Regular Legendrian knots and the HOMFLY polynomial of immersed plane curves

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Abstract

We show that every unframed knot type in $ST^*\mathbf{R}^2$ has a representative obtained by the Legendrian lifting of an immersed plane curve. This is the positive answer to the question asked by V.I.Arnold in [3]. The Legendrian lifting lowers the framed version of the HOMFLY polynomial [18] to generic plane curves. We prove that the induced polynomial invariant can be completely defined in terms of planes curves only. Moreover it is a geniune, not Laurent, polynomial in the framing variable. This provides an estimate on the Bennequin-Tabachnikov number of a Legendrian knot.

According to Arnold [2], theory of regular plane curves is a kind of non-commutative knot theory. It is not so difficult to see the "commutative" part there: this is the theory of plane curve invariants which change only in homotopies involving direct self-tangencies, that is when the tangent branches have coinciding orientations. Indeed, one can rise a generic plane curve to a Legendrian knot in the solid torus $ST^*\mathbf{R}^2$ or, if the winding number of the curve is zero, to $\mathbf{R}^3$. Such a knot will experience cross-changings only at the instants of the above self-tangencies.

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It has been observed that the theory of regular plane curves without direct self-tangencies possesses a far-going parallel with the theory of framed knots. For example, the space of Vassiliev type invariants is the same in both cases [10, 11]. Of course, this does not ensure that any framed knot can be represented by a Legendrian lift of an immersed plane curve equipped with the canonical Legendrian framing. If fact, this is not true in this generality: the Bennequin's inequality [5] tells that the twisting numbers of the canonical framings of Legendrian representatives of a fixed unframed knot type are bounded from one side. On the other hand, while the classical result in the area claims that any unframed knot type in the standard contact solid torus or 3-space has a Legendrian representative (see, e.g., [12]), the canonical projection to the plane of such a representative may have cusps.

In the present paper we are trying to make the parallel between knots and regular plane curves more explicit. We show that, in fact, the Legendrian reperesentatives can be chosen to be the lifts of regular curves. We also investigate restrictions on the Legendrian framings of such lifts. We show that there is another estimate on these framings which is often stronger than the Bennequin's inequality (cf. [9]). Our estimate comes from the HOMFLY polynomial of a knot in a solid torus. Other similar estimates provided by Legendrian lowerings of the other polynomial knot invariants to regular plane curves and plane curves with cusps will be discussed in [7].

1 Legendrian realisation

1.1 Standard contact spaces

We recall a few basic notions.

A contact element at a point of a plane is a line in the tangent plane. Its *coorientation* is a choice of one of two half-planes into which it divides the tangent plane. The manifold M of all cooriented contact elements of the plane is the spherisation $ST^*\mathbf{R}^2$ of the cotangent bundle of the plane. It is diffeomorphic to the solid torus $\mathbf{R}^2 \times S^1$: the coorienting normal vector is defined by the angle $\varphi \bmod 2\pi$ which it makes with a fixed direction on the plane. Manifold M has the standard contact structure defined as zeros of the 1-form $\alpha = (\cos\varphi)dx + (\sin\varphi)dy$, where (x, y) are coordinates on $\mathbf{R}^2$ with the positive direction of the x-axis being that fixed above (see Fig.1). We equip M with the orientation $dx \wedge dy \wedge d\varphi = -\alpha \wedge d\alpha$.

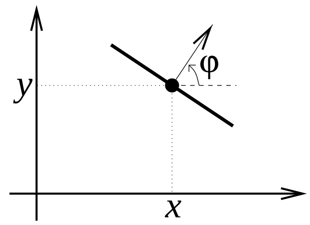

Figure 1: *Coordinates in the solid torus $ST^*\mathbf{R}^2$.*

A generic oriented curve C in $\mathbf{R}^2$ is an immersed circle whose only singularities are transverse double points. Such a curve lifts to a knot L_C in the solid torus M by setting φ to be the direction of the normal which gives a positive frame on the plane when followed by the orientation of C. The knot L_C will be called *a regular Legendrian knot*. It is everywhere tangent to the contact structure.

Along with the solid torus M we will also be considering its universal cover $\widetilde{M} \simeq \mathbf{R}^3$, with the orientation induced from that of M. Its standard contact form is given by the same formula as α with the only difference that now the angular coordinate φ is not reduced $\bmod 2\pi$. A generic closed plane curve lifts to a Legendrian knot in $\widetilde{M}$ only if its winding number (that is the number of rotations made by the coorienting vector during one complete walk along the curve) is zero.

1.2 Knots in $\mathbf{R}^3$

Theorem 1.1 *Any unframed oriented knot type in $\widetilde{M} \simeq \mathbf{R}^3$ has a regular Legendrian representative.*

Proof. We have to construct a regular Legendrian knot in $\widetilde{M}$ of a given topological type.

Let $K \subset \widetilde{M}$ be an oriented non-Legendrian knot which represents this type and is generic with respect to the canonical projection $\widetilde{p} : \widetilde{M} \to \mathbf{R}^2$. The plane curve $D = \widetilde{p}(K)$ is generic. Equipping it with the information about the over- and under-crossings we get the knot diagram $D(K)$ of K. We are going to make minor corrections of D to obtain a curve whose Legendrian lifting to $\widetilde{M}$ is topologically equivalent to K.

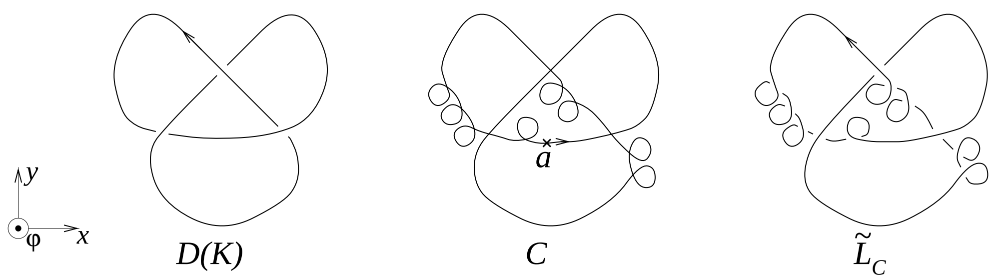

Figure 2: *A knot diagram $D(K)$ of the right-handed trefoil and its adjustment to get a generic plane curve C whose Legendrian lifting $\widetilde{L}_C$ to $\widetilde{M} \simeq \mathbf{R}^3$ is the same trefoil.*

Choose any non-double point a of D (see Fig.2). Start the lifting procedure from it sending a to any point of $\widetilde{M}$ that corresponds to the direction of the normal to D at a which agrees with our lifting orientation convention (the ambiguity of the choice is a shift by a multiple of 2π along the fibre of $\widetilde{p}$). Follow D in the direction of its orientation lifting it to $\widetilde{M}$ until nearly the first second-time visit to a double point. Here we have to bother about the type of crossing in $D(K)$: the phase $\varphi \in \mathbf{R}$ which we have gained by this moment may be forcing us to make the crossing of a wrong type. But we can easily decrease or increase the phase inserting a certain number of extra small curls (either all clockwise or all counter-clockwise) before our second-time visit and pass the double point in the right way, as prescribed by $D(K)$.

We continue our lifting trip along D in the same fashion adjusting the curve before second-time visits to double points if needed. Just before coming back to the initial point a we may also need to insert a few small curls to make the winding number of the adjusted curve zero. We end up with a regular plane curve C whose Legendrian lifting $\widetilde{L}_C$ to $\widetilde{M}$ has the topological type of K. $\square$

1.3 Knots in the solid torus

Theorem 1.2 *Any unframed oriented knot type in the solid torus $M = ST^*\mathbf{R}^2$ has a regular Legendrian representative.*

Proof. Take a generic representative $K \subset M$ of an oriented topological knot type which we have to realise by a regular Legendrian knot. Let $p : M \to \mathbf{R}^2$ be the canonical projection and $D = p(K)$. The curve D is a generic plane curve. As in the proof of the previous theorem we are going to make some changes in D so that its Legendrian lift to M would be topologically equivalent to K.

We can assume that K is transversal to the section $\varphi = 0$ of M and none of the points of the set $V = K \cap \{\varphi = 0\}$ projects to a double point of D.

Let us first make K "looking in the regular Legendrian way" around the set V. By a homotopy that fixes V and is trivial outside a small neighbourhood of V in M we can make the velocities of D at all the points of the set $p(V)$ vertically upward. Moreover, we can choose our homotopy so that the p-images of the local (around V) branches of K along which φ is increasing (respectively decreasing) are lying to the left (respectively to the right) of the above vertical velocities (Fig.3).

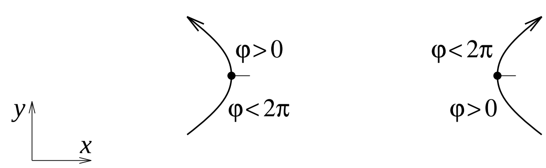

Figure 3: *Projections of increasing and decreasing branches of a knot normalised in a neighbourhood of the section $\varphi = 0$.*

Now we cut M along the section $\varphi = 0$ and represent it as the direct product $\mathbf{R}^2 \times [0, 2\pi] \subset \mathbf{R}^2_{x,y} \times \mathbf{R}_\varphi = \widetilde{M}$ (Fig.4). The knot K becomes a tangle T in $\mathbf{R}^2 \times [0, 2\pi]$. Projection $\widetilde{p} : \widetilde{M} \to \mathbf{R}^2$ sends T onto the curve D. The points of the boudary ∂T of T are glued in pairs to the become points of $p(V)$.

The pair $(D, p(V))$, with the additional information about the over- and under-crossings of the tangle T, is the tangle diagram $D(T)$ of T. The way to break D at the points of $p(V)$ to restore the boundary of the tangle is encoded in the local pictures of D shown in Fig.3.

Let us adjust D and lift the adjusted plane curve C to a Legendrian curve $\widetilde{L}_C \subset \widetilde{M}$ with boundary, such that $\widetilde{p}(\partial \widetilde{L}_C) = p(V)$ and $\widetilde{L}_C$ closes after the canonical projection to M to become a knot equivalent to K .

The adjustment is very similar to that of the previous subsection (see Fig.4).

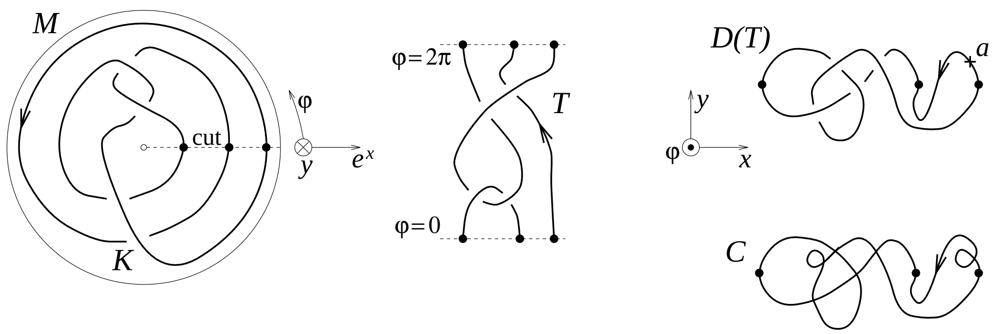

Figure 4: *Breaking a knot in the solid torus M into a tangle T and a modification of the tangle diagram $D(T)$ providing a regular curve C whose Legendrian lift to M is topologically equivalent to K.*

We start the straightforward lifting of D to $\widetilde{M}$ at an arbitrary generic point a and go in the direction of the orientation of D. The value of the coordinate $\varphi \in \mathbf{R}$ changes continuously according to the change of the direction of the normal until we arrive at a point of $p(V)$. Having arrived to such a point along an increasing branch (Fig.3), we subtract 2π from the current phase and continue our further lifting from this reduced φ. Having arrived along a decreasing branch, we add 2π to the current value of φ.

As in the proof of Theorem 1.1, just before a second-time visit to a double point of D we may have to insert some extra curls into D to guarantee the type of the crossing prescribed by the tangle diagram $D(T)$. Now we want to be a bit more accurate than in the case of $\mathbf{R}^3$: we make the absolute value of the difference between the phases of two visits to the same double point less than 2π (unnecessary extra curls, like the one the reader can find in Fig.2, are not allowed now).

On the final step we may have to insert some more curls into the adjusted D to close the Legendrian curve in $\widetilde{M}$ above a.

We end up with a modification C of the curve D and its Legendrian lift $\widetilde{L}_C \subset \widetilde{M}$ with boundary. Reduction of φ modulo 2π projects $\widetilde{L}_C$ onto the closed regular Legendrian curve $L_C \in M$. We claim that L_C is a knot topologically equivalent to K.

Indeed, the condition on the difference of the phases at a double point guarantees that L_C is an embedded curve. Moreover, the same condition implies that there exists a smooth function f, such that the curve $\widetilde{L}_C$ is lying in the slice $f(x,y) \leq \varphi \leq f(x,y) + 2\pi$ of $\mathbf{R}^2_{x,y} \times \mathbf{R}_\varphi$ with only the boundary $\partial\widetilde{L}_C$ being on the boundary of the slice.

Homotop the above slice to $\mathbf{R}^2 \times [0, 2\pi]$ along fibres of $\widetilde{p}$ putting the function $(1-t)f$ instead of f into the inequalities. This homotopy sends $\widetilde{L}_C$ to a tangle whose boundary and topological type coincide with those of T. Our homotopy lowers to a family of diffeomorphisms of the solid torus M which therefore sends the knot L_C to a knot equivalent to K. $\Box$

Remark 1.3 The link versions of Theorems 1.1 and 1.2 are also valid. In the case of links in $\mathbf{R}^3$ one has to be slightly patient: starting lifting a component of a link diagram one has to make it clear to which particular φ-level in $\widetilde{M}$ this point is lifted. This equips a starting point on each component of the curve collection C with a real number.

Remark 1.4 Theorem 1.2 was proved simultaneously and independently by A. Shumakovich. The method he used is not very much different from ours.

2 Framed knots

2.1 The Bennequin-Tabachnikov number

Legendrian knots in a 3-manifold with a cooriented contact structure are canonically framed by a transversal shift in the direction of the coorientation of the structure.

For a regular Legendrian knot in the standard $\mathbf{R}^3 = \widetilde{M} = \mathbf{R}^2 \times \mathbf{R}$ this framing is exactly the framing blackboard with respect to the projection to the base $\mathbf{R}^2$. The writhe β of it is called the Bennequin number [5].

The analog of the Bennequin number for the standard solid torus $M = ST^*\mathbf{R}^2$ was defined by Tabachnikov [17]. He set it to be the intersection number of a Legendrian knot shifted in the direction of the canonical framing and a 2-film which realises homology between the unshifted knot and the multiple of the fibre over a sufficiently distant point of the plane. To calculate the Tabachnikov number, one can consider the knot diagram of the projection of the Legendrian knot from $M \simeq \mathbf{R}^2_{x,y} \times S^1_\varphi$ to the plane with polar coordinates e^x, φ. In terms of this diagram, the Tabachnikov number is the usual writhe of a knot with a generic (not necessary blackboard) framing (Fig.5).

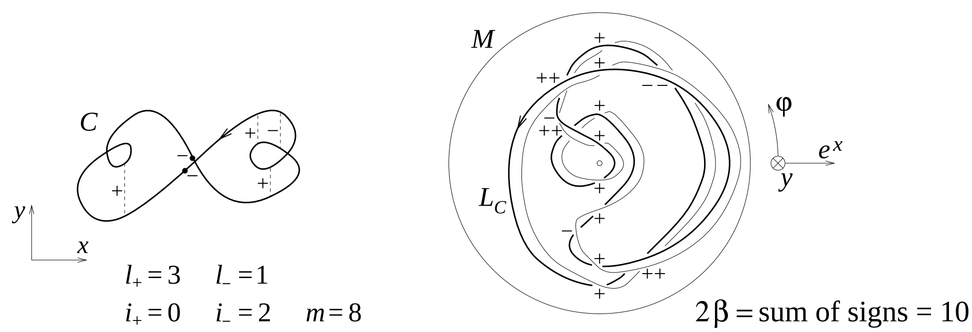

Figure 5: *Example of calculation of the Bennequin-Tabachnikov number β of a regular Legendrian knot in $ST^*\mathbf{R}^2$.*

Fig.5 illustrates the following algorithm to evaluate β on a regular Legendrian knot L_C in M.

Take a cartesian coordinate system on $\mathbf{R}^2$ generic with respect to the curve C.

There are finitely many lines $x = \text{const}$ which intersects C at two points with the velocities of the same direction (dashed lines in Fig.5). Call such a line positive (respectively negative) if the curvature κ of C at the upper of these two points is greater (respectively less) than that at the lower one (the derivative of the unit orienting vector with respect to the natural parameter on C is $-\kappa\nu$, where ν is the unit normal whose direction φ is used for the Legendrian lifting). Let ℓ_+ and ℓ_- be the numbers of all such positive and negative lines respectively.

Now consider an inflection point a of C. Assume that the phase $\varphi(a)$ is either in the 1st or in the 3rd quater. If the phase achieves its local maximum (minimum) at a, call this inflection positive (negative). Use the opposite terminology for the 2nd and 4th quaters. Let i_+ and i_- be the numbers of positive and negative inflections respectively.

Let m be the number of extrema of function y on C.

Proposition 2.1 $2\beta(L_C) = 2(\ell_+ - \ell_-) + (i_+ - i_-) + m$.

Proof. Indeed only "vertical" pairs of points with the velocities of the same direction, inflection points and extrema of function y on C contribute to the writhe of the diagram of L_C. The signs we have attached to these local events are easily seen to be their contributions to the writhe. □

Remark 2.2 a) The numbers ℓ_+ and ℓ_- can be split in an obvious way to provide all the coefficients of Arnold's and Aicardi's polynomials [4, 1].

b) Other formulas to calculate β can be found in [17] and [8].

2.2 The two invariants of unframed knots

Not any framed knot type in M or $\widetilde{M}$ can be represented by a canonically framed Legendrain knot. The Bennequin-Tabachnikov number β is bounded from one side (according to the chosen orientation) on a set of all Legendrian knots of the same unframed topological type [5]. For our choice of the orientation the number is bounded from below. Indeed, insertion of a small fragment containing two curls with opposite directions of rotation into a generic plane curve C does not affect the unframed type of the Legendrian knot $L_C \subset M$. On the other hand, this operation increases $\beta(L_C)$ by 2.

On a regular Legendrian knot β is odd [3, 4] (see also Proposition 3.5 below). To increase $\beta(L_C)$ by 1 within the same unframed knot type in M or $\widetilde{M}$ one can insert into the curve C a small non-self-intersecting fragment with two cusps and zero winding number. In the representation of the solid torus M used in Fig.5, such a fragment provides a small smooth curl with the blackboard framing of writhe 1.

Thus we arrive at two appriori different characteristics of an unframed knot in M or $\widetilde{M}$. Those are the minimal Bennequin-Tabachnikov numbers of Legendrian knots of the same unframed knot type K realised as Legendrian liftings of either regular plane curves or plane curves with cusps (the latter corresponds to arbitrary Legendrian knots). We denote them by $\beta_{min,reg}(K)$ and $\beta_{min}(K)$ respectively. Of course, $\beta_{min,reg}(K) \geq \beta_{min}(K)$.

Example 2.3 For the left-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$, β_{min} is known to be -1 (see, e.g., [15, 9]). It is easy to achieve the minimum in a regular way (Fig.6).

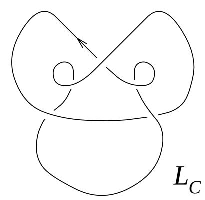

Figure 6: *A Legendrian left-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$ with the minimal Bennequin number -1.*

Example 2.4 For the right-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$, $\beta_{min} = 6$ [15, 9]. We show the corresponding extreme realisation with cusps in Fig.7. The best regular Legendrian realisation of the right-handed trefoil we know has $\beta = 9$ (Fig.7).

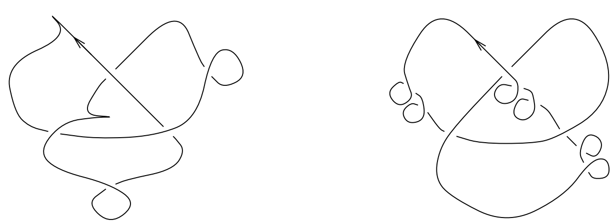

Figure 7: *A Legendrian right-handed trefoil knot in $\widetilde{M} \simeq \mathbf{R}^3$ with the minimal Bennequin number* 6 *and the minimal known example of a regular Legendrian right-handed trefoil knot with $\beta = 9$.*

Thus the number $\beta_{min,reg}(K)$ does not seem to be completely defined by only the parity argument correction of $\beta_{min}(K)$.

The main goal of the rest of the paper is to obtain an estimate on $\beta_{min,reg}(K)$ in M and $\widetilde{M}$. In fact, the lower bound we get there works for $\beta_{min}(K)$ too [7].

3 HOMFLY polynomial

3.1 Legendrian lowering of the polynomial to plane curves

In a generic 1-parameter family of regular plane curves there can appear triple points and points of self-tangency. A self-tangency can be either *direct* (the two velocity vectors have the same directions) or *inverse* (the directions are opposite). Topology of a regular Legendrian knot L_C in M or $\widetilde{M}$ can change only under direct self-tangency perestroikas of the underlying regular curve C.

We call an invariant of collections of regular plane curves a *J^+-type invariant* if it does not change under homotopies which involve no direct self-tangencies. Our terminology follows the name of the first invariant of this type introduced by Arnold in [2]. Arnold's invarinat J^+ of a one-component regular plane curve is basicly the Bennequin-Tabachnikov number of its lifting to the solid torus: in [3, 4] Arnold shows that $J^+(C) = 1 - \beta(L_C)$.

J^+-type invariants can be induced via Legendrian lifting from invariants of knots in M or $\widetilde{M}$. In [6] this approach was used to define polynomial invariants of plane fronts. Now we do the same for regular plane curves.

In [18] Turaev defined the HOMFLY polynomial of an unframed oriented link in a solid torus. This is an element of $\mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \xi_{\pm 1}, \xi_{\pm 2}, \ldots]$. Similar polynomial of a framed oriented link belongs to the same ring and is uniquely defined by the relations and initial data of Fig.8. The links L' and L'' there are mutually unlinked.

For example, on an unknot with the trivial framing $P = (x - x^{-1})/y$.

Definition 3.1 *The HOMFLY polynomial of a plane curve collection C* is that of the Legendrian link L_C in the solid torus $ST^*\mathbf{R}^2$: $P(C) = P(L_C)$.

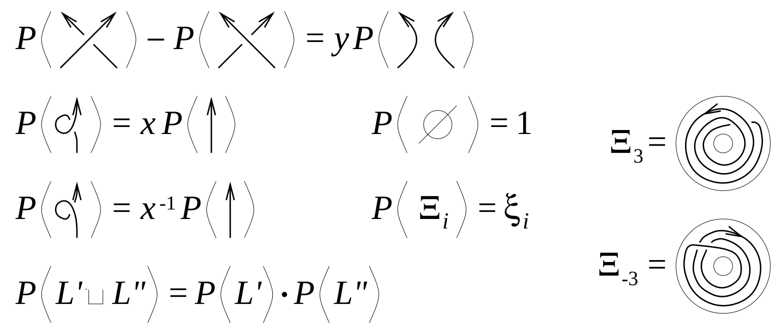

Figure 8: *Definition of the framed version of the HOMFLY polynomial for oriented links with the blackboard framing in a solid torus.*

Thus the Legendrian lifting lowers the polynomial to generic collections of plane curves. Translation of the definition of Fig.8 to that case is given in Fig.9. The collections C' and C'' of the last line are lying in disjoint half-planes. According to the second rule of Fig.8, the relation between the Legendrian generators z_i we are using now and the blackboard generators ξ_i is $z_i = x^{|i|}\xi_i$: it is easily seen that $L_{Z_i} = \Xi_i$ as unframed knots in the solid torus, and the canonical framing of L_{Z_i} differs from the blackboard one of Ξ_i by $2|i|$ positive half-twists similar to those on the vertical line through the centre of the annulus in Fig.5.

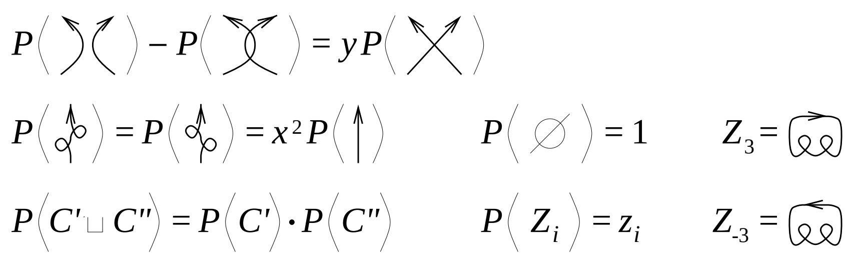

Figure 9: *Legendrian lowering of the definition of Fig.8 to generic collections of regular oriented plane curves.*

Theorem 3.2 *There exists a unique J^+–type invariant $P(C) \in \mathbf{Z}[x^2, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots]$ of a generic collection C of oriented plane curves satisfying the relations and initial data of Fig.9.*

Thus the HOMFLY polynomial of a regular plane curve turns out to be a genuine polynomial in x, not a Laurent one. Moreover, only even powers of x occur in it.

3.2 Basic curves

Example 3.3 Consider classes of framed knots represented by the knots Ξ_i of Fig.8 with the blackboard framing. Since their polynomials are $x^{-|i|} z_i$, Theorem 3.2 implies that they do not have any canonically framed regular Legendrian representatives.

Moreover, according to Theorem 3.2, the Bennequin-Tabachnikov number of the curve L_{Z_i} is the minimum of all such numbers on the set of all regular Legendrian representatives of the unframed knot type of Ξ_i. This minimal number is, thus, $2|i| - 1$. In fact, it is possible to show that $\beta_{min,reg}(\Xi_i) = \beta_{min}(\Xi_i)$ [7].

Example 3.4 Calculations of Fig.10 show that the polynomial of the figure-eight curve is $(x^2 - 1)/y$. Indeed, the curve lifts to the Legendrian unknot in M with $\beta = 1$, so its HOMFLY polynomial should be that of an unknot with the trivial framing times x.

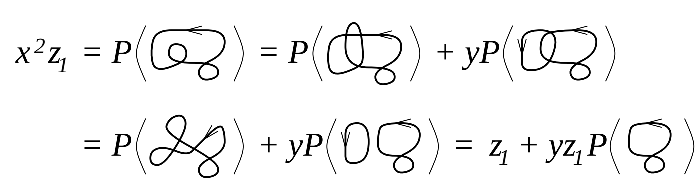

Figure 10: *Calculations of the HOMFLY polynomial of the figure-eight curve.*

In what follows, we will denote the figure-eight curve by Z_0.

The oddness of the $\beta(L_{Z_i})$ implies

Proposition 3.5 *The Bennequin-Tabachnikov number of a regular Legendrian knot in the solid torus $M = ST^*\mathbf{R}^2$ is odd.*

Proof. By the Whitney-Graustein theorem [19], any regular plane curve may be deformed by a regular homotopy to one of the curves Z_i, $i \in \mathbf{Z}$. In a generic regular homotopy, the Bennequin-Tabachnikov number of the corresponding regular Legendrian knot changes only under direct self-tangency perestroikas. Each time the change is ± 2.

Example 3.6 In some cases, it is convenient to use a different system of generators, w_i, instead of z_i. The w_i is $P(W_i)$, where W_i is the circle equipped with outer $|i|+1$ α-shaped curls and the orientation providing it with the winding number i (see Fig.11). For example, $W_0 = Z_0$ and $w_{\pm 1} = z_{\pm 1}$, since the curves $W_{\pm 1}$ can be homotoped without direct self-tangencies to the embedded circles.

The way the two systems of generators are related is shown in Fig.12. There and below we write the relations on polynomials as relations on the corresponding curves.

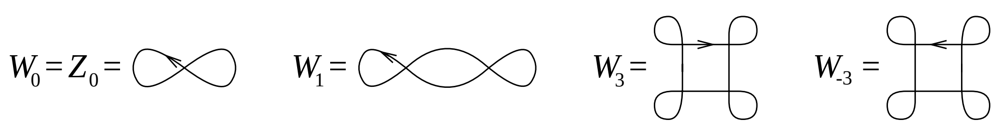

Figure 11: *The curves W_i.*

Definition 3.7 A *simple curl* of a curve collection is an α-shaped loop that contains no fragments of the collection in its interior.

Example 3.8 A figure-eight curve that has no intersection with other components of a curve collection passes through such a collection freely, with no effect on the polynomial. Indeed, the homotopy of Fig.13 does not involve any direct self-tangencies.

A circle can also pass through a line, but at the expense of a certain change in P (see Fig.14).

In general, a basic curve Z_i makes a similar pass generating many extra summands in P (see Figs. 15 and 16). The crucial point for our further

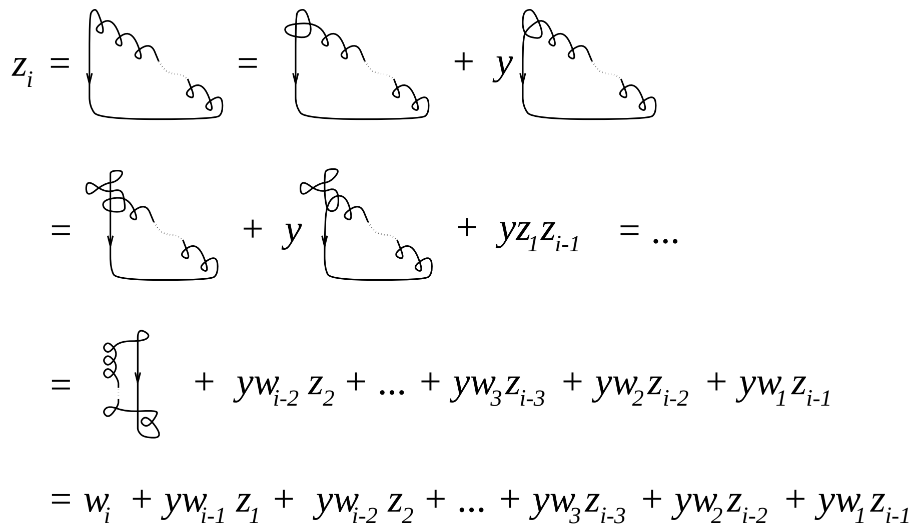

Figure 12: *Recursive relation between the generators* z_i *and* w_i*.*

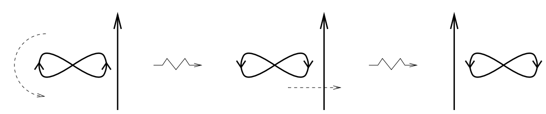

Figure 13: *The figure-eight curve as a neutrino.*

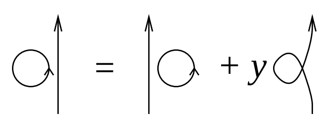

Figure 14: *A circle passes through a line.*

considerations is that all these summands can finally be expressed as the polynomials of curve collections that have nothing on that side of the line from which Z_i has been removed and have only basic curves Z_j on the other side. The involved part of the line receives only a number of additional simple curls.

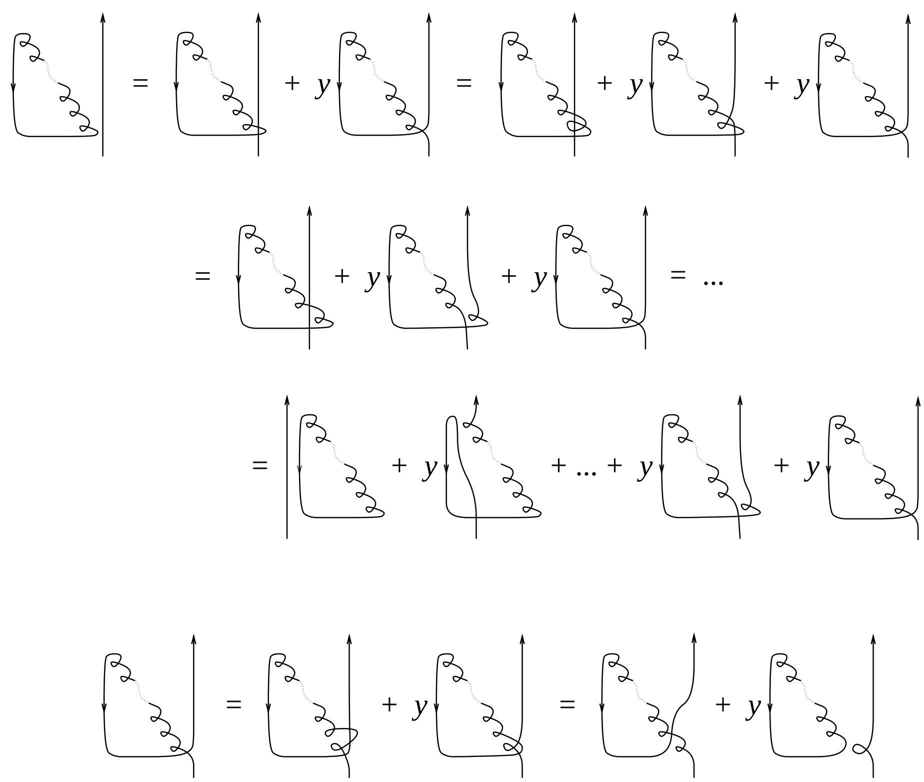

Figure 15: *Removing a basic curve from one side of a line generates a "cloud" of basic curves on the other side.*

4 Proof of Theorem 3.2

The existence of an invariant is guaranteed by [18]. We need only to show that the system of the rules of Fig.9 is sufficient to define the polynomial of

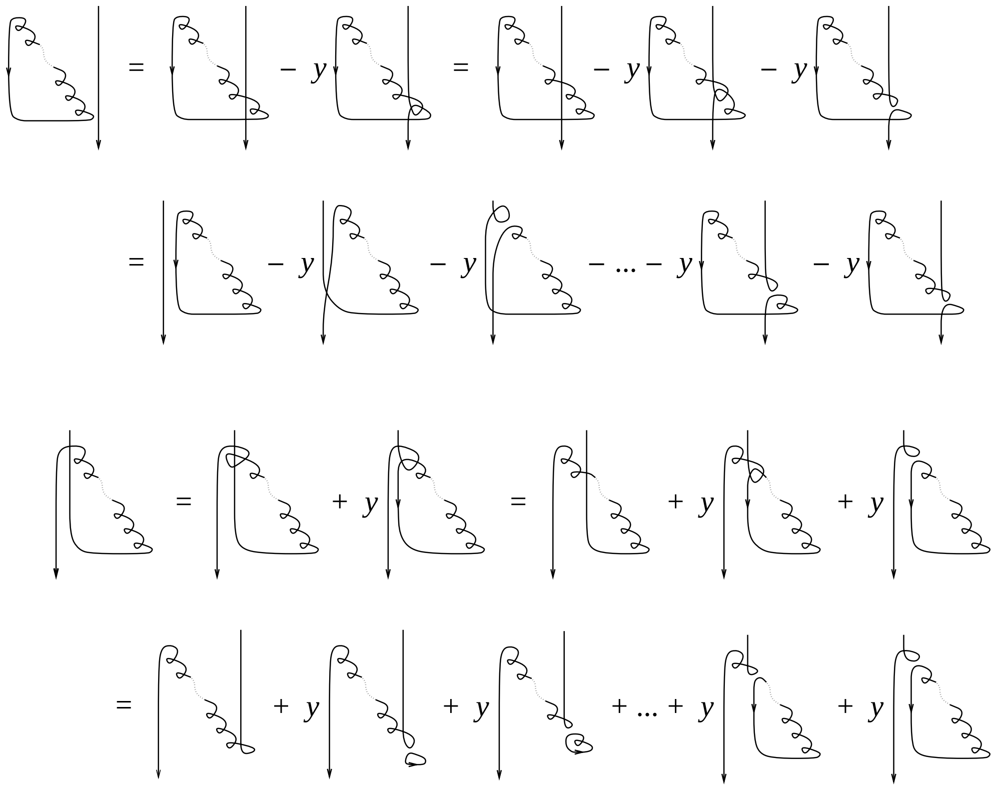

Figure 16: *Removing a basic curve in the case of the other orientation of a line.*

any curve collection uniquely. The restriction on the powers of x that are allowed to appear in the polynomials will immediately follow from the way in which the proof of the uniqueness will be carried out.

To measure a complexity of a curve collection we introduce

Definition 4.1 The double point of a simple curl is called a *simple double point.* An *essential double point* is that which is not simple.

We prove Theorem 3.2 by induction on the number of essential double points. Some ideas of [13] will be useful for us.

4.1 The base of induction

If all the double points of a curve collection are simple, the collection is basicly a nest of curves Z_i and W_j, $i, j \in \mathbf{Z}$. More precisely, calculation of the polynomial of the collection reduces to the calculation for such a nest by omitting a certain number of pairs of simple curls with opposite orientations (the move of the second line of Fig.9).

Now Example 3.6 makes a further reduction to the calculations for nests with only the curves Z_i being the innermost ones. After this, Example 3.8 reduces the depth of a nest. Iterating this procedure, we finally end up with collections of basic curves which bound disjoint domains. The polynomial of the initial nest is easily seen to contain only even non-negative powers of x.

Thus, we have proved Theorem 3.2 in absense of essential double points.

The rest of the proof shows that calculation of the polynomial of any curve collection reduces, via the rules of Fig.9, to polynomials of collections with smaller numbers of essential double points. The aim is either to make one of such points simple or to create a situation in which a self-tangency perestroika would be able to kill two double points. In the next subsection we are looking for some elementary domains within which such transformations could occur.

4.2 Search for a minimal 0- or 1-gon

Definition 4.2 An *embedded circular component* of a curve collection is called *simple* if its interior does not contain any other fragments of the collection.

Consider an arbitrary generic curve collection C. Smooth out all its simple curls. Omit all the simple circles of the result. Let C' be the final result of the two-stage operation (see Fig.17).

Definition 4.3 The collection C' is called the *essential part* of the collection C.

Definition 4.4 A closed disc D' is called an *n-gon* of the collection C' if its boundary $\partial D'$ is contained in C' and has exactly n vertices, that is double points of C' where $\partial D'$ fails to be differentiable.

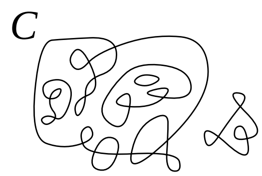

Figure 17: *A curve collection C and its essential part C'.*

Definition 4.5 A 0- or 1-gon D' of C' is called *minimal* if there are neither 0- nor 1-gons of C' inside D'.

Intersection of the interior of D' with C' may be non-empty.

Let us find such a minimal 0- or 1-gon.

Start at any generic point of C' and walk along C' until the first second-time visit to some point. Take the closed path we have traced. It bounds a closed 0- or 1-gon D'. Let us try to reduce it.

Start a similar walk as before from a point of $C' \cap D'$, but with a restriction not to leave D'. If, during this trip, we are able to make a closed loop different from $\partial D'$ it will provide us with a 0- or 1-gon contained in D'. Take it for a new, reduced disc.

If we are not able to make a closed loop, do similar try starting at another topologically different point of $C' \cap D'$.

In a finite number of steps we will not be able to make any further reduction.

It is easy to see that if we end up with a minimal 1-gon it must be α-shaped, not heart-shaped.

4.3 Reduction of the number of essential double points for minimal 0- and 1-gons of different types

We denote by D the polygon of the initial collection C from which a minimal 0- or 1-gon D' of C' is obtained by smoothing some vertices and simple curls out. Our stragtegy to simplify C inside D depends on the type of D'.

4.3.1 The boundary of a minimal 0-gon is a simple circle

This means that the disc D contains a certain number of unnested curves all of whose double points are simple. It is sufficient to assume these curves to be the Z_i. Following Example 3.8 we remove them out of D. This does not provide us with collections that would have less essential double points than C. But the essential parts of these new collections differ from C' exactly by the absense of $\partial D'$.

4.3.2 The boundary of a minimal 1-gon is a simple curl

Modulo Example 3.8 and omitting pairs of successive curls of opposite orientations, we may assume that the disc D is bounded by one of the loops of Fig.18. The same figure shows how to make the essential double point of such a loop simple so that the additional collections appearing in the skein relation would have less essential double points than C.

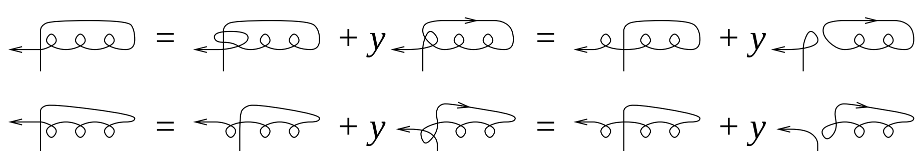

Figure 18: *Making a simple double point of the essential part of a curve simple on the curve itself.*

4.3.3 The boundary of a minimal 1-gon is a non-simple curl

This is the most complicated case.

As earlier, we may assume that all the connected components of C contained in D are basic curves Z_i and remove them out of D following Example 3.8.

We can also assume that there are no simple curls inside D as well as on its boundary. Indeed, due to the relation of Fig.19, a simple curl move through an essential double point changes the polynomial by the summand corresponding to a collection having less essential double points.

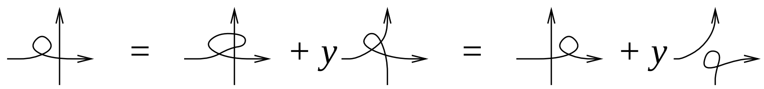

Figure 19: *A simple curl passes through an essential double point.*

By the way, altogether our assumptions mean that the reduction of C to its essential part C' makes no changes inside $D = D'$.

Now we consider several subcases which cover all possible situations. Notice that neither branch of $C \cap D$ has a self-intersection inside D.

a) There are no double points of C in the interior of D.

Then D contains a 2-gon adjacent to the boundary ∂D with no other branches of C inside it. A self-tangency move kills the 2-gon either with no effect on the polynomial (for the inverse self-tangency) or representing $H(C)$, by the main skein relation, as the combination of the polynomials of collections with the number of essential double points reduced.

b) Each pair of branches of C inside D has at most one point of intersection.

We also assume that there is no a 2-gon adjacent to ∂D similar to that killed in a).

Lemma 4.6 *The disc D contains a 3-gon Δ with exactly one of its sides on ∂D and with no fragments of C in the interior of Δ.*

Pushing the inner vertex of Δ through ∂D by the triple-point move we reduce the number of double points of C inside D. Iteration of the process finally reduces the situation to that of a).

Proof of the Lemma. Let B_1 be a branch of $C \cap D$ that intersects some other branches inside D. We may assume that the double point of ∂D and all the branches inside D which do not intersect B_1 are on the same side of B_1. This is a sort of a minimality condition on B_1.

Let $P \in B_1$ be the double point closest to an endpoint N of B_1 (see Fig.20). Let B_2 be the other branch passing through P. One of its endpoints, Q, is a vertex of a 3-gon NPQ based on ∂D. This 3-gon may be non-minimal:

there can be some other double points of C on the side PQ (due to the minimality of B_1 this is the only possible obstruction to the minimality of NPQ). Choose the one, R, closest to Q. Consider the branch B_3 through R. It cuts the corner piece QRS of NPQ. This is guaranteed by the fact that neither pair of the branches has more than 1 point of intersection.

Now, if QRS is still not minimal, we iterate the descending procedure.

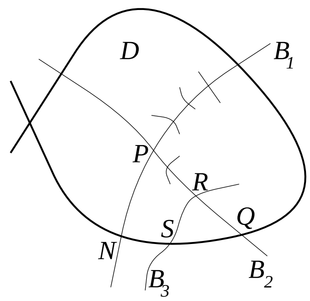

Figure 20: *Search for a minimal 3-gon.*

c) There are two branches of $C \cap D$ having at least two points of intersection inside D.

We again assume that there is no empty 2-gon adjacent to ∂D.

Let B^1 and B^2 be branches with more than 1 common point. Then there exists a 2-gon $T \subset D$ whose boundary is lying on these branches and whose vertices are two successive intersections of B^1 and B^2. We may assume the following minimality properties of T:

1) endpoints of any branch of $C \cap T$ are on the different sides of T;

2) any pair of branches of $C \cap T$ has at most one common point.

If there are any double points of C inside T, we remove them out using the triple-point moves as in b). After this we remove all the branches of C out from T by the triple-point moves across the vertices of T. Now the 2-gon T is empty and we apply a self-tangency move to kill it.

4.3.4 The boundary of a minimal 0-gon is a non-simple circle

Gathering all the simple curls of the boundary of D in a small neighbourhood of some point $P \in \partial D$, we reduce this case to the case of subsection 4.3.3

with the point P playing the role of the double point of the curl.

This finishes the proof of Theorem 3.2.

5 Other versions of the HOMFLY polynomial for regular plane curves

5.1 Regular Legendrian links in the standard 3-space

There exists an obvious analog of Theorem 3.2 that corresponds to links in $\widetilde{M} \simeq \mathbf{R}^3$.

Definition 5.1 A J^+-type invariant of generic one-component plane curves of winding number zero is called a *J_0^+-type invariant* if it changes only under direct self-tangency perestroikas in which the winding numbers of the two subcurves into which the self-tangency point breaks the curve are zero.

This corresponds to a change of the topological type of the lifted Legendrian knot in $\widetilde{M}$.

A multi-component oriented regular Legendrian link in $\widetilde{M}$ is defined by a collection of oriented plane curves in which each of the components has the winding number zero. According to Remark 1.3, on each of the components there should be a point marked by an integer number whose reduction modulo 2π is the angle φ of the corresponding normal. The markings define phases $\varphi \in \mathbf{R}$ at all the points of the collection.

Definition 5.2 A J^+-type invariant of the above marked oriented curve collections is called a *J_0^+-type invariant* if it changes only under self-tangencies in which the difference of the phases is zero.

Similar to Theorem 3.2 we have

Theorem 5.3 *There exists a unique J_0^+-type invariant $P_0(C_0) \in \mathbf{Z}[x^2, y^{\pm 1}]$ of generic collections C_0 of marked oriented plane curves of winding numbers zero satisfying the relations and the initial data of Fig.21.*

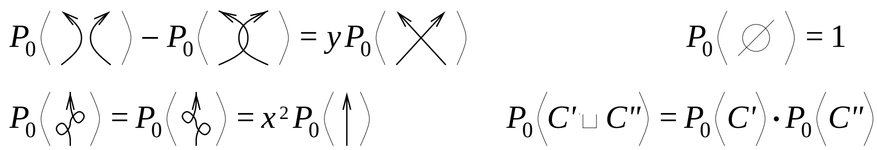

Figure 21: *Legendrian lowering of the framed version of the HOMFLY polynomial of links in $\widetilde{M} \simeq \mathbf{R}^3$ to a J_0^+-type invariant of generic collections of plane curves of winding numbers zero. The phases of the two interacting branches in the main skein relation coincide.*

5.2 The polynomials of unframed links and the Bennequin-Tabachnikov number estimates

Let β be the Bennequin or Tabachnikov number of an oriented regular Legendrian link $\widetilde{L}_{C_0} \in \widetilde{M}$ or $L_C \in M$. The traditional, unframed versions of the HOMFLY polynomials [14, 18] of these links, in terms of the underlying plane curves, are

$$P_{0,u}(C_0) = x^{-\beta} P_0(C_0) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$$

and

$$P_u(C) = x^{-\beta} P(C) \in \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, z_{\pm 1}, z_{\pm 2}, \ldots].$$

Those are topological invariants of the links.

The J^+-type invariant P_u is calculated by the rules of Fig.22. Omitting the initial data of this figure related to the curves Z_i, one gets the rules to calculate the J_0^+-type invariant $P_{0,u}$. As earlier, the systems of the rules define the plane curve polynomials inambiguously.

Theorems 3.2 and 5.3 immediately imply

Theorem 5.4 *Let $\mathcal{L}$ be an oriented unframed link in the standard contact manifold $\widetilde{M} \simeq \mathbf{R}^3$ or $M = ST^*\mathbf{R}^2$. Let x^k be the minimal power of the framing variable x in the corresponding unframed version of the HOMFLY polynomial of $\mathcal{L}$. Then*

$$\beta_{min,reg}(\mathcal{L}) \geq -k.$$

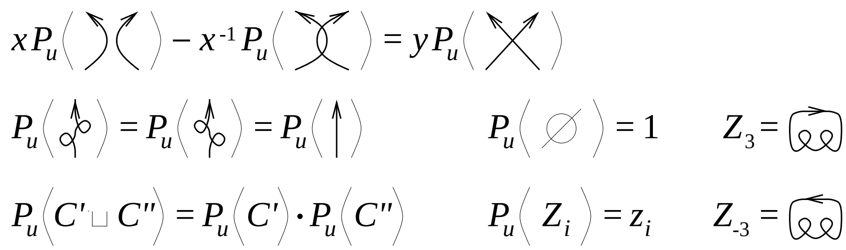

Figure 22: *Legendrian lowering of the unframed version of the HOMFLY polynomial of links in a solid torus to generic plane curve collections.*

Remark 5.5 For $\mathbf{R}^3$ this is guaranteed by the theorem of Fuchs and Tabachnikov [9] which derives the same estimate for $\beta_{min}(\mathcal{L})$ from comparison of the results of [5] and [16].

5.3 HOMFLY polynomials of curves with few double points

In Fig.23 we give the results of calculations of the polynomial P for Arnold's list [2, 3] of all the plane curves with at most 3 double points. We set there $z_0 = (x^2 - 1)/y$. The orientations of the curves with non-zero winding numbers are chosen so that these numbers are positive. The change of the orientation is covered by the following

Proposition 5.6 *Let C be a generic collection of oriented regular plane curves whose polynomial $P(C)$ is $p(x, y, z_1, z_{-1}, z_2, z_{-2}, \ldots)$. Let C^- (respectively C^r) be the collection obtained from C by the change of the orientations of all of its components (respectively by the reflection of the plane). Then*

$$P(C^-) = P(C^r) = p(x, y, z_{-1}, z_1, z_{-2}, z_2, \ldots) \ .$$

Proof. One can calculate $P(C^-)$ and $P(C^r)$ following the chain of calculations of $P(C)$. All the curves collections appearing in this chain should be respectively either eqipped with the opposite orientations or reflected. The chain for $P(C)$ ends up with disjoint collections of the curves Z_i, $i \in \mathbf{Z}$. Both the considered operations send a curve Z_i to Z_{-i}.

Corollary 5.7 $P((C^-)^r) = P(C)$.

An illustration to this is seen in the 5th line of Fig.23.

There is one more rather obvious observation which follows from the coincidence of the total winding numbers of all three curve collections participating in the main skein relation for P.

Proposition 5.8 *The sum of indices of all the z-variables appearing in a particular monomial of $P(C)$ is equal to the winding number of the curve collection C.*

The table of Fig.23 contains the Bennequin-Tabachnikov numbers of the corresponding regular knots in the solid torus [2, 3, 4]. They do not depend on the orientations of the curves and the plane.

Most of the polynomials of Fig.23 which have no obvious reason to be divisible by x^2 (those are polynomials of the curves with no pairs of simple curls of opposite orientations) are not divisible by it. Non-divisibility of the $P(C)$ by x^2 means that the Bennequin-Tabachnikov number of the knot $L_C \subset ST^*\mathbf{R}^2$ is the minimal possible among all the regular knots of the same topological type: $\beta(L_C) = \beta_{min,reg}(L_C)$.

The inverse does not seem to be true. For example, for the last curve in the 4th line, $P = x^2(\frac{x^2-1}{y} + yz_{-1}z_1)$, but there seems to exist no regular plane curve whose polynomial is that in the brackets of this formula. Another similar example is the first curve of the 5th line. Arnold's tables in [2] contain some other curves of the same nature. All of them are certain modifications of those two of Fig.23. This indicates that the estimate of Theorem 5.4 may not be exact in all the cases. Perhaps, there are some special bounds for powers of x in coefficients of various products of z-variables in the HOMFLY polynomials of regular plane curves.

The polynomials $P_0(C)$ of the curves C of Fig.23 of winding number zero are trivial: each of them is obtained from the $P(C)$ by the formal setting $y = 0$ everywhere except for the relation $z_0 = (x^2 - 1)/y$. Thus, for a table curve, $P_0(C) = x^\alpha z_0$, where $\alpha + 1$ is the Bennequin number of the corresponding Legendrian knot in $\widetilde{M}$. Of course, such a reduction is not correct in general.

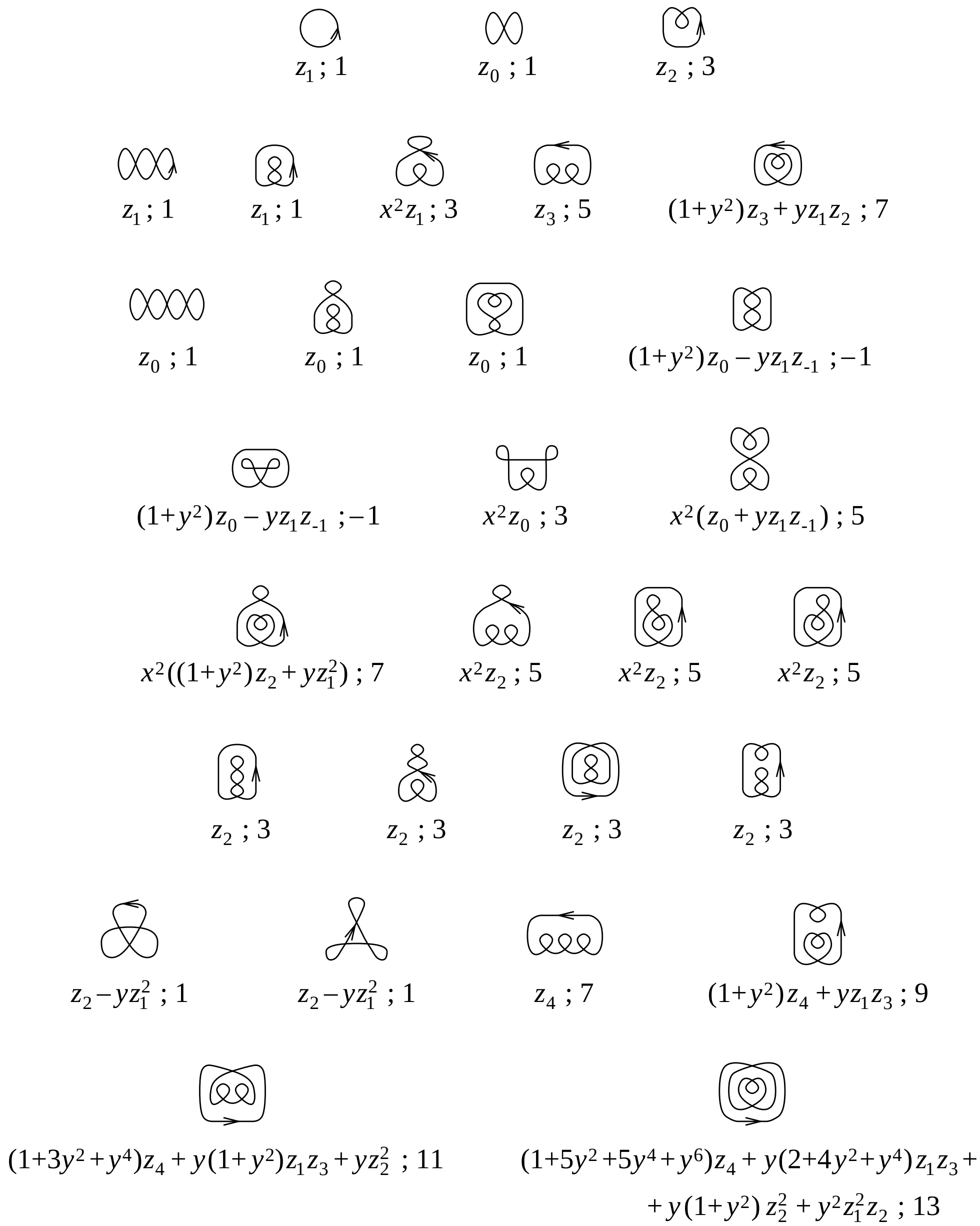

Figure 23: *The HOMFLY polynomials and Bennequin-Tabachnikov numbers of plane curves with at most 3 double points.*

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