On stability of projections of Lagrangian varieties

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Abstract

We show that Lagrangian and Legendre varieties associated with matrix singularities and singularities of composite functions are stable in a sense which is a natural modification of Givental's notion of stability of Lagrangian projections.

The study of singular Lagrangian and Legendre varieties was initiated about twenty five years ago by Arnold when he was investigating singularities in the variational problem of obstacle bypassing [1]. The first examples of such varieties, open swallowtails, were related to the discriminants of the non-crystallographic Coxeter groups [4, 8]. Incorporating these examples into a general context, Givental [4] introduced the notion of stability of Lagrangian and Legendre varieties with respect to perturbations of symplectic structure and Lagrangian or, respectively, contact structure and Legendre projection only, keeping the diffeomorphic type of the variety fixed.

Later, in [7], it was shown that this stability notion has an explicit geometrical meaning in terms of generating families, versal deformations of function singularities and inducing mappings.

The interest in theory of singular Lagrangian and Legendre varieties has been growing recently due to its possible applications to Frobenius structures, D-modules and other areas.

In this paper we extend the results of [7] to a natural modification of Givental's stability notion and show that the stability condition holds for a wide class of Lagrangian and Legendre varieties associated with matrix singularities (see [2, 3, 6]) and singularities of composite functions [5].

1 0-stability

In this section we recall some standard notions and introduce their modifications we shall use later.

1.1 The Lagrangian setup

A singular *Lagrangian (sub)variety* L of a symplectic space M^{2n} is an n-dimensional analytic subset of M which is Lagrangian in the ordinary sense at all its regular points. A *Lagrangian projection* π is a projection $\pi : M \to B^n$ defining a fibre bundle whose fibres are Lagrangian.

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Fibres of any Lagrangian fibration posses a well-defined affine structure. Indeed, local coordinates on the base rise to regular functions on the total space, which are pairwise in involution. Hence their Hamiltonian vector fields do not vanish, commute and are tangent to the fibres.

The restriction $\pi|_L$ of the Lagrangian projection π to a Lagrangian subvariety $L \subset M$ is called a *Lagrangian mapping*.

Two Lagrangian mappings, of Lagrangian subvarieties L' and L'', are called *equivalent* if there exists a symplectomorphism of the ambient symplectic spaces sending L' to L'' and fibres of one Legendrian projection to fibres of the other. In particular, L' and L'' are symplectomorphic.

The germ of a Lagrangian map $\pi|_L$ of a variety L at its point m is called *stable* if the germ of any Lagrangian map $\widetilde{\pi}|_L$ close to $\pi|_L$ at any point $\widetilde{m}$ close to m is equivalent to the germ of $\pi|_L$ at a point near m. Notice that only the fibration π is allowed to vary in this context while the subvariety L is fixed.

According to Givental [4], the stability introduced is essentially equivalent to the following versality of the map-germ $\pi|_L$.

Let $\mathcal{O}_L$ be the algebra of regular functions on L and $\mathbf{m}_{B,m}$ the maximal ideal in the algebra $\mathcal{O}_{B,m}$ of function-germs on the base B at the point $\pi(m)$. We define the local algebra of the germ of $\pi|_L$ at m as

$$Q_m = \mathcal{O}_L / \Big((\pi|_L)^* (\mathbf{m}_{B,m}) \Big) \mathcal{O}_L \, .$$

The algebra Q_m is the algebra of restrictions of functions on L to the intersection of L with the fiber $F_{\pi(m)} = \pi^{-1}(\pi(m))$.

Denote by $\mathcal{A}_m$ the subspace of affine (with respect to the corresponding affine structure) functions on the fibre $F_{\pi(m)}$ and by $r : \mathcal{A}_m \to Q_m$ the restriction homomorphism sending a function on the fibre to its restriction to $L \cap F_{\pi(m)}$.

The germ of $\pi|_L$ at $m \in L$ is called *versal* if r is surjective, and *miniversal* if r is an isomorphism.

Let p, q be local Darboux coordinates on M about m: $p(m) = p_0$, $q(m) = 0$ and $\pi(p,q) = q$. The Weierstrass preparation theorem implies that the versality of $\pi|_L$ at m is equivalent to the existence of a representation of any analytic function-germ φ on M at $m = (p_0, 0)$ in the form

$$\varphi(p,q) = \psi(p,q) + \sum_{j=1}^{n} a_j(q) p_j + a_0(q) \, , \tag{1}$$

where the a_j, $\;j \geq 0$ are analytic function-germs on the base B, and the function-germ ψ vanishes on L.

Remark. The decomposition means that any function-germ on M at m is a sum of a function vanishing on L and a function affine on the individual fibres. Therefore, any Hamiltonian vector field near m is a sum of a Hamiltonian vector field tangent to L and a Hamiltonian vector field preserving the fibration π. Hence the homotopy method implies that any symplectomorphism-germ of M at m close to the identity is a composition of a symplectomorhism preserving L and a symplectomorphism preserving the standard projection π. Since any perturbation of the germ of π in the class of Lagrangian projections is a composition of π with an appropriate symplectomorphism, the versality implies stability. See [4] for more details.

We now turn to a restricted version of the above setup. Namely, we take $M = T^*B$ to be the cotangent bundle of a manifold B^n, and distinguish the zero section T_0 of T^*B. Let $Sym_0(M)$ be the subgroup of symplectomorphisms of M preserving T_0.

Two Lagrangian mappings of Lagrangian subvarieties of a cotangent bundle are called *0-equivalent* if they are equivalent via a symplectomorphism from $Sym_0(M)$.

Replacement of the equivalence by the 0-equivalence in the stability definition yields a definition of the 0-stability of Lagrangian map-germs.

The zero section T_0 determines a linear structure on fibres of a cotangent bundle. Replacing the space $\mathcal{A}_m$ of affine functions on F_m by its well-defined subspace $\mathcal{A}_m^0$ of linear functions, we obtain the definition of the *0-versality* which is equivalent to the existence of the representation of any function-germ φ on M at m such that $\varphi|_{T_0} = 0$ in the form

$$\varphi(p, q) = \psi(p, q) + \sum_{j=1}^{n} a_j(q) p_j \,, \tag{2}$$

where the germs a_j and ψ are similar to those in (1) and we assume that the Darboux coordinates p vanish on the zero section T_0.

Like before, the 0-versality implies the 0-stability.

The *multiplicity* μ of a 0-miniversal germ of a Lagrangian map, that is the rank of its local algebra as a linear space, is n. It is at most n if the germ is 0-versal.

For the benefit of the exposition, we continue now with the complex case only. Everything below transfers absolutely straightforwardly to the real situation.

Lemma 1. *The projection $\pi : T^*\mathbf{C}^n \to \mathbf{C}^n$, $(p, q) \mapsto q$, of a Lagrangian germ L at the origin is* 0*-stable if and only if the germs of the products $p_i p_j$, $i, j = 1, \ldots, n$, have decompositions*

$$p_i p_j = \psi_{ij}(p, q) + \sum_{k=1}^{n} c_{ij}^k(q) p_k \tag{3}$$

in which the function-germs ψ_{ij} and c_{ij}^k are holomorphic, and the ψ_{ij} vanish on L.

Proof. The "only if" part is obvious. To prove the "if" part, we notice that the ideal I generated by all the quadratic polynomials $P_{ij}(p) = p_i p_j - \sum c_{ij}^k(0) p_k$, $i, j = 1, \ldots, n$, in the space of all holomorphic function-germs on the fibre F_0 is of finite codimension. Modulo I, any function-germ on F_0 is an affine function in p. After the projection to the local algebra Q_0, that is after a further reduction modulo the functions vanishing on L (more precisely on $L \cap F_0$), such a function is still affine in p. Hence, the 0-versality condition holds. □

For the stability (rather than 0-stability) version of the lemma see [4].

The *suspension* of a Lagrangian fibration $\pi : M \to B$ is its direct product

$$\widehat{\pi} = (\pi, \pi_0) : \widehat{M} = M \times T^*\mathbf{C} \to B \times \mathbf{C}$$

with the canonical projection $\pi_0 : T^*\mathbf{C} \to \mathbf{C}$.

A *suspension* of a Lagrangian variety $L \subset M^{2n}$ is an $(n+1)$-dimensional Lagrangian variety $\widehat{L} \subset M \times T^*\mathbf{C}$ which is the product of L with the line $\ell = \{p_{n+1} = const \neq 0\}$ in $T^*\mathbf{C}$ endowed with the standard Darboux coordinates p_{n+1}, q_{n+1}.

The propositions below follow immediately from the definitions.

Proposition 2. *A map-germ $\pi|_L$ at $m \in M$ is (mini)versal if and only if its suspension $\widehat{\pi}|_{\widehat{L}}$ is 0-(mini)versal at a point of the line $m \times \ell$ (hence at all the points of this line) in $\widehat{M}$.*

Example. A germ of the standard projection π of a Lagrangian submanifold $L \subset T^*\mathbf{C}^n = \{p, q\}$ determined by a generating family $f = f(x, q)$ with parameters $q \in \mathbf{C}^n$ and variables $x \in \mathbf{C}^k$,

$$L = \{(p, q) | \exists x : \partial f/\partial x = 0, p = \partial f/\partial q\} ,$$

is stable if and only if the family-germ $f(\cdot, \cdot)$ is an $\mathcal{R}^+$-versal deformation of the function-germ $f(\cdot, 0)$. The projection is 0-stable if $f(\cdot, \cdot)$ is an $\mathcal{R}$-versal deformation of $f(\cdot, 0)$.

Proposition 3. *Consider a germ of a Lagrangian subvariety L in $\widehat{M} = T^*\mathbf{C}^n \times T^*\mathbf{C}$ at a point not in the zero section. Assume $\widehat{\pi}|_L$ is 0-versal and L belongs to a regular hypersurface in $\widehat{M}$ transversal to the $\partial_{p_{n+1}}$-direction. Then $\widehat{\pi}|_L$ is 0-equivalent to a suspension of a versal map-germ $\pi|_{L'}$ of a Lagrangian subvariety $L' \subset M = T^*\mathbf{C}^n$.*

1.2 The Legendre case

A singular n-dimensional subvariety of a contact space is called *Legendre* if at all its regular points it is Legendre in the ordinary sense. For standard (and equivalent) local models of contact $(2n + 1)$-spaces we use the projectivised cotangent bundle $PT^*\mathbf{C}^{n+1}$ and the space $J^1(\mathbf{C}^n, \mathbf{C}) = \{p, q, z\}$ of one-jets of functions on $\mathbf{C}^n$ endowed with the contact form $\alpha = dz - pdq$. The definitions of Legendre mappings, stability and others are analogous to the Lagrangian case (see also [4]).

Symplectisation and contactisation functors relate Lagrangian and Legendre germs as follows.

A. The projection $\rho : (p, q, z) \mapsto (p, q)$ maps a Legendre variety $\Lambda \subset J^1(\mathbf{C}^n, \mathbf{C})$ to the Lagrangian variety $\rho(\Lambda) \subset T^*\mathbf{C}^n$.

B. Local Lagrangian fibration and its zero section determine uniquely the Liouville primitive form $\alpha = pdq$ of the symplectic form $\omega = d\alpha$. Given a Lagrangian germ $L \subset T^*\mathbf{C}^n$ at a point m, denote by $L_{0,m}$ the subset of points $s \in L$ such that the integral of α along some path γ on L joining m and s vanishes.

For simplicity we assume the values of the integral do not depend on the local path γ, that is the cohomology class of α vanishes on L (see [4] for examples of the opposite).

If $L_{0,m}$ does not meet the zero section of $T^*\mathbf{C}^n$, then its projectivisation is a Legendre (or isotropic) variety in $PT^*\mathbf{C}^n$. Its projection $W_0(L, m) = \pi(L_{0,m}) \subset \mathbf{C}^n$ is called the *0-wave front* of L.

C. For a Lagrangian germ $L \subset T^*\mathbf{C}^n$ at a point m, the set $\Lambda_{L,m} \subset J^1(\mathbf{C}^n, \mathbf{C})$ of points (p, q, z) such that $s = (p, q) \in L$ and the integral of α along a path in L joining m and s equals z is a Legendre variety in $J^1(\mathbf{C}^n, \mathbf{C})$.

A germ of a symplectomorphism $\theta \in Sym_0(T^*\mathbf{C}^n)$ preserving π preserves α. Hence if $\theta(L') = L''$ then $\theta(L'_{0,m}) = L''_{0,\theta(m)}$. In Darboux coordinates, θ has the form:

$$\theta : (p, q) \mapsto (P, \check{\theta}(q)) ,$$

where $\check{\theta}$ is the underlying diffeomorphism of the base and $P = (\check{\theta}^{-1})^* p$ is the value at p of the linear operator on the fibres dual to the inverse of the derivative of θ. In particular, $\check{\theta}(W_0(L', m)) = W_0(L'', \theta(m))$.

The proof of the following statement is straightforward.

Proposition 4. *Consider a Legendre germ $\Lambda \subset J^1(\mathbf{C}^n, \mathbf{C})$. Assume the variety $\widehat{\rho(\Lambda)}$ does not meet the zero section and that its standard Lagrangian projection is* 0*-stable. Then the projection of* Λ *to* $J^0(\mathbf{C}^n, \mathbf{C})$ *is Legendre stable.*

Conversely, if the projection of Λ *to* $J^0(\mathbf{C}^n, \mathbf{C})$ *is Legendre stable and* Λ *is quasihomogeneous with positive weights then* $\widehat{\rho(\Lambda)}$ *is* 0*-stable.*

2 Stability of induced mappings

2.1 The critical-value theorem

The images under a Lagrangian mapping $\pi|_L$ of singular points of the Lagrangian variety L along with the images of critical points of $\pi|_L$ on the regular part of L form the *caustic* Σ_L of the Lagrangian mapping.

The caustic of a Lagrangian germ L at a point m of finite multiplicity μ is a proper analytic subset of the base B of codimension at least 1.

For $q \notin \Sigma_L$ close to the distinguished point $\pi(m)$, the inverse image $\pi^{-1}(q) \bigcap L$ consists of μ distinct points m_i close to m. We can assume that locally π is the standard fibration $T^*B \to B$. This allows us to introduce the *Maxwell set* $M_L \subset B$ as the closure of the set of the points $q \notin \Sigma_L$ for which the μ values of z on $\Lambda_{L,m} \bigcap (\rho \circ \pi)^{-1}(q)$ are not all distinct. If μ is finite, the Maxwell set is a germ of a proper analytic subset of the base. The union of the caustic and Maxwell set is called the *bifurcation diagram* $\mathrm{Bif}(\pi, L)$ of the Lagrangian projection.

Consider the Lagrangian projection $\pi : T^*\mathbf{C}^n \to \mathbf{C}^n$ of a Lagrangian variety-germ L. Let $g : \mathbf{C}^k \to \mathbf{C}^n$ be a germ of a smooth mapping. If the choice of the base points of the germs is consistent, we define the *induced Lagrangian mapping* $g^*(\pi|_L)$ as the projection of $g^*(L) \subset T^*\mathbf{C}^k$ to $\mathbf{C}^k$.

Theorem 5. *Assume the germs* $\pi|_L$ *at* $m, m \notin T_0$ *and* $g^*(\pi|_L)$ *are 0-miniversal and* 0*-stable respectively. Then the critical value set* Ξ_g *of the mapping* g *belongs to the union* $W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$.

Here we consider a source point of a mapping as critical if the derivative at the point is not surjective. In particular, all the source is critical if its dimension is less than that of the target, in which case the theorem implies that g maps $\mathbf{C}^k$ into $W_0(L, m) \bigcup \mathrm{Bif}(\pi, L)$.

The stability analog of the theorem was proved in [7].

Proof. Take a point $q_0 \in \mathbf{C}^n \setminus \Sigma_L$ close to the base point. Its $\pi|_L$-inverse image consists of n distinct points $m_1, \ldots, m_n \in F_{q_0}$, all different from the origin. The multi-germ of $\pi|_L$ at the finite set $\{m_1, \ldots, m_n\}$ is 0-versal (the decomposition (2) holds for multi-germs). This is equivalent to the m_i being linearly independent in the fibre F_{q_0}: the restriction of any function from the fibre to this set coincides with the restriction of a linear function.

Consider now $\lambda_0 \in g^{-1}(q_0)$. Let $I \subset T_{q_0}\mathbf{C}^n$ be the image of the derivative $g_* : T_{\lambda_0}\mathbf{C}^k \to T_{q_0}\mathbf{C}^n$. The pullback mapping $g^* : F^n_{q_0} \to F^k_{\lambda_0}$ between the fibres of the cotangent bundles is a composition of the factorisation *pr* of F_{q_0} by the subspace $I^\vee$ of covectors annihilating I and an embedding. Assume the dimension r of $I^\vee$ is positive, that is λ_0 is a critical point of g. The 0-stability of $g^*(\pi|_L)$ implies that the *pr*-images of the linearly independent points

$m_1, \dots, m_n \in F_{q_0}$ form a linearly independent set in the $(n-r)$-dimensional space $F_{q_0}/I^\vee$ (the image points counted without the multiplicities). As a result, the vertex set $\{m_0 = 0, m_1, \dots, m_n\}$ of the n-simplex $\sigma \subset F_{q_0}$ is mapped to the vertex set $\{m'_0 = 0, m'_1, \dots, m'_{n-r}\}$ of an $(n-r)$-simplex in $F_{q_0}/I^\vee$. In particular, the rank r subspace $I^\vee$ is spanned by all the differences $m_i - m_j$ such that $pr(m_i) = pr(m_j)$, that is by the vectors in all the faces of σ contracted by pr to points (the sum of the dimensions of such faces is r).

Near each of the m_i, $i = 1, \dots, n$, the Lagrangian variety L is locally the graph of the differential of a function $z = \psi_i(q)$, $\psi(q_0) = 0$. The linearly independent points $m_i \in F_{q_0}$ are the differentials of the ψ_i at q_0.

For any pair $i \neq j$, denote by $\Delta_{ij} \subset T_{q_0}\mathbf{C}^n$ the hyperplane tangent to the hypersurface $\psi_i(q) - \psi_j(q) = 0$.

For any ℓ, let $\Delta_\ell \subset T_{q_0}\mathbf{C}^n$ be the hyperplane tangent to the hypersurface $\psi_\ell(q) = 0$. The hyperplanes Δ_ℓ and Δ_{ij} are dual to the directional lines of the 1-dimensional faces of the simplex $\sigma \subset F_{q_0}$.

The condition for the multi-germ $g^*(\pi|_L)$ to be 0-stable at the points of F_{λ_0} is equivalent to the subspace I being the intersection of all the Δ_ℓ such that $pr(m_\ell) = 0$ and all the Δ_{ij} such that $pr(m_i) = pr(m_j) \neq 0$. Hence I is the intersection of the subspaces in $T_{q_0}\mathbf{C}^n$ dual to certain faces of the simplex σ. Since I belongs to the tangent cone at q_0 of the critical value set Ξ_g, the regular strata of Ξ_g near q_0 coincide with the integrable manifolds of the distributions defined similarly to I in the spaces $T_q\mathbf{C}^n$ by subsets of the faces of the relevant n-simplices in the fibres F_q.

According to [7] (items 7.1. and 7.2), among such integrable manifolds, those having the highest dimension and containing $\pi(m)$ in their closures are the regular strata of the caustic, Maxwell set, and , as it is easy to see, wavefront $W_0(L,m)$. Hence $\Xi_g \subset W_0(L,m) \bigcup \mathrm{Bif}(\pi, L)$. $\square$

Theorem 6. *If g is the germ of a proper mapping between spaces of the same dimension, then the 0-stability of $g^*(\pi|_L)$ is equivalent to g being a ramified covering with the ramification locus contained in $W_0(L,m) \bigcup \mathrm{Bif}(\pi, L)$.*

Proof. In this case the regular strata of Ξ_g are $(n-1)$-dimensional. By Theorem 5, the 0-stability implies the ramification property. To prove the converse it is sufficient to notice that outside the ramification locus the induced map $g^*(\pi|_L)$ is 0-miniversal. Also it is versal at points of the regular strata of the ramification set, as it can be seen from the pullback mapping g^* action on the corresponding symplex in the fiber. Hence any holomorphic function-germ $\varphi(p,q)$ possesses a decomposition (2) with the coefficients $a_j(q)$ uniquely determined on the complement of the analytic subset of codimension at least 2. Now Hartogs' theorem extends the decomposition to an entire neighbourhood of the base point. $\square$

Remark. Assume the Lagrangian variety-germ L at $m \in T^*\mathbf{C}^n$ is a suspension of a Lagrangian germ L' at a point not contained in the zero section of $T^*\mathbf{C}^{n-1}$. The base $\mathbf{C}^n$ of the suspended Lagrangian fibration contains a distinguished coordinate function, let it be q_n, corresponding to the second factor of the decomposition $L \simeq L' \times \mathbf{C}$. The caustic and Maxwell set for L are also isomorphic to the products of the caustic and Maxwell set for L' with a line, the q_n-axis. On the contrary, the hyperplane tangent to the wavefront $W_0(L,m)$ at m is $dq_n = 0$.

If, under the conditions of Theorem 6, the ramification locus Ξ_g contains an $(n-1)$-dimensional component of the caustic or of the Maxwell stratum then the direction ∂_{q_n} belongs to the image I of the differential of g at points arbitrary close to m. Hence the composition $q_n \circ g$ is not singular at the base point. On the other hand, if the ramification locus contains an $(n-1)$-dimensional component of the wavefront $W_0(L,m)$, then the composition $q_n \circ g$ must be singular at the base point. Otherwise, the hyperplanes tangent to Ξ_g near the base point are not close to the hyperplane $dq_n = 0$.

2.2 Composite functions

An interesting class of 0-stable Lagrangian projections is provided by versal deformations of composite mappings [6].

Given a function-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ consider the group $\mathcal{K}_f$ (see [6]) which consists of diffeomorphism-germs Θ of the product space $(\mathbf{C}^m \times \mathbf{C}^n, (0,0))$ fibred over the projection to the first factor $\Theta : (x,y) \mapsto (X(x), Y(x,y))$, $x \in \mathbf{C}^m$, $y \in \mathbf{C}^n$, and such that $f(Y(x,y)) = f(y)$ for any (x,y).

The group $\mathcal{K}_f$ acts naturally on the space of map-germs $\varphi : (\mathbf{C}^m, 0) \to (\mathbf{C}^n, 0)$ sending the graph of one map to the graph of the other.

Assume a map-germ φ at the origin has a finite Tjurina number τ with respect to the group $\mathcal{K}_f$. Let $\Phi(x,\lambda) = \varphi(x) + \sum \lambda_s \varphi_s(x)$, $\lambda \in \mathbf{C}^\tau$, be a $\mathcal{K}_f$-miniversal deformation of φ. Introduce the composition $F = f \circ \Phi$.

Theorem 7. *The Lagrangian projection defined by the generating family-germ $F(x,\lambda)$ is* 0*-stable.*

Proof. Let $t \in (\mathbf{C}, 0)$ be an additional parameter. Consider the deformation

$$F_{ij} = f \circ (\Phi + t \frac{\partial F}{\partial \lambda_j} \varphi_i)$$

of the composite function $f \circ \varphi$. Since $F_{ij}|_{t=0} = F$ and Φ is K_f-versal, there exists a family of K_f-equivalences depending on t and inducing F_{ij} from F:

$$F_{ij}(x,\lambda,t) = f \circ \left(\varphi(X(x,\lambda,t)) + \sum_{s=1}^{\tau} \Lambda_s(\lambda,t) \varphi_s(X(x,\lambda,t)) \right).$$

Moreover, we chose the family so that for $t = 0$ the mapping $(x,\lambda) \mapsto (X,\Lambda)$ is the identity mapping.

Differentiating this equality with respect to t at $t = 0$ we obtain

$$\frac{\partial F}{\partial \lambda_i} \frac{\partial F}{\partial \lambda_j} = \sum \frac{\partial F}{\partial x_r} \frac{\partial X_r}{\partial t} + \sum \frac{\partial F}{\partial \lambda_k} \frac{\partial \Lambda_k}{\partial t}.$$

Since $\partial F / \partial \lambda_i = p_i$ and $\partial F / \partial x_r = 0$ on the Lagrangian variety defined by the generating family F, this means that the 0-stability criterium of Lemma 1 holds for it. □

Assume the germ at the origin of the composed function $h = f \circ \varphi$ has a finite multiplicity μ. The deformation $F = f \circ \Phi$ of h is induced from an $\mathcal{R}$-miniversal deformation H of h by a map-germ $g : (\mathbf{C}^\tau, 0) \to (\mathbf{C}^\mu, 0)$ between the deformation bases.

Corollary 8. *If $\tau = \mu$ and the inducing mapping g is proper, then g is a covering ramified over the* 0*-wavefront of the* 0*-stable Lagrangian manifold defined by the generating family H.*

Proof. The claim is trivial when the function-germ f is regular (if so, the mapping g is a diffeomorphism). So we may assume that f has a critical point at the origin. In this case the composition of g with the projection $\mathbf{C}^\mu \to \mathbf{C}$ along the hyperplane tangent to the discriminant of h at the origin is singular at $0 \in \mathbf{C}^\tau$. Now Theorems 7, 6 and the Remark after Theorem 6 imply the result.

Remarks. 1. The covering mapping inducing the determinantal function of a versal matrix deformation of a simple matrix singularity from a versal deformation of the determinantal function of the unperturbed matrix (see [6]) is a particular case of Corollary 8.

2. The space of linear functions on a fibre F_q of the cotangent bundle $T^*B \to B$ is the tangent space T_qB. So the functions c_{ij}^k defined in (3) for a 0-versal Lagrangian map-germ determine a point-wise associative multiplication on the germs of vector fields on the base.

3. Under the conditions of Corollary 8, the $\mathcal{K}_f$-discriminant of φ is a free divisor. We postpone a proof of this to a forthcoming paper.

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