which is additionally $\mathbf{Z}_2$-symmetric or anti-symmetric.

Combining this way the approaches of Sections 4 and 2.2 we obtain, for the two rank 2 subspaces $H_{e^{\pm 2\pi i/3},-1}$, bases which consist of one long and one short cycle each. The self-intersection index of the short cycle is that given by Proposition 4.1 and of the long one is twice as much (unlike Proposition 4.1, this is not zero for an odd-dimensional Milnor fibre). The cycles can be chosen so that their intersection index is $6/(\chi - 1)$. The $B_2^{(3,3)}$ graph of Figure 3 can be respectively interpreted as the intersection diagram in this basis.

The Picard-Lefschetz operator on $H_{\chi,-1}$ corresponding to the short cycle acts by the formula of Proposition 4.3 ($m = 3$). The operator corresponding to the long cycle acts by the formula similar to that, with $2m = 6$ in the denominator (that is, differs form the long χ-cycle operators of Proposition 4.3). Both operators are of order 3.

Passing to multiples of the above two cycles, one can reduce the intersection matrix to that of Section 3. Therefore, for even n, the monodromy group acting on each of the subspaces $H_{e^{\pm 2\pi i/3},-1}$ is $G_5 = B_2^{(3,3)}$.

Naturally we call the discussed singularity $B_2^{(3,3)}$. It turns out to be the only elliptic codimension >1 singularity with similar direct product symmetry of order >4.

5.5 The lattice property

The distinguished bases of subspaces H_χ of the elliptic cyclic singularities possess a lattice property analogous to that of root systems of Weyl groups. Namely, the image of an element a of such a basis under the unitary reflection in element b is $a + \ell(\chi)b$, where $\ell(\chi)$ is a linear function of the character with *integer* coefficients. The coefficients do not depend on χ: if elements $a, b \in H_\chi$ and $a', b' \in H_{\chi'}$ are appropriate lifts of the same vanishing (semi-)cycles from (V, V_0), then the reflection in b' sends a' to $a' + \ell(\chi')b'$.

The same happens for the skew-Hermitian versions of the groups under consideration.

To demonstrate the lattice property we list in the table below irreducible two-dimensional subgroups generated by various pairs $e_1, e_2 \in H_\chi$ of elements of the standard distinguished bases of elliptic cyclic singularities in both even and odd cases, along with the corresponding intersection matrices.

The $A_2^{(m)}$ entry of the table shows that the monodromy $A_k^{(m)}$ is just the Burau representation of the braid group on $k+1$ threads at the root of unity (or at its negative in the odd case).

Stable equivalence of the singularities, which adds two new variables, takes the intersection matrix to its negative and does not change the Picard-Lefschetz operators. This way the table provides the information for $n \equiv 2, 3 \bmod 4$ as well.

5.6 Fusion of diagrams

A versal cyclic deformation of a cyclic singularity is a subdeformation of a versal deformation of the corresponding absolute singularity (for example,

graph	intersection matrix and the monodromy operators h_1, h_2 for $n \equiv 0 \bmod 4$	intersection matrix and the monodromy operators h_1, h_2 for $n \equiv 1 \bmod 4$
$A_2^{(m)}$	$\begin{pmatrix} m & -\frac{m}{1-\chi} \\ -\frac{m}{1-\bar{\chi}} & m \end{pmatrix}$ $\begin{pmatrix} \chi & -\chi \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & \chi \end{pmatrix}$	$\begin{pmatrix} m\frac{1+\chi}{1-\chi} & -\frac{m}{1-\chi} \\ \frac{m}{1-\bar{\chi}} & m\frac{1+\chi}{1-\chi} \end{pmatrix}$ $\begin{pmatrix} -\chi & \chi \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & -\chi \end{pmatrix}$
$B_2^{(m)}$	$\begin{pmatrix} m & -m \\ -m & 2m \end{pmatrix}$ $\begin{pmatrix} \chi & 1-\chi \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} m\frac{1+\chi}{1-\chi} & -m \\ m & 0 \end{pmatrix}$ $\begin{pmatrix} -\chi & \chi-1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
A_2 long cycles	$\begin{pmatrix} 2m & -m \\ -m & 2m \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$B_2^{(3,3)}$	$\begin{pmatrix} 3 & -\frac{6}{1-\chi} \\ -\frac{6}{1-\bar{\chi}} & 6 \end{pmatrix}$ $\begin{pmatrix} \chi & -2\chi \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & \chi \end{pmatrix}$	$\begin{pmatrix} 3\frac{1+\chi}{1-\chi} & -\frac{6}{1-\chi} \\ \frac{6}{1-\bar{\chi}} & 6\frac{1+\chi}{1-\chi} \end{pmatrix}$ $\begin{pmatrix} -\chi & 2\chi \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & -\chi \end{pmatrix}$

$B_2^{(m)}$ in A_{km-1}). The monodromy groups are similarly embedded one into the other. For $\mathbf{Z}_2$-symmetric singularities this has an interpretation as the folding of the canonical Dynkin diagrams (Figure 2). This operation generalises to the higher symmetry as follows.

At an elementary degeneration $B_1^{(m)}$ of a $\mathbf{Z}_m$-symmetric function one short χ-cycle vanishes. As an absolute singularity this is an A_{m-1} critical point. Therefore the action of the Picard-Lefschetz operator $h = \oplus_{\chi^m=1} h_\chi$ on the homology of the symmetric Milnor fibre is that of the classical monodromy operator of A_{m-1} (which is the appropriate product of the $m-1$ Picard-Lefschetz operators of a non-symmetric morsification of the function). The short χ-cycle is an eigenvalue χ eigenvector of the classical monodromy in the linear space spanned by the $m-1$ non-symmetric vanishing cycles.

Similarly, the Picard-Lefschetz operator $h = \oplus_{\chi^m=1} h_\chi$ corresponding to m long χ-cycles vanishing on the same critical level of a $\mathbf{Z}_m$-symmetric perturbation of a cyclic singularity is the product of the m commuting non-symmetric Picard-Lefschetz operators. The linear subspace in the homology spanned by the m long χ-cycles coincides with that spanned by the corresponding m non-symmetric vanishing cycles.

This shows that one can produce diagrams of the unitary groups (Figure 3) from appropriate diagrams of the ambient Weyl groups (last column of the table of Theorem 5.2) by fusing A_{m-1} subdiagrams to "short" vertices and gluing m-tuples of disconnected vertices to "long" ones. Additionally, there must be a check if the required relations between the generators (or the Hermitian products of the obtained roots) are satisfied, that is, if the fusing-folding operation works properly on the edges.

The results of the suggested algorithm are shown in Figure 8. We assume the dimension of a Milnor fibre to be divisible by 4. The diagrams of the absolute singularities are rather non-canonical Dynkin diagrams of the Weyl groups. As usual, all their vertices have square 2 and a solid edge means that the scalar product is -1. A dashed edge denotes scalar product 1.

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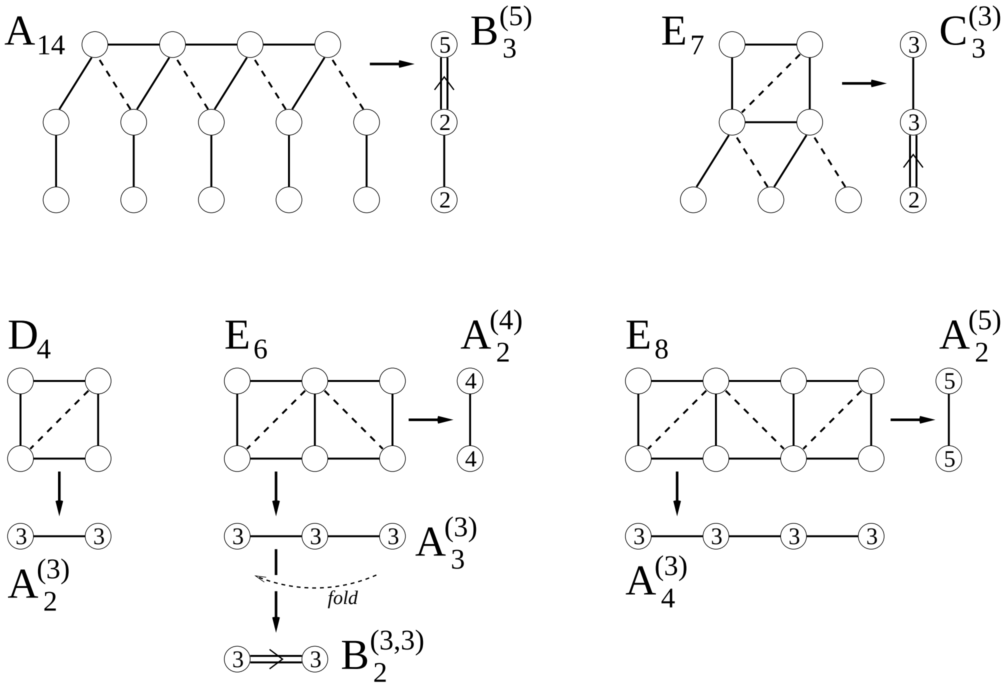

Figure 8: *Fusing Dynkin diagrams of Weyl groups to diagrams of unitary reflection groups.*

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