Vassiliev invariants of knots in R^3 and in a solid torus

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Abstract

We give a chord-diagram description of finite type invariants of framed and unframed knots in a solid torus. The relation is established via appropriate versions of the universal Vassiliev-Kontsevich invariant. The framed case is treated from the singularity theory point of view that involves knots with degenerate framings.

1 Introduction

The major aim of this paper is to introduce a necessary basis for a part of the theory of Vassiliev type invariants of regular plane curves.

Consideration of such invariants was started recently by Arnold. In [1, 2] he defined three order 1 invariants which are dual to the three generic bifurcations in families of regular plane curves. While singularities of a generic curve are only transverse double points, in 1-parameter families there appear triple points and two types of self-tangencies: direct (when the two velocity vectors at the self-tangency point have the same direction) and inverse (when the directions are opposite).

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Considering invariants that do not change during triple-point and inverse self-tangency transformations one immediately arrives to invariants of framed knots in a solid torus. Indeed, a regular plane curve with no direct self-tangencies lifts to a Legendrian knot in the solid torus $ST^*\mathbf{R}^2$. This knot has a natural framing. Appearance of direct self-tangencies corresponds to singular knots, with double points. So the theory of Vassiliev type invariants of regular plane curves without direct self-tangencies looks parallel to that of framed knots in a solid torus. Thus a chord-diagram interpretation (similar to [7, 11]) of the latter is very desirable.

Obviously, such interpretation should be constructed from two pieces: that for framed knots in $\mathbf{R}^3$ and that for unframed knots in a solid torus.

i) The chord-diagram description of finite type invariants of framed knots in 3-space has already been given. In [11] Kontsevich mentioned that the space of such real-valued invariants is dual to the real linear space spanned by circular chord diagrams modulo the 4-term relation. Later this was proved by Lê and Murakami [12, 13] who adjusted the method used by Kontsevich in the unframed case.

The approach by Lê and Murakami (their regularisation of the Kontsevich's integral) works only for the blackboard framing. Unfortunately, this is not sufficient for the study of plane curves. Indeed, the canonical framing of the Legendrian lift of a regular plane curve is blackboard only with respect to the projection which is not very convenient to consider if one wants to construct Vassiliev type theory for plane curves.

In the present paper we fill this gap modifying the definition of the universal Vassiliev-Kontsevich invariant so that it serves knots with any framings.

ii) The first attempt to construct a chord-diagram interpretation for Vassiliev type theory for unframed knots in a 3-manifold was done by Kalfagiani [10]. But, since the case considered by her was rather general, there was no obvious way to complete the theory by, say, a definition of a corresponding Kontsevich's integral. In our special case of a solid torus the integral is defined straightforwardly: almost the only difference with the original Kontsevich's idea [11] is that now we use the decomposition $\mathbf{C} \times S^1$ of the solid torus instead of the decomposition $\mathbf{C} \times \mathbf{R}^1$ of $\mathbf{R}^3$.

Those are two basic constructions of the paper. The third one is the singularity theory approach to invariants of framed knots. This is very close to the original Vassiliev's idea to consider singular knots instead of non-singular ones passing from embeddings of a circle into 3-space to arbitrary smooth mappings. Any of numerous equivalent definitions of a framing leaves obvious room to make a framing singular. We introduce one more definition of our own and trace the framing degenerations. The considered bifurcations show, for example, what happens with the 1-term framing-independence relation of invariants of unframed knots: it does not disappear but gets new terms which reflect the framing degenerations.

The contents of the paper is briefly as follows.

In Section 2, we introduce knots with degenerate framings and the extension of invariants of non-singular framed knots to those with elementary singularities. We construct the diagram theory for framed knots in 3-space which basicly coincides with that of Kontsevich [11, 12, 13] (some difference appears only for $\mathbf{Z}_2$-valued invariants).

In Section 3, we define the universal Vassiliev-Kontsevich invariant for knots in $\mathbf{R}^3$ that have arbitrary framings and reprove the result of Le-Murakami in this general setting.

In Section 4, we consider Vassiliev-Kontsevich type theory for unframed knots in a solid torus. We show that the graded space of complex-valued finite order invariants in this case is dual to the graded linear space generated by marked chord diagrams modulo marked 1- and 4-term relations. The marking is defined by the fundamental group of the solid torus.

In Section 5, we obtain the similar result for framed knots in a solid torus. We also show that all the coefficients of the version of the HOMFLY polynomial for framed knots in a solid torus are in fact Vassiliev invariants of finite order (cf. [7]).

Remark 1.1 Paper [9] establishes the isomorphism between the theory of Vassiliev invariants for framed knots in a solid torus and that for regular plane curves with no direct self-tangencies.

Remark 1.2 The constructions of the present paper are very convenient to built up spectral sequences (similar to the Vassiliev's one [15]) to calculate cohomology of spaces of framed knots in $\mathbf{R}^3$ and unframed and framed knots in a solid torus. This will be the topic of some other paper.

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2 Framed knots from the singularity point of view

2.1 Framed knots as mappings

A smooth unframed knot in 3-space is the image of a smooth embedding of a circle into $\mathbf{R}^3$. So, in singularity theory, an oriented knot is treated as an element of the set Ω of all C^∞-mappings of an oriented circle into $\mathbf{R}^3$. Formally, *a non-singular unframed oriented knot* in 3-space is a connected component of the subset of Ω that consists of all the embeddings.

In the theory of Vassiliev invariants of knots in $\mathbf{R}^3$ the key role is played by so-called singular knots. Namely, consider the subset of Ω of all the immersions whose images have only n double points with non-tangent branches and no other singularities. *An unframed oriented knot with n singular points* is a connected component of this subset. *A singular unframed knot* is a knot with a finite number of singular points.

We introduce now similar notions for framed knots.

Let $S^1 \subset \mathbf{R}^2$ be an oriented C^∞-embedded circle and $U \subset \mathbf{R}^2$ an open (annular) neighbourhood of S^1. Consider a C^∞-mapping $g : U \to \mathbf{R}^3$. It defines the mapping Tg from the restriction $T_{S^1}\mathbf{R}^2$ of the tangent bundle $T\mathbf{R}^2$ to the tangent bundle $T\mathbf{R}^3$.

Definition 2.1 Two mappings $g_i : U_i \to \mathbf{R}^3$, $i = 1, 2$, are *equivalent* if the mappings $Tg_i : T_{S^1}\mathbf{R}^2 \to T\mathbf{R}^3$, $i = 1, 2$, coincide.

We denote by Ω_f the set of all C^∞-mappings $g : U \to \mathbf{R}^3$ modulo this equivalence. The class of a mapping g in Ω_f will be denoted by g as well.

Definition 2.2 Consider the set of all the equivalence classes of mappings $g \in \Omega_f$ such that the restriction $Tg : T_{S^1}\mathbf{R}^2 \to T\mathbf{R}^3$ is an embedding. *A non-singular oriented framed knot* in $\mathbf{R}^3$ is a connected component of this set.

The $Tg : T_{S^1}\mathbf{R}^2 \to T\mathbf{R}^3$ being an embedding guarantees the mapping $g : S^1 \to \mathbf{R}^3$ being an embedding too.

The image $g(S^1)$ will be called the *core* of the mapping.

Consider a subset of Ω_f which consists of all the equivalence classes of mappings g such that:

i) the mapping $g : S^1 \to \mathbf{R}^3$ represents an unframed knot with $n \geq 0$ singular points;

ii) for all $s \in S^1$, except $k \geq 0$ points none of which is mapped to a double point of the core $g(S^1)$, the mapping Tg has rank 2 on $T_s\mathbf{R}^2$;

iii) for the remaining k points $s \in S^1$, the mapping Tg has rank 1 on $T_s\mathbf{R}^2$.

Definition 2.3 *An oriented framed knot* in $\mathbf{R}^3$ *with* $n + k$ *singularities* is a connected component of the above subset. *A singular oriented framed knot* is an oriented framed knot with a finite number of singularities.

Condition i) implies that the kernel of Tg, that appears in iii), is not tangent to S^1.

2.2 The framed equivalence

The set Ω_f of framed curves in $\mathbf{R}^3$ splits into orbits of the natural equivalence group which we denote by $\mathcal{F}$ (for "framed"). This is an analog of the group of left-right equivalence of mappings (cf. [5, 4]). Namely, we consider a representative $g : U \to \mathbf{R}^3$ of an element of Ω_f modulo:

i) orientation-preserving diffeomorphisms of the target $\mathbf{R}^3$;

ii) diffeomorphisms of the source pair (U, S^1) preserving the orientations of the circle and its neighbourhood;

iii) terms of order greater than 1 in the direction in U that is transversal to S^1.

There is an obvious local version of the $\mathcal{F}$-equivalence for germs of mappings $g : (\mathbf{R}^2, \mathbf{R}^1, 0) \to (\mathbf{R}^3, 0)$. This has the following coordinate description.

Let x and y be coordinates on the source plane with the $\mathbf{R}^1$ being the x-axis. We use the notations:

$\mathcal{O}_{x,y}$ for the space of all real-valued C^∞-function-germs on $(\mathbf{R}^2, 0)$;

$\mathcal{O}^3_{x,y}$ for the space of all C^∞-map-germs of $(\mathbf{R}^2, 0)$ to $\mathbf{R}^3$;

$\mathcal{O}^3_g$ for the space of C^∞-map-germs from the target copy of $(\mathbf{R}^3, 0)$ to $\mathbf{R}^3$ pulled back to $(\mathbf{R}^2, 0)$ by the germ g.

The tangent space to the $\mathcal{F}$-orbit of a map-germ $g \in \mathcal{O}^3_{x,y}$ is

$$T_g(\mathcal{F}g) = \mathcal{O}^3_g + \mathcal{O}_{x,y}\langle \partial g/\partial x, y\partial g/\partial y\rangle + y^2\mathcal{O}^3_{x,y} \ ,$$

the middle summand being a module on the two generators.

Example 2.4 A non-singular germ $(\mathbf{R}^2, \mathbf{R}^1, 0) \to \mathbf{R}^3$ can be reduced to

$$(x, y) \mapsto (x, y, 0).$$

The tangent space to its $\mathcal{F}$-orbit is the whole of $\mathcal{O}^3_{x,y}$ (so, the germ is stable).

2.3 Bifurcation diagrams of framed curve-germs

As usual, *an $\mathcal{F}$-miniversal deformation of g* (cf. [5, 4]) is a minimal transversal to its $\mathcal{F}$-orbit so long as the tangent space $T_g(\mathcal{F}g)$ has a finite codimension in $\mathcal{O}^3_{x,y}$.

The base of an $\mathcal{F}$-miniversal deformation of a map-germ g contains *the bifurcation diagram* $\Sigma_{\mathcal{F}}(g)$. That is the set of the values of the deformation parameters λ for which the corresponding perturbed mappings $Tg_\lambda : T_{\mathbf{R}^1}\mathbf{R}^2 \to T\mathbf{R}^3$ are not embeddings. There are two options to achieve a degeneration:

1) either the mapping $g_\lambda : \mathbf{R}^1 \to \mathbf{R}^3$ is not an embedding,

2) or, for some point $s \in \mathbf{R}^1$, the differential Tg_λ is not of rank 2 on $T_s\mathbf{R}^2$.

Thus the diagram $\Sigma_{\mathcal{F}}(g)$ has two components which we denote by $\Sigma'_{\mathcal{F}}(g)$ and $\Sigma''_{\mathcal{F}}(g)$ respectively. Both are hypersurfaces. Forgetting the framing and considering g as a mapping of the line alone, we stay with the component $\Sigma'_{\mathcal{F}}(g)$ only.

Example 2.5 A local normal form for the simplest singular framing on a smooth curve in $\mathbf{R}^3$ is

$$h_0 : (x, y) \mapsto (x, yx, 0).$$

The differential of this mapping has rank 1 at the origin.

For a one-parameter miniversal deformation one can take

$$h_\alpha : (x, y) \mapsto (x, yx, \alpha y),$$

where α is the deformation parameter. In Fig.1, above the parameter line, we show the corresponding framed curves. The bold line there is the core,

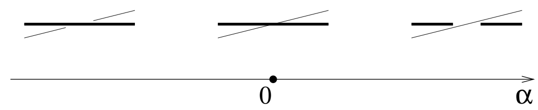


Figure 1: *The simplest framing degeneration.*

that is the image of the x-axis. The thin line is the framing. It represents the Th_α-image of a section of $T_{\mathbf{R}^1}\mathbf{R}^2$ which contains the generator ∂_y of the kernel of $T_0 h_0$. The bifurcation diagram is $\Sigma_{\mathcal{F}}(h_0) = \Sigma''_{\mathcal{F}}(h_0) = \{\alpha = 0\}$.

The bifurcation diagram of a framed curve-germ is coorientable in the base of an $\mathcal{F}$-versal deformation at its regular points. Namely, to coorient the component $\Sigma''_{\mathcal{F}}$ we say that the local bifurcation of Fig.1, for *decreasing* α is done in *the positive direction.* To coorient $\Sigma'_{\mathcal{F}}$ we say, as usual, that the bifurcation of Fig.2 is *positive.* We assume here and further on that the right orientation of $\mathbf{R}^3$ is fixed. In both cases the positive move increases the writhe of the framed curve.

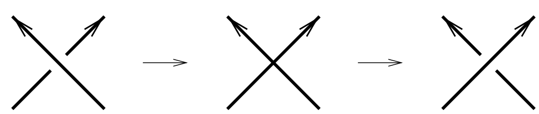


Figure 2: *Positive crossing of the stratum of non-embedded curves.*

Example 2.6 The simplest local singularity of a mapping $\mathbf{R}^1 \to \mathbf{R}^3$ has a normal form

$$x \mapsto (x^2,\ \ x^3,\ \ 0).$$

Equipping this map-germ with a generic framing we arrive to normal forms

$$(x, y) \mapsto (x^2,\ \ x^3 \pm yx,\ \ y)$$

and 2-parameter miniversal deformations

$$(x^2,\ \ x^3 \pm yx + \alpha x,\ \ y + \beta x),$$

with the parameters α, β. Bifurcations in the (+)-family are shown in Fig.3. The cooriented bifurcation diagram of the (−)-family is absolutely the same.

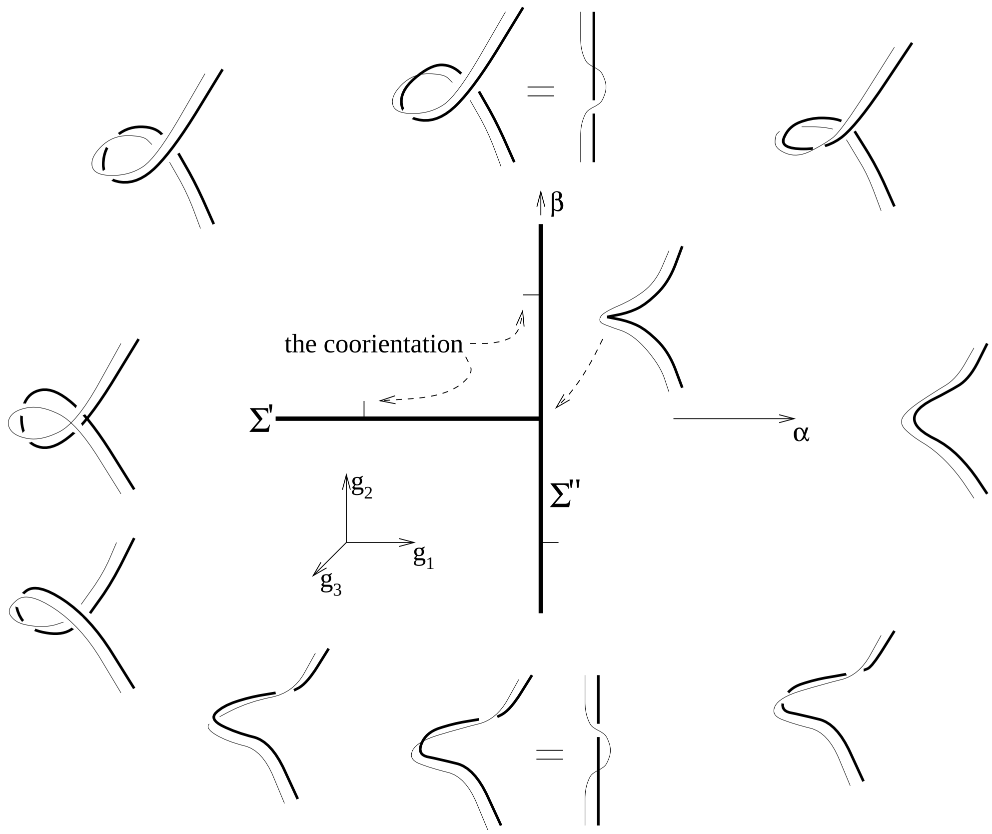


Figure 3: *A miniversal deformation of a framed curve with a generic singular point.*

Example 2.7 Another particular case useful for our further considerations is the generic degeneration of the framing at a double point of the core. Its $\mathcal{F}$-miniversal deformation is the 3-parameter family of bigerms

$$(x_1, y_1) \mapsto (x_1, y_1, 0),$$
$$(x_2, y_2) \mapsto (\alpha y_2, x_2 y_2 + \beta + \gamma y_2, x_2).$$

Cooriented bifurcation diagram of this family, within the assumption that each of the two curve-germs is oriented by the increase of the corresponding x-coordinate, is shown in Fig.4.

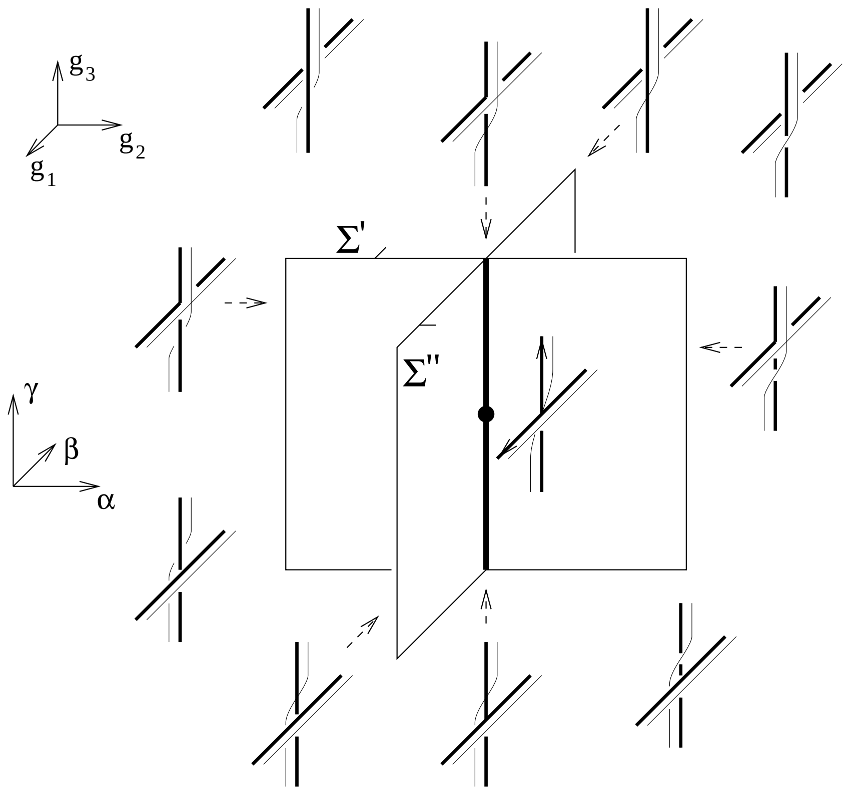


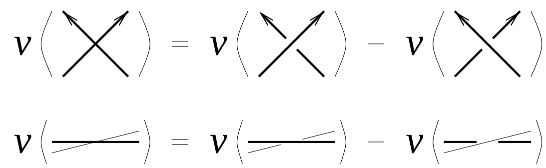
Figure 4: *Bifurcations of a double point with a degenerate framing.*

2.4 Extended invariants

The obvious global version, in the space Ω_f, of the bifurcation diagram will be called *the discriminant* of Ω_f and denoted by Σ_f. This is the union of the two hypersurfaces Σ'_f and Σ''_f. We coorient the discriminant by local means, using the above coorientation of bifurcation diagrams.

An invariant of oriented framed knots is an element of the group $H^0(\Omega_f \setminus \Sigma_f)$ (with any coefficients).

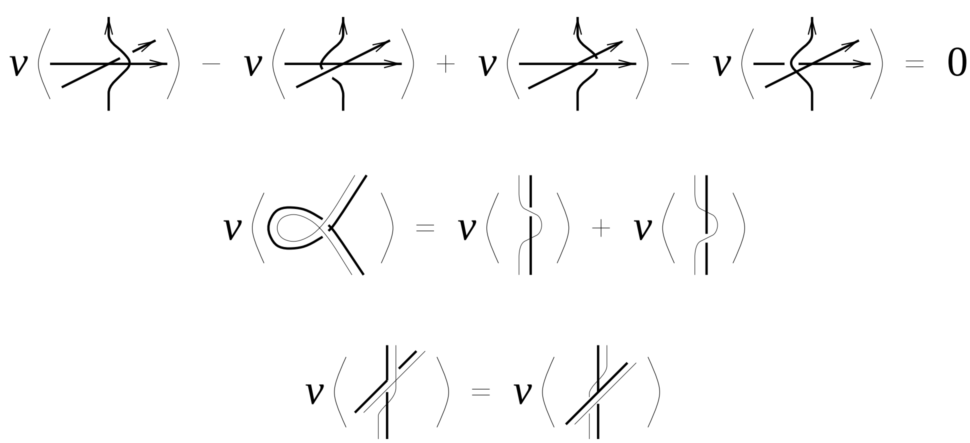
A regular point of m-tuple self-intersection of Σ_f represents a singular oriented framed knot. We extend an invariant v of non-singular framed knots to the singular ones by the recursive setting:



Here and below we assume that all the framed curves that enter one and the same equality coincide modulo the shown curve and framing fragments. The lower line uses the bifurcation in the normal form of Example 2.5.

Application of the recursive definition to certain special degenerations of framed knots imply

Proposition 2.8 *The values of an invariant on singular framed knots are subject to the 4-term, 3-term and commutativity relations:*



Proof. The 4-term relation here is in fact the one that is induced from the Vassiliev theory of invariants of unframed knots by omitting the framing. It follows from the bifurcations of a generic triple point of the core. To prove the 4-term relation one can follow [8] resolving the double points by the definition (Fig.5).

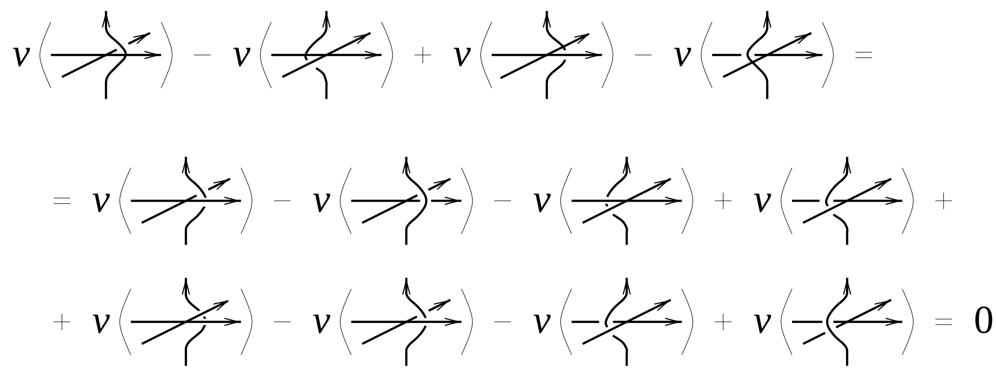


Figure 5: *Proof of the 4-term relation.*

The two other relations are easily read from the bifurcations of Examples 2.6 and 2.7 in the similar way (Fig.6). □

2.5 Chord diagrams with distinguished points and invariants of finite order

Consider $2n+k$ distinct points on an oriented circle. Join $2n$ of them in n non-ordered pairs. Consider such objects up to diffeomorphisms of the circle preserving its orientation. Each equivalence class will be called *an n-chord diagram with k distinguished points* or, shortly, *an (n,k)-diagram.*

We associate an (n,k)-diagram to a singular framed oriented knot $g : U \to \mathbf{R}^3$:

i) the circle is the source $S^1 \subset U \subset \mathbf{R}^2$ (we take it to be a standard counter-clockwise oriented circle on a plane and never mention this orientation in our figures);

ii) a pair of points is the inverse image of a double point;

iii) at a distinguished point the rank of the differential Tg is 1.

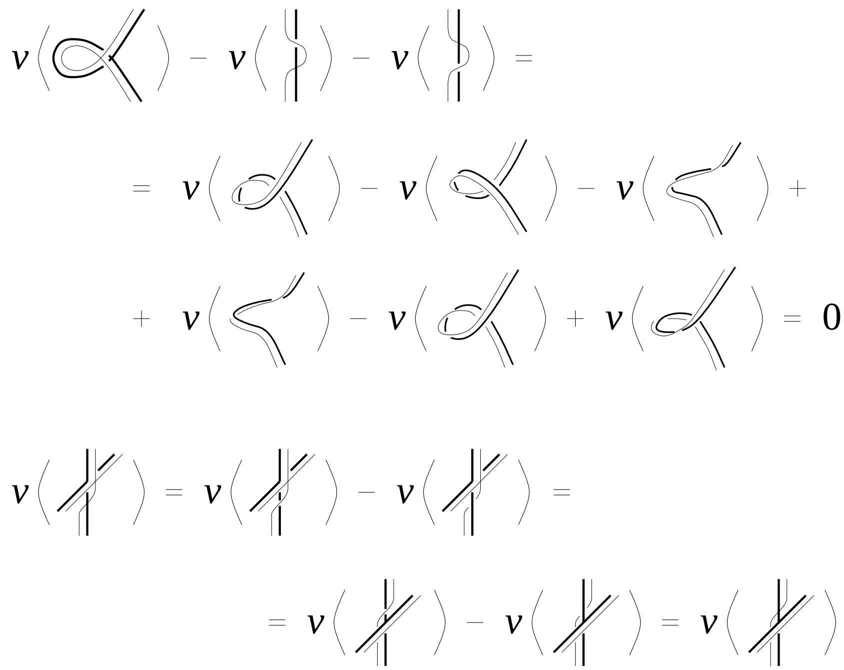


Figure 6: *Proof of the 3-term and commutativity relations.*

Definition 2.9 An invariant of framed oriented knots in $\mathbf{R}^3$ is *a (Vassiliev) invariant of order less than* m if it vanishes on all oriented framed knots with m singularities.

We denote the linear space of all invariants of order less than m by V^f_{m-1}.

Take an invariant of order m, that is an element $v \in V^f_m \setminus V^f_{m-1}$. Its restriction to the set of all oriented framed knots with m singularities is called *the symbol* of v.

Proposition 2.10 *The symbol of an order* m *invariant is a well-defined function on the set of all* (n, k)*-diagrams,* $m = n + k$.

Proof. Consider the set $\Omega(D) \subset \Omega_f$ of all parametrizations $g : U \to \mathbf{R}^3$ of singular framed knots with the same (n, k)-diagram D. We need to show that any two elements $g_1, g_2 \in \Omega(D)$ can be deformed one into another without change of the value of the symbol.

On the first step we deform g_1 into $g_2' \in \Omega(D)$ that has the same core as g_2. This can be done by a homotopy that stays almost all the time in $\Omega(D)$ and, at the remaining finitely many instances, passes transversally through the set of parametrizations with $n+1$ double points on the core.

Up to homotopies in $\Omega(D)$, we can assume that g_2' and g_2 have the same k points of generic framing degeneration and coincide on small neighbourhoods of these distinguished points. So, the two parametrizations differ only by the rotation of the framings along the intervals between distinguished points. Now we deform the framing of g_2' to that of g_2 by a homotopy that stays almost all the time in $\Omega(D)$ and, at the remaining finitely many instances, passes transversally through the set of parametrizations with $k+1$ points of framing degeneration.

During the constructed homotopy the value of the symbol could change only on the two sets of the above mentioned finitely many instances. The increments are the values of the invariant on framed knots with $n+k+1 = m+1$ singularities which are zeros. □

The relations of Proposition 2.8 immediately imply

Proposition 2.11 *The values of a symbol are subject to the 4-term, 2-term and floating-point relations of Fig. 7.*

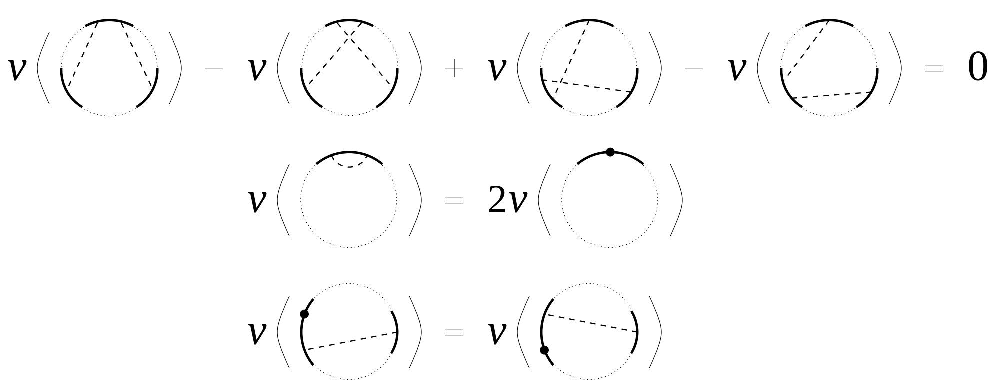


Figure 7: *4-term, 2-term and floating-point relations for symbols.*

In Fig.7 and further on we show all the distinguished points and chords based on the solid arcs and none of those based on the dotted arcs. All the diagrams entering the same relation are assumed to differ only by their parts based on the solid arcs.

Remark 2.12 In the Vassiliev theory of unframed knots, invariants and their symbols are subject to the 1-term relations [15, 3, 7, 11, 6] (Fig.8). These relations follow from Fig.3 with all the framings omitted. Propositions 2.8 and 2.11 show what happens with the 1-term relations when we pass to the framed setting. For example, for symbols of $\mathbf{Z}_2$-valued invariants the 1-term relation still holds in the framed case.

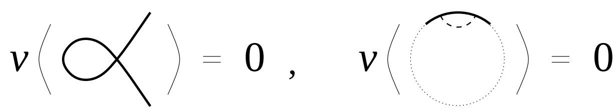


Figure 8: *The 1-term relations for invariants of unframed knots and their symbols.*

Remark 2.13 Twice the floating-point relation follows from the two others of Proposition 2.11. Indeed, consider the 4-term relation in the case when all its 7-through-1-o'clock arcs are solid. The two middle terms cancel one another. Expressing each of the two remaining terms by means of the 2-term relation we obtain what has been promised.

Let $\dot{\mathcal{A}}_m$ be the linear space spanned by all (n,k)-diagrams, with $m = n+k$, modulo the three relations of Proposition 2.11 considered now as relations on the diagrams rather than functions on them. Proposition 2.11 embeds the space V^f_m/V^f_{m-1} of symbols of order m invariants into the space $\dot{\mathcal{A}}^*_m$ dual to $\dot{\mathcal{A}}_m$. Set $\dot{\mathcal{A}} = \oplus_{m\geq 0}\dot{\mathcal{A}}_m$.

Similar to [11, 6], the 4-term and floating-point relations imply that the connected sum of two chord diagrams with distinguished points is a well-defined element in $\dot{\mathcal{A}}$, that is the sum does not depend on the location of the connecting surgery. So

Proposition 2.14 *$\dot{\mathcal{A}}$ is an algebra with respect to the connected summation of diagrams.*

If the ground field is not of $char = 2$, the graded algebra $\dot{\mathcal{A}}$ is isomorphic to the graded algebra $\mathcal{A} = \oplus_{m\geq 0}\mathcal{A}_m$ whose mth direct summand is spanned by all m-chord diagrams (with no distinguished points at all) modulo the 4-term relation only. The operation on $\mathcal{A}$ is the connected summation as well.

As it was mentioned in [11, 6], the 4-term relation on chord diagrams with no distinguished points turns out to be the only relation for real-valued invariants of framed knots in $\mathbf{R}^3$. Namely, indicating the real versions of the spaces by the subscript $\mathbf{R}$, we have

Theorem 2.15 $\quad V^f_{m,\mathbf{R}}/V^f_{m-1,\mathbf{R}} = \mathcal{A}^*_{m,\mathbf{R}}$.

This fact was proved in [12, 13] by introduction of a version of the universal Vassiliev-Kontsevich invariant for knots with the blackboard framing. In the next section we define the universal invariant that serves knots with arbitrary framings and reprove Theorem 2.15. Our approach is distinct from that of [12, 13].

3 The universal Vassiliev-Kontsevich invariant for framed knots in $\mathbf{R}^3$

3.1 The invariant of Morse knots

We represent the Euclidean 3-space as a direct product $\mathbf{C}\times\mathbf{R}$ with the complex coordinate z and the real coordinate t.

An unframed knot in $\mathbf{C}\times\mathbf{R}$ is called *a Morse knot* if t is a Morse function on it. *A framed knot* in $\mathbf{C}\times\mathbf{R}$ is called *Morse* if its core is a Morse knot.

A. Consider a non-singular oriented framed Morse knot K^f parametrized by a mapping $g : (U, S^1) \to \mathbf{R}^3$. Let K be its core $g(S^1)$. Fix a decomposition $(U, S^1) = S^1\times(\mathbf{R}, 0)$ of the annulus. Let y be the coordinate along the second factor. Denote by u the vector field $Tg(\partial_y)$ on K. For small $\varepsilon > 0$, we shift the core K in the direction of u:

$$(z, t) \mapsto (z, t) + \varepsilon u(z, t).$$

For all sufficiently small ε, the result K_ε of the shift is a Morse knot that does not intersect K. We orient K_ε by the orientation inherited from K.

B. In order to have a good definition of a chord diagram later on, we slightly adjust the Morse link $K \cup K_\varepsilon$. Near a local maximium of the function t on K, t has the local maximum on K_ε as well. We take the lowest of the two critical levels and remove the small arc of $K \cup K_\varepsilon$ that is locally above this level. In the similar way, we remove the small arc that is locally below the highest of the two critical levels near a local minimum of t on K. After the surgery at all the local extrema, we remain with the subsets $\widehat{K} \subseteq K$ and $\widehat{K}_\varepsilon \subseteq K_\varepsilon$.

The shift along the framing field u provides the one-to-one correspondence between the sets of intervals of monotonicity of the function t on K and K_ε. For each non-critical point $(z', t) \in \widehat{K}_\varepsilon$ this correspondence correctly defines its unique *neighbour* $(z'', t) \in \widehat{K}$ on the same t-level.

C. Now we take m different non-critical levels $t_{min} < t_1 < t_2 < \ldots < t_m < t_{max}$, where t_{min} and t_{max} are the global extreme values of t on $\widehat{K} \cup \widehat{K}_\varepsilon$. In each section $t = t_j$ of $\widehat{K} \cup \widehat{K}_\varepsilon$, we choose an *ordered* pair of points $(z_j, z'_j) = (z_j, z'_j)(t_j) \in \widehat{K} \times \widehat{K}_\varepsilon$. Let P be a set of m such pairs, one pair per level.

The set P defines the m-chord diagram $D(P)$ as follows (see Fig.9).

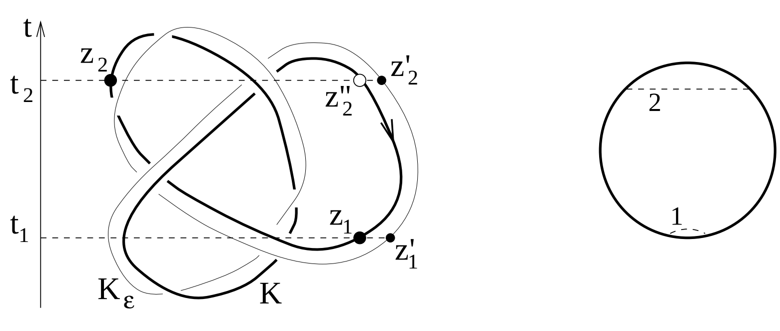


Figure 9: *A pairing on a framed knot and the chord diagram of the pairing.*

In each pair we substitute $z'_j \in \widehat{K}_\varepsilon$ by its neighbour $z''_j \in \widehat{K}$. The core K is the image of the embedding of the oriented circle S^1 that we again take to be a standard counter-clockwise oriented circle on the plane. If $z_j \neq z''_j$, we join the preimages of the points z_j and z''_j on the source circle by the chord. If $z_j = z''_j$, we draw a small chord between two arbitrary points on the circle which are very close to the preimage of z_j (so that on the small arc subtended by this chord there are no endpoints of any of the $m-1$ chords corresponding to the other distinguished t-levels).

D. Let $\mathcal{A}_{m,\mathbf{C}}$ be the complex linear space generated by all m-chord diagrams modulo the 4-term relation.

We introduce

Definition 3.1 $\hat{Z}_m(K, K_\varepsilon) =$

$$= \frac{1}{(2\pi i)^m} \int\limits_{t_{min} < t_1 < t_2 < \ldots < t_m < t_{max}} \sum_{P = \{(z_j, z'_j)(t_j)\}} (-1)^{P_\downarrow} \bigwedge_{j=1}^{m} \frac{dz_j - dz'_j}{z_j - z'_j} \mathcal{D}(P) \in \mathcal{A}_{m,\mathbf{C}} ,$$

where P runs through all possible pairings on $\widehat{K} \cup \widehat{K}_\varepsilon$, $P_\downarrow$ is the number of points in the m pairs at which the function t is decreasing along the oriented link $K \cup K_\varepsilon$, and $\mathcal{D}(P)$ is the class of the diagram $D(P)$ in $\mathcal{A}_{m,\mathbf{C}}$.

Definition 3.2 $Z_m(K^f) = \lim_{\varepsilon \to 0} \hat{Z}_m(K, K_\varepsilon)$.

Theorem 3.3 (cf. [11]) *i) The limit that defines $Z_m(K^f)$ is finite.*

ii) $Z_m(K^f)$ does not depend on the decomposition $(U, S^1) = S^1 \times (\mathbf{R}, 0)$ of the annulus used in the definition.

iii) $Z_m(K^f)$ is invariant under the homotopy in the class of framed Morse knots.

iv) $Z_m(K^f)$ is an invariant of order less than $m+1$.

Statement iv) concerns the extension (in the sense of subsection 2.4) of the invariant to singular framed Morse knots none of whose singular points is a local extremum of t.

The proof of the theorem occupies the next two subsections.

3.2 Proof of the convergence

The divergence of the limit could arise from the two *dangerous* types of pairs (chords):

infinitesimal pairs that correspond to the case $z_j = z_j''$ in the definition of $D(P)$;

short pairs whose elements $z_j \neq z_j''$ are lying on two successive intervals of monotonicity of the function t on K, so that no chord connects the two semicircles into which z_j and z_j'' cut K.

In both cases, the diagram $D(P)$ is obtained from a diagram with fewer chords by insertion of an isolated chord. Due to the 4-term relation, the corresponding generator $\mathcal{D}(P)$ of $\mathcal{A}_{m,\mathbf{C}}$ does not depend on the location of the insertion. We are going to exploit this independence and group together the terms which correspond to one and the same generator of such type so that their individual divergences kill one another. The grouping is mainly based on the following.

Example 3.4 Consider the family of all possible pairings on $\widehat{K} \cup \widehat{K}_\varepsilon$ which have only two pairs involving points from a neighbourhood of some local maximum of the function t on K (Fig.10), with the upper pair being dangerous and the lower one not.

Let $\hat{t}$ be the local maximum value of t on $\widehat{K} \cup \widehat{K}_\varepsilon$. Let $t_1 < \hat{t}$ be the level of a *long* pair (z_1, z_1') that joins a point inside our neighbourhood with a point outside.

Pairings in the family have only one infinitesimal or short pair in the slice $t_1 < t < \hat{t}$ of the neighbourhood. Integration, within these limits, of the sum of the four 1-forms corresponding to the four dangerous pairs gives the logarithm of the cross-ratio:

$$\ln \frac{(z_2 - z_2')(z_3 - z_3')}{(z_2 - z_3')(z_3 - z_2')} \Bigg|_{t_1}^{\hat{t}} .$$

The upper bound evaluation gives zero.

The divergence of the lower bound terms $\ln(z_i - z_i')$, $i = 2, 3$, for $\varepsilon \to 0$, is cancelled by the integration along the pairings in which the pair (z_i, z_i') dives under the level $t = t_1$.

In the remaining lower bound term $\ln((z_2 - z_3')(z_3 - z_2'))$, z_i' tends to z_i as $\varepsilon \to 0$, $i = 2, 3$. At the same time, $\hat{t}$ tends to the local maximum value t^0 of the function t on K (unless $\hat{t} = t^0$ from the very beginning). Now the integral

$$\int_{const}^{t^0} \ln(z_2(t_1) - z_3(t_1)) \, \frac{dz_1(t_1) - dz_1''(t_1)}{z_1(t_1) - z_1''(t_1)}$$

along the levels of the long pair converges at $t_1 = t^0$.

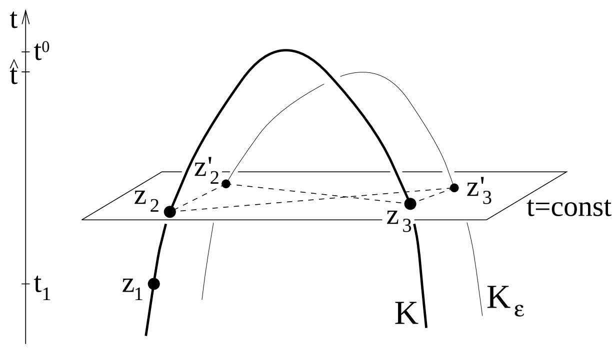


Figure 10: *Cancellation of divergencies.*

In a general case we follow the above order of integrations and passing to the limit.

We integrate first in the prelimit setting along the t-levels of infinitesimal and short chords for all the other levels fixed.

Consider two intervals, $I \subset \widehat{K}$ and $I_\varepsilon \subset \widehat{K}_\varepsilon$, of monotonicity of the function t that correspond to each other by the shift along the framing field u.

We say that *an infinitesimal pair* (z_j, z'_j) *is based on* $I \cup I_\varepsilon$ if the points z_j and z'_j are lying on these intervals.

We say that *a short pair* (z_j, z'_j) *is based on* $I \cup I_\varepsilon$ if that of the points z_j and z'_j which lies on $I \cup I_\varepsilon$ cannot be moved continuously along $\widehat{K} \cup \widehat{K}_\varepsilon$ to the neighbouring couple of intervals of monotonicity on which the other member of the pair lives. This applies in the case when the surgery that adjusted $K \cup K_\varepsilon$ to $\widehat{K} \cup \widehat{K}_\varepsilon$ was non-trivial in the neighbourhood of the local extremum of t on K. If, on the contrary, the local surgery was trivial (that is the framing vector u at the extremum has zero t-component) the base-interval assignment is arbitrary (but should be fixed before we start the integration).

Example 3.5 In Fig.10, the pairs (z_2, z'_3) and (z_3, z'_2) are based on the left and right couples of the intervals of monotonicity respectively.

Now, assume that in the expression for $\widehat{Z}_m(K, K_\varepsilon)$ we are considering an m-pairing P that has exactly r dangerous pairs based on a certain couple $I \cup I_\varepsilon$ of intervals. Consider this particular pairing as a member of the entire family of all the pairings that have exactly r dangerous pairs based on $I \cup I_\varepsilon$ and whose remaining $m - r$ pairs $\widetilde{P}$ are exactly the same as in P. The chord diagrams of all these m-pairings define the same element in $\mathcal{A}_{m,\mathbf{C}}$.

In this family, the t-levels of the short pairs, that are based on $I \cup I_\varepsilon$ and join the points of this couple of intervals with the points of the other couple of intervals that is adjacent to $I \cup I_\varepsilon$ via the two local maxima of t on $K \cup K_\varepsilon$, are bounded from below by the level $t = t_a$ of the corresponding long pair. There is also the similar bound $t < t_b$ for the levels of the short pairs located by the two local minima of the function t on $K \cup K_\varepsilon$.

A small exercise in elementary calculus and combinatorics shows that integration in $\widehat{Z}_m(K, K_\varepsilon)$ along the described family provides the form

$$\frac{1}{(2\pi i)^m} (-1)^{\widetilde{P}_\downarrow} \bigwedge_{j=1}^{m-r} \frac{dz_j - dz'_j}{z_j - z'_j} \; \frac{1}{r!} \; \ln^r \frac{z(t_a) - z'(t_a)}{z(t_b) - z'(t_b)} \; \mathcal{D}(P) \, ,$$

where after the reodering of the pairings of the family we set $\widetilde{P} = \{(z_1, z'_1), \ldots, (z_{m-r}, z'_{m-r})\}$. The evaluations of the differences under the logarithm are done on the two short pairs based on $I \cup I_\varepsilon$ and lying on the corresponding t-levels. The obtained form is to be integrated along the various t-levels of the pairings $\widetilde{P}$.

After similar integration for all the couples of intervals is done, we do not finish the integration in $\widehat{Z}_m(K, K_\varepsilon)$ but immediately pass to the limit for $\varepsilon \to 0$ to get $Z_m(K^f)$. This means the substitution of z''_j (and z'') for z'_j (and z') everywhere in the above $(m - r)$-form or in the lower degree form that has emerged from it.

So obtained limiting integral is absolutely convergent since its only singularities (cf. [11, 6]) are estimated by constant multiples of the integrals like

$$\int_0^{\text{const}} (\int_0^{x_s} \ldots (\int_0^{x_2} (\int_0^{x_1} \ln^r x_0 dx_0) dx_1) \ldots dx_{s-1}) dx_s,$$

that are convergent at 0.

3.3 Invariance of the limit under horizontal moves

The prelimit integrals $\widehat{Z}_m(K, K_\varepsilon)$ do not need to be invariant under horizontal perturbations (that is when each point stays in its t-level) of the link $K \cup K_\varepsilon$. On the contrary, the limit $Z_m(K^f)$ is invariant under horizontal isotopies of the framed link. Let us show this assuming, at first, that none of the critical levels of the function t on K moves.

From the previous subsection we see that it is enough to assume that the framing is such that, out of sufficiently small neighbourhoods of critical points of t on K, the framing field u is lying in the levels $t = const$ and is of length 1. So the distance between a point $z'_j \in \widehat{K_\varepsilon}$ and its neighbour $z''_j \in \widehat{K}$ is ε:

$$z'_j - z''_j = \varepsilon e^{i\varphi_j}, \;\; \varphi_j \in \mathbf{R}/2\pi\mathbf{Z}\,.$$

Consider a slice $a < t < b$ such that the closed interval $[a, b]$ contains no critical values of t on K. The part of K in this slice consists of, say, r branches going upwards and r branches going downwards. Consider a horizontal isotopy of our framed knot which is non-trivial only in $a < t < b$. Absolutely similar to [11, 6], the invariance of the elements $Z_m(K^f) \in \mathcal{A}_{m,\mathbf{C}}$, $m \geq 0$, under such isotopy is implied, according to Stokes' formula, by flatness of the following Knizhnik-Zamolodchikov type connection.

This formal connection, which we denote by $\Theta^f_{r,r}$, is defined on the direct product of two spaces. One of them is the set of all ordered $2r$-tuples $(z_1, \dots, z_{2r})$ of pairwise distinct complex numbers. The other is the $2r$-dimensional torus with the coordinates $\varphi_p \in \mathbf{R}/2\pi\mathbf{Z}$.

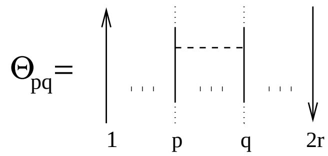
We set

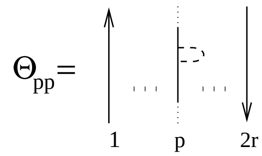
$$\Theta^f_{r,r} = \sum_{1 \leq p,q \leq 2r} s_p s_q \Theta_{pq} \omega_{pq}\,,$$

where

s_p is 1 for $p \leq r$ and -1 otherwise;
$\omega_{pq} = \frac{dz_p - dz_q}{z_p - z_q}$ when $p \neq q$;
$\omega_{pp} = d\varphi_p$;

Θ_{pq} and Θ_{pp} are the 1-chord diagrams based on $2r$ ordered parallel arrows, first r of which point upwards and the others downwards:





The 1-form $\Theta^f_{r,r}$ is closed.

Now, the product of two chord diagrams based on our $2r$ ordered arrows is, as usual, drawing the first of them below the second. Formal linear combinations of chord diagrams are considered modulo the 4-term relation. For example, the special "diagonal" case of the 4-term relation says that Θ_{pp} commutes with any Θ_{pq} (this is actually twice the floating-point relation). Within this understanding the fact $\Theta^f_{r,r} \wedge \Theta^f_{r,r} = 0$ is obvious (cf. [6]).

Thus connection $\Theta^f_{r,r}$ is flat.

The proof of invariance of $Z_m(K^f)$ under all other moves which preserve the class of Morse framed knots is almost word-to-word repetition of subsection 4.3.3 of [6].

Part ii) of Theorem 3.3 is a paricular case of part iii).

Finally, as in [11], part iv) of Theorem 3.3 is obvious.

3.4 The universal invariant

Similar to the unframed case [11, 6], the integrals $Z_m(K^f)$ are not invariant under the move that cancels two neighbouring local extrema of t on K. We fix the problem exactly in the same way as it was done in [11, 6].

Set

$$Z(K^f) = \sum_{m \geq 0} Z_m(K^f) \in \overline{\mathcal{A}}_{\mathbf{C}} \ ,$$

where $\overline{\mathcal{A}}_{\mathbf{C}} = \prod_{m \geq 0} \mathcal{A}_{m,\mathbf{C}}$.

Let $\mathcal{U}^f$ be the curve of Fig.11 lying in the plane $\mathrm{Im} z = 0$ and equipped with the trivial framing $i\partial_z$.

The series $Z(\mathcal{U}^f) \in \overline{\mathcal{A}}_{\mathbf{C}}$ is invertible since it starts with $1 \in \mathcal{A}_{0,\mathbf{C}}$.

Let c be the number of critical points of the function t on the core of a framed Morse knot K^f.

Definition 3.6 The element

$$\tilde{Z}(K^f) = Z(K^f) \times Z(\mathcal{U}^f)^{1-\frac{c}{2}} \in \overline{\mathcal{A}}_{\mathbf{C}}$$

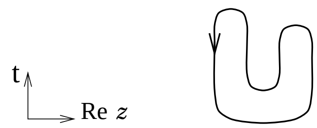


Figure 11: *The curve $\mathcal{U}$.*

is called *the universal Vassiliev-Kontsevich invariant* of the framed Morse knot K^f.

Example 3.7 Let $\omega \in \mathcal{A}_{1,\mathbf{C}}$ be the 1-chord diagram. Consider an unknot with the framing that makes one positive rotation around it. The value of $\widetilde{Z}$ on this unknot is $\exp(\omega)$.

Theorem 3.8 (cf. [11, 6]) *For any framed Morse knot K^f, $\widetilde{Z}(K^f)$ depends only on the topological type of K^f. The degree m component $\widetilde{Z}_m(K^f) \in \mathcal{A}_{m,\mathbf{C}}$ of $\widetilde{Z}(K^f)$ is an invariant of order less than $m+1$.*

Since the proof completely repeats that in the unframed case [11, 6], we omit it here.

The lowest order term of $\widetilde{Z}(K^f)$ for a singular framed knot with n double points and k points of degeneration of the framing is easily seen to be $2^n\mathcal{D}(K)$, where $\mathcal{D}(K) \in \mathcal{A}_{n+k,\mathbf{C}}$ is the chord diagram of K^f (we use here the 2-term relation to treat an (n,k)-diagram as the $(n+k)$-chord diagram). As in [11, 6], this fact implies the claim $V^f_{m,\mathbf{R}}/V^f_{m-1,\mathbf{R}} = \mathcal{A}^*_{m,\mathbf{R}}$ of Theorem 2.15 on the description of the space of symbols of order m real-valued invariants of framed knots in $\mathbf{R}^3$.

Remark 3.9 Similar to Exercise 4.5 of [6], it is easy to see that the series $\widetilde{Z}(K^f)$ is real.

4 Unframed knots in a solid torus

4.1 Marked chord diagrams

Starting with the space Ω_{ST} of C^∞-mappings of an oriented circle to a solid torus (ST), we construct the theory of Vassiliev type invariants of oriented unframed knots in ST in the obvious way. We get notions of non-singular and singular knots, extended invariants, chord diagrams, etc. The main new feature here is that the chord diagram of a singular knot possesses a natural integer marking (cf. [10]).

Namely, let us fix a generator of the fundamental group $\pi_1(\mathrm{ST}) = \mathbf{Z}$. A double point of an oriented singular knot in ST cuts the knot into two subloops each of which has its class in $\pi_1(\mathrm{ST})$. We write the corresponding fundamental integer on the side of the corresponding chord in the diagram that faces the preimage of the corresponding subloop (Fig.12). For convenience we also mark the circle of the diagram with the fundamental class of the whole knot. The sum of the two markings on each chord is equal to the marking of the circle.

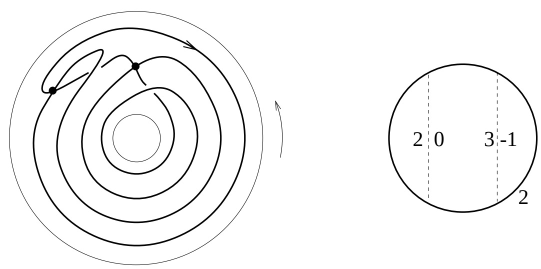


Figure 12: *The marked chord diagram of a singular knot in a solid torus.*

Let $\Delta_n \subset \Omega_{ST}$ be the set of parametrizations of knots with exactly n generic double points. We say that two elements of Δ_n are *related* (cf. [15]) if they can be joined by a C^∞-homotopy that stays almost all the time in Δ_n and, at the remaining finitely many instants, crosses Δ_{n+1} transversally. We also say that *two singular knots are related* if their representatives are related.

We have evident

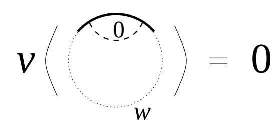
Proposition 4.1 *Two singular knots in a solid torus are related if and only if their marked chord diagrams coincide.*

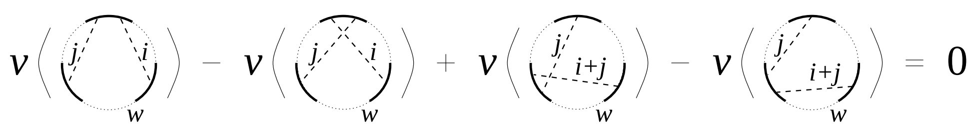
Recall that we consider chord diagrams up to diffeomorphisms of the circle which preserve the orientation. In the marked case diffeomorphisms should preserve the markings as well.

4.2 Marked relations

Of course, the values of invariants on oriented singular knots in a solid torus are subject to the 4-term relation of Proposition 2.8 and the framing-independence 1-term relation of Fig.8. On the other hand, due to Proposition 4.1, the values of a symbol of order n invariant on singular knots in ST are in fact functions on marked n-chord diagrams. This implies

Proposition 4.2 *The values of a symbol v on marked chord diagrams are subject to the marked 1- and 4-term relations:*

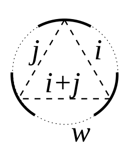




Here we give only partial markings that allow us to restore the complete ones.

Proof. Indeed, the marked 1-term relation corresponds to the contraction of a small subloop of a singular knot which is unlinked with the remaining part of the knot (see Fig.3 and omit all the framings there). The fundamental class of a contractible subloop is 0.

The marked 4-term relation comes from the bifurcation of a triple point. So the partial markings are inherited from that of the marked chord diagram of a triple point:



□

4.3 Module of marked chord diagrams

An abstract marked n-chord diagram is an n-chord diagram with integer two-side marking of its chords and integer marking of the circle such that the sum of the two markings of each chord is the making of the circle.

Let $\mathcal{M}^0$ be the linear space of finite linear combinations (with some fixed coefficients) of all abstract marked chord diagrams modulo the marked 1- and 4-term relations (that is, once again, the relations of Proposition 4.2 for diagrams themselves rather than for functions on them). We denote by $\mathcal{M}^0_n$ the order n part of $\mathcal{M}^0$ generated by n-chord diagrams. Unlike the non-marked case, $\mathcal{M}^0_n$ is infinite-dimensional for any $n \geq 0$.

In the non-marked case the linear space $\mathcal{A}^0$ generated by all chord diagrams modulo the 1- and 4-term relations is an algebra with respect to the connected sum operation [11, 6] ($\mathcal{A}^0$ is the quotient-algebra of the algebra $\mathcal{A} = \oplus_{m\geq 0}\mathcal{A}_m$ considered above). It is pretty obvious that $\mathcal{M}^0$ does not have similar algebra structure: in general, the markings of chords of the connected sum of two marked chord diagrams depend on the arcs on which the connecting surgery is done. The marking on the side of a chord that faces the surgery increases by the marking of the circle of the added diagram (Fig.13) as this should be for the connected sum of singular knots in a solid torus.

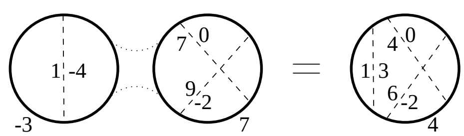


Figure 13: *Connected sum of marked chord diagrams.*

Nevertheless the connected sum is a well-defined operation on $\mathcal{M}^0$ in the following special case.

Consider the embedding $\zeta : \mathcal{A}^0 \to \mathcal{M}^0$ that assings identically zero marking to a non-marked diagram.

Theorem 4.3 *The connected summation defines on $\mathcal{M}^0$ the structure of a module over $\zeta(\mathcal{A}^0)$.*

The proof, based on the marked 4-term relation, repeats the proof of the fact that $\mathcal{A}^0$ is an algebra with respect to the same operation (Lemma 2.1 of [11]). One needs only to be slightly attentive to the markings.

4.4 The universal invariant for unframed knots in a solid torus

We consider a solid torus (ST) as a direct product $\mathbf{C} \times S^1$ with the complex coordinate z and circular coordinate θ mod 2π. We take a generator of $\pi_1(\mathrm{ST})$ being a loop that runs once around the torus in the direction of increase of θ.

A knot in ST is *a Morse knot* if θ is a Morse function on it.

Let K be an oriented non-singular Morse knot in ST.

Take n different non-critical levels $0 < \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$. In each section $\theta = \theta_j$ of K, choose an *unordered* pair of points $(z_j, z'_j) = (z_j, z'_j)(\theta_j)$. The set P of n such pairs, one pair per level, defines the n-chord diagram in the obvious way. The diagram is marked. Its circle is marked by the class of K in $\pi_1(\mathrm{ST})$. The marking on a chord is given by the fundamental classes of the two loops obtained by a homotopy of K in ST that glues together the points of the corresponding pair and is the identity outside a small neighbourhood of the θ-level of the pair.

We denote by $\mathcal{D}(P) \in \mathcal{M}^0_{n,\mathbf{C}}$ the class of the obtained marked diagram.

Using the obvious notations we introduce

Definition 4.4 $\quad Z_n^{ST}(K) =$

$$= \frac{1}{(2\pi i)^n} \int\limits_{0<\theta_1<\theta_2<\ldots<\theta_n<2\pi} \sum_{P=\{(z_j,z'_j)(\theta_j)\}} (-1)^{P_\downarrow} \bigwedge_{j=1}^{n} \frac{dz_j - dz'_j}{z_j - z'_j} \mathcal{D}(P) \in \mathcal{M}^0_{n,\mathbf{C}} \; .$$

Absolutely similar to [11, 6], adding only the locality of the marked 4- and 1-term relations to trace the markings in homotopies, one gets

Theorem 4.5 *i) The integral that defines $Z_n^{ST}(K)$ is absolutely convergent.*

ii) $Z_n^{ST}(K)$ does not depend on the choice of the zero-level of the circular coordinate θ on the solid torus.

iii) $Z_n^{ST}(K)$ is invariant under the homotopy in the class of Morse knots in the solid torus.

iv) $Z_n^{ST}(K)$ is an invariant of Morse knots of order less than $n+1$.

We set $Z^{ST}(K) = \sum_{n\geq 0} Z^{ST}_n(K) \in \overline{\mathcal{M}}^0_{\mathbf{C}}$, where $\overline{\mathcal{M}}^0_{\mathbf{C}} = \prod_{n\geq 0} \mathcal{M}^0_{n,\mathbf{C}}$.

Consider the curve $\mathcal{U}$ of subsection 3.4, this time *unframed* and lying in a sector of our solid torus (the coordinate t is replaced by θ). The series $Z^{ST}(\mathcal{U})$ belongs to the subspace $\zeta(\overline{\mathcal{A}}^0_{\mathbf{C}}) \subset \overline{\mathcal{M}}^0_{\mathbf{C}}$ spanned by chord diagrams with identically zero markings. As we have already mentioned, $\overline{\mathcal{M}}^0_{\mathbf{C}}$ is a module over the algebra $\zeta(\overline{\mathcal{A}}^0)$. The element $Z^{ST}(\mathcal{U})$ is invertible in $\zeta(\overline{\mathcal{A}}^0)$.

Definition 4.6 Let c be the number of critical points of the function θ on an oriented non-singular Morse knot K in the solid torus. The element

$$\tilde{Z}^{ST}(K) = Z^{ST}(K) \times (Z^{ST}(\mathcal{U}))^{1-\frac{c}{2}} \in \overline{\mathcal{M}}^0_{\mathbf{C}}$$

is called *the universal Vassiliev-Kontsevich invariant of K.*

Theorem 4.7 *The element $\tilde{Z}^{ST}(K)$ depends only on the topological type of the Morse knot K in the solid torus. The degree n component $\tilde{Z}^{ST}_n(K) \in \mathcal{M}^0_{n,\mathbf{C}}$ of $\tilde{Z}^{ST}(K)$ is an invariant of order less than $n+1$.*

The proof of the statement completely repeats that for knots in $\mathbf{R}^3$ given in [11, 6].

For any abstract marked chord diagram D one can find a singular knot in ST whose marked diagram is exactly D. Calculation based on the definition of the Vassiliev type extension of an invariant of non-singular knots shows that the lowest degree term of the series $\tilde{Z}^{ST}$ for such a knot is exactly D (cf. subsection 4.4.2 of [6]). This provides

Theorem 4.8 *The space of symbols of complex-valued order n Vassiliev invariants of oriented unframed knots in a solid torus coincides with the space of all complex-valued functions on the set of all marked n-chord diagrams subject to the marked 1- and 4-term relations.*

5 Framed knots in a solid torus

5.1 Finite type invariants in terms of marked chord diagrams

This case is the obvious symbiosis of the two above cases of framed knots in $\mathbf{R}^3$ and unframed knots in a solid torus. We do it very sketchy.

We extend invariants of non-singular oriented framed knots in ST to those singular following the two recursive settings of subsection 2.4.

Making the mixture of Definitions 3.1 and 4.4, for a non-singular oriented Morse framed knot K^f in ST we obtain the elements $\widehat{Z}_m^{ST}(K, K_\varepsilon) \in \mathcal{M}_{m,\mathbf{C}}$. The $\mathcal{M}_{m,\mathbf{C}}$ are the degree m components of the $\mathbf{C}$-linear space $\mathcal{M}_{\mathbf{C}}$ generated by all marked chord diagrams modulo the marked 4-term relation. Passing to the limit for $\varepsilon \to 0$, we define the elements $Z_m^{ST}(K^f)$. Similar to Theorems 3.3 and 4.5, we have

Theorem 5.1 *i) The limit element $Z_m^{ST}(K^f)$ is a finite element of $\mathcal{M}_{m,\mathbf{C}}$.*

ii) $Z_m^{ST}(K^f)$ is invariant under homotopies in the class of framed Morse knots in the solid torus.

iii) $Z_m^{ST}(K^f)$ is an invariant of framed Morse knots of order less than $m+1$.

Making the mixture of Definitions 3.6 and 4.6, we define the element $\widetilde{Z}^{ST}(K^f) \in \overline{\mathcal{M}}_{\mathbf{C}}$. This time we take the curve $\mathcal{U}^f$ of Fig.11 lying in a sector of the annulus $\mathrm{Im} z = 0$ in the solid torus and equipped with the framing $i\partial_z$. We also use the fact that $\overline{\mathcal{M}}_{\mathbf{C}}$ is a module over its subspace generated by chord diagrams with all the markings zero.

We have (cf. Theorems 3.8 and 4.7)

Theorem 5.2 *The element $\widetilde{Z}^{ST}(K^f) \in \overline{\mathcal{M}}_{\mathbf{C}}$ depends only on the topological type of the Morse framed knot K^f in the solid torus.*

The final classification result now is

Theorem 5.3 *The space of symbols of complex-valued order m Vassiliev invariants of oriented framed knots in a solid torus coincides with the space of all complex-valued functions on the set of all marked m-chord diagrams subject to the marked 4-term relation.*

Remark 5.4 The space of symbols of arbitrarily-valued order m Vassiliev invariants of oriented framed knots in ST embeds into the space of functions (with the same values as invariants) on the set of all marked (n, k)-diagrams. Symbols satisfy the marked 4-term relation of Proposition 4.2 as well as the marked 2-term and floating-point relations of Fig.14.

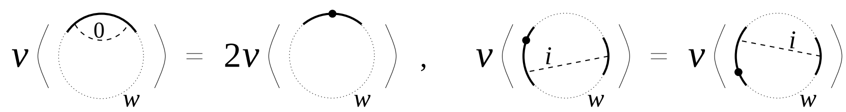


Figure 14: *The marked 2-term and floating-point relations.*

5.2 Coefficients of the polynomial invariants as invariants of finite order

As in the case of knots in 3-space [7], coefficients of the polynomial invariants of knots in a solid torus, when properly understood, turn out to be invariants of finite order. To illustrate this we consider in detail the framed version of the HOMFLY polynomial.

In the definition of the HOMFLY polynomial of framed knots and links in ST, we follow the definiton of [14] given for the unframed setting.

This time we consider the solid torus $I \times I \times S^1$, where I is the interval $[0, 1]$. The polynomial is defined on representatives of framed knots and links whose framing is blackboard with respect to the projection of the solid torus that forgets the first factor I.

Definition 5.5 For a non-singular link $L \subset I^2 \times S^1$ the polynomial $H(L) \in \mathbf{Z}[x, x^{-1}, y, y^{-1}, z_1, z_{-1}, z_2, z_{-2}, \ldots]$ is defined by the recursive and initial data of Fig.15.

In the last relation of Fig.15 the links L' and L'' are lying in the solid tori $[0, 1/2) \times I \times S^1$ and $(1/2, 1] \times I \times S^1$ respectively. The curves $L_{\pm 3}$ show the pattern for the whole of the basic series $\{L_j\}_{j=\pm 1, \pm 2, \ldots}$.

The results of [14] imply

Theorem 5.6 *Function H is a well-defined function on the isotopy classes of framed links in ST.*

Example 5.7 The value of H on an unknot with the blackboard framing is $(x - x^{-1})/y$. Participation in this fraction is the only way for y^{-1} to enter the polynomial of any link.

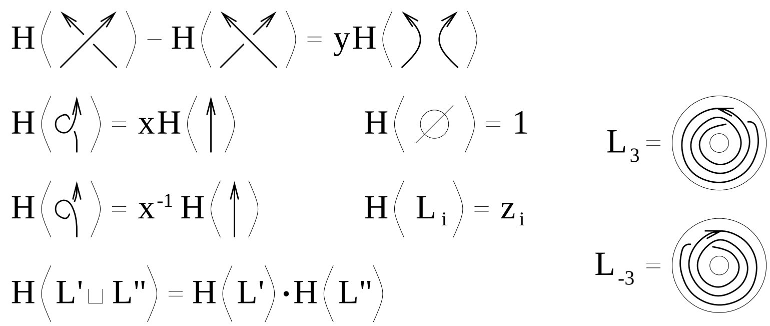


Figure 15: *Definition of the framed version of the HOMFLY polynomial for links with the blackboard framing in a solid torus.*

Fix an integer $n \neq -1$ and set

$$x = e^{(n+1)t/2}, \qquad y = e^{t/2} - e^{-t/2}$$

in $H(L)$. Since y^{-1} enters $H(L)$ only in the combination $(x - x^{-1})y^{-1}$, the result $W_n(L)$ of the substitution is an element of $\mathbf{Q}[z_{\pm 1}, z_{\pm 2}, \ldots]\{t\}$. Consider the expansion in powers of t:

$$W_n(L) = \sum_{m=0}^{\infty} w_{n,m}(L) t^m ,$$

where the $w_{n,m}(L)$ are polynomials in the variables z_i.

Theorem 5.8 *For a knot K each polynomial $w_{n,m}(K)$ is a framed knot invariant of order not greater than m.*

Proof (cf. [7, 11]). Extend the function H to the set of singular framed knots in ST via the two recursive relations of subsection 2.4. Consider the value of H on a framed knot K_s with a double points and b points of framing degeneration. We can assume that one of the 2^{a+b} non-singular resolutions of K_s used for the calculation of $H(K_s)$ is a knot with the blackboard framing and all the others are obtained from that one by the local moves of Fig.16.

The differences of the values of H on the resolutions that define the value $H(K_s)$ are given by the first line of Fig.15 and by Fig.17.

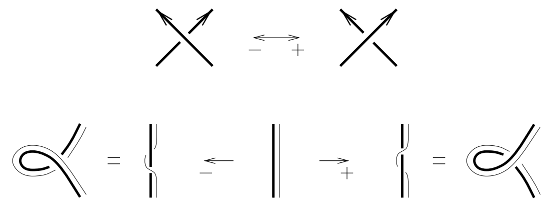


Figure 16: *Resolutions of local singularities of a singular framed knot in the blackboard setting.*

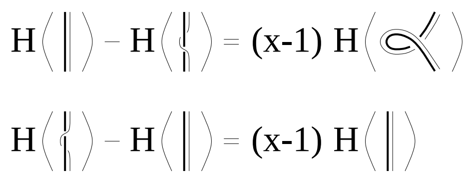


Figure 17: *Calculation of the polynomial of a knot with a degenerate framing.*

Thus the value $H(K_s)$ is $y^a(x-1)^b$ times a polynomial in $x, x^{-1}, y, (x - x^{-1})/y, z_{\pm 1}, z_{\pm 2}, \ldots$. Since both y and $x-1$ vanish at $t=0$, the series $W_n(K_s)$ is divisible by t^{a+b}. $\square$

Example 5.9 Assume that the class of the knot K in $\pi_1(\mathrm{ST})$ coincides with that of the basic loop L_i. Then the t-free term $w_{n,0}(K)$ is $z_i = H(L_i)$. For a contractible K the t-free term is $n+1$.

The statements analogous to Theorem 5.8 hold for the Kauffman polynomial and the unframed version of the HOMFLY polynomial [14].

References

[1] V.I.Arnold, *Plane curves, their invariants, perestroikas and classifications*, In: *Singularities and curves*, Adv.Sov.Math. **21** (1994), AMS, Providence, RI, 33–91

[2] V.I.Arnold, *Topological invariants of plane curves and caustiques*, University Lecture Notes **5** (1994), AMS, Providence, RI.

[3] V.I.Arnold, *Vassiliev's theory of discriminants and knots*, In: *First European Congress of Mathematics, Paris, July 1992*, vol.1, Progress in Math. **119** (1993), Birkhäuser, 3–28.

[4] V.I.Arnold, V.V.Goryunov, O.V.Lyashko and V.A.Vassiliev *Singularities II*, Encyclopaedia of Mathematical Sciences, v. 39 (1993), Springer Verlag, Berlin.

[5] V.I.Arnold, S.M.Gusein-Zade and A.V.Varchenko *Singularities of differentiable mappings*, v.I, Birkhäuser, Boston, 1985.

[6] D.Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[7] J.Birman and X.-S.Lin, *Knot polynomials and Vassiliev type invariants*, Inv.Math. **111** (1993), 225–270.

[8] S.V.Chmutov and S.V.Duzhin, *An upper bound for the number of Vassiliev knot invariants*, J. of Knot Theory and its Ramifications **3** (1994), 141–151.

[9] V.V.Goryunov, *Vassiliev type invariants in Arnold's J^+-theory of plane curves without direct self-tangencies*, Preprint 2-95, The University of Liverpool, June 1995.

[10] E.Kalfagiani, *Finite type invariants for knots in three-manifolds*, preprint, Columbia University, New York, 1993.

[11] M.Kontsevich, *Vassiliev's knot invariants*, Adv.Sov.Math. **16** (1993), part 2, AMS, Providence, RI, 137–150.

[12] T.Q.T.Lê and J.Murakami, *Representation of the category of tangles by Kontsevich's iterated integral*, preprint, Max-Planck-Institut für Mathematik, Bonn, 1993.

[13] T.Q.T.Lê and J.Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, preprint, Max-Planck-Institut für Mathematik, Bonn, 1994.

[14] V.G.Turaev, *The Conway and Kauffman modules of the solid torus with an appendix on the operator invariants of a tangle*, LOMI preprint E-6-88, Leningrad, 1988.

[15] V.A.Vassiliev, *Cohomology of knot spaces*, In: *Theory of Singularities and its Applications* (V.I.Arnold ed.), Adv.Sov.Math. **1** (1990), AMS, Providence, RI, 23–69.

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