UNITARY REFLECTION GROUPS AND AUTOMORPHISMS OF SIMPLE HYPERSURFACE SINGULARITIES

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Abstract. We list the automorphisms of simple hypersurface singularities, which allow symmetric smoothings. The monodromy group acting on the character subspaces in the homology of a symmetric smoothing is shown to be a unitary reflection group.

In paper [7], generalising Arnold's approach to boundary singularities, we studied smoothings of simple hypersurfaces invariant under a unitary reflection of finite order. The reflection splits the homology with complex coefficients of a symmetric Milnor fibre into a direct sum of the character subspaces H_χ. The monodromy in the space of hypersurfaces with the same symmetry preserves the splitting. It was observed in [7] that the monodromy on each of the H_χ is a finite group generated by unitary reflections [13]. This way unitary reflection groups, $G(m,1,k)$ and seven exceptional groups, made their first appearance in singularity theory addressing one of the long-standing questions of Arnold on realisations of the Shephard-Todd groups [1].

Later, in [8], some other Shephard-Todd groups were shown to be the monodromy groups of simple function singularities equivariant with respect to the action of finite order elements of $SU(2)$.

In the present paper we continue this programme and study arbitrary finite order automorphisms of the zero levels of simple function-germs whose action can be extended to some of the smoothings. This generalises the approaches of [7, 8] and, as it is explained by Slodowy in his paper [14] in this volume, is directly related to the classification of Springer's regular elements in Coxeter groups [15]. We show that the monodromy in the space of the symmetric smoothings is still a Shephard-Todd group. In addition to [7, 8], we obtain the group G_{10} (new in singularity theory) and two other realisations of the series $G(m,1,k)$. The relation between the two realisations of $G(m,1,k)$ is analogous to the relation between the Weyl groups B_k and C_k. The realisation of $G(m,1,k)$ obtained in [7] was of the B-type.

For the cases not considered in [7, 8], we construct distinguished sets of generators of the H_χ. We show how symmetric Dynkin diagrams of the simple functions can be folded into diagrams of the equivariant singularities. We also describe the skew-Hermitian analogues of the unitary reflection groups under consideration.

We finish the paper by giving a singularity theory interpretation of the rank 2 groups G_{12}, G_{20} and G_{22}, and showing how their Dynkin diagrams can be obtained by folding those of E_6 and E_8.

1. The unitary reflection groups

A *complex reflection* in $\mathbf{C}^k$ is a unitary transformation identical on a hyperplane, which is called the *mirror* of the reflection. The complete list of finite irreducible groups generated by complex reflections was obtained by Shephard and Todd [13]. It contains the Coxeter groups as a proper subset.

The Shephard-Todd list consists of three infinite series (Weyl groups A_k, cyclic $\mathbf{Z}_m$ and three-index $G(p,q,k)$) and 34 exceptional groups. Now we shall briefly recall the description of the groups we will deal with in detail in this paper. In our considerations a mirror is identified by its normal, which we call a *root*.

1.1. GROUPS $G(P,Q,K)$

The group $G(p,q,k)$ (all the parameters are natural numbers, q divides p, and $k \geq 2$) is a subgroup of $U(k)$. It is generated by the rotation through $2\pi q/p$ corresponding to the root u_1 (the u_i are mutually orthogonal unit coordinate vectors) and by k reflections of order 2 defined by the roots

$$u_2 - u_1\,,\ u_3 - u_2\,,\ \dots, u_k - u_{k-1} \text{ and } u_2 - e^{2\pi i/p}u_1\,.$$

For example, in two cases, when either $q = p$ or $q = 1$, just k reflections are sufficient to generate the group.

The series contains Coxeter groups:

$$G(2,2,k) = D_k\,, \qquad G(2,1,k) = B_k\,, \qquad G(p,p,2) = I_2(p)\,.$$

Information about generating reflections of a group can be represented by a graph analogous to a Dynkin diagram of a Coxeter group (cf. [4, 11]). Our conventions are as follows:

- a vertex of a graph represents a root;
- the Hermitian square of a root is written beside the vertex (square 2 is omitted);
- the order of the reflection is written at the vertex (order 2 is omitted);
- the Hermitian product (v_1, v_2) of the roots is written on an oriented edge $v_1 \to v_2$;
- there is no edge between two orthogonal roots;
- the product -1 is not written;
- orientation of an edge equipped with a real number is omitted.

Figure 1 shows graphs of the groups $G(m,1,k)$ and $G(2m,2,k)$.

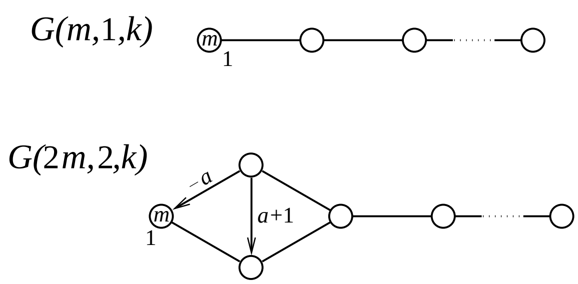

Figure 1. Graphs of the groups $G(m,1,k)$ (k vertices) and $G(2m,2,k)$ ($k+1$ vertices). Notation: $a = e^{\pi i/m}$.

Remark 1.1 It is convenient to include the groups $G(1,1,k)$ in the series. According to the above description, the group $G(1,1,k)$ is the permutation group of the coordinates in $\mathbf{C}^k$. Hence $\mathbf{C}^k$ splits into the standard representation of A_{k-1} on the hyperplane $x_1 + \ldots + x_k = 0$ and the one-dimensional trivial representation.

It is also natural to set $G(m,1,1) = G(mq,q,1) = \mathbf{Z}_m$.

The orbit space of any Shephard-Todd group is smooth. It contains the *discriminant* Σ of the group, that is, the space of its irregular orbits.

The basic invariants of $G(p,q,k)$ have degrees $p, 2p, \ldots, (k-1)p, kp/q$ (hence the order of the group is $p^k k!/q$). In particular, for $q = 1$ and $p > 1$, the degrees are proportional to those of the Weyl group B_k. Moreover, explicit consideration of the invariants easily implies that the discriminants of $G(p,1,k)$ and B_k are isomorphic.

Remark 1.2 Usually an exceptional Shephard-Todd group is denoted G_s, where s is its number in the original list in [13] (the second line in the list is occupied by a series, therefore, the Weyl group $G_2 = G(6,6,2)$ provides no confusion). Of course, such a notation is not very illustrative. For this reason, we used mainly the notation of [3] in [7]: $G(m,1,k) = B_k^{(m)}$, $G_4 =$

$A_2^{(3)}$, $G_5 = B_2^{(3,3)}$, $G_8 = A_2^{(4)}$, $G_{16} = A_2^{(5)}$, $G_{25} = A_3^{(3)}$, $G_{26} = C_3^{(3)}$ and $G_{32} = A_4^{(3)}$ (see also the table in the next section). For example, in $A_k^{(m)}$, the order 2 of the standard generators (transpositions) of the group $A_k = S_{k+1}$ is changed to m. See [7] for further details.

1.2. GROUPS G_{10} AND G_{20}

Both groups are of rank 2. The group G_{10} is generated by two reflections, one of order 3 and the other of order 4. The degrees of its basic invariants are 12 and 24, and $\Sigma(G_{10}) \simeq \Sigma(B_2)$. As an abstract group, G_{10} is a quotient group of Brieskorn's [2] braid group associated to the Weyl group B_2 (that is, generated by two elements, a and b, subject to the relation $(ab)^2 = (ba)^2$) by the relations $a^3 = id$ and $b^4 = id$. Therefore, it is natural to denote G_{10} by $B_2^{(4,3)}$.

The group G_{20} is generated by two order 3 reflections in $\mathbf{C}^2$. The degrees of the basic invariants are 12 and 30. The discriminant is isomorphic to that of $I_2(5)$, which is a 5/2 parabola. The relations defining G_{20} are $ababa = babab$ (coming from the $I_2(5)$ braids) and $a^3 = b^3 = id$. In a natural sense, $G_{20} = I_2(5)^{(3)}$.

Consider the general case of a finite rank 2 group generated by two reflections, a and b, subject to the relations $abab\ldots = baba\ldots$ (q factors on each side) and $a^r = b^s = id$. It has a graph as shown in Figure 2. One of the possible choices of the real weight w is

$$w = -\sqrt{2\frac{\cos(\frac{\pi}{r} - \frac{\pi}{s}) + \cos\frac{2\pi}{q}}{\sin\frac{\pi}{r}\sin\frac{\pi}{s}}}\,, \tag{1}$$

see [5].

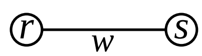

Figure 2. The standard diagram of a group generated by two reflections.

For G_{10}, another choice of w exists, with $\cos\frac{5\pi}{12}$ in (1) instead of $\cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos\frac{\pi}{12}$. The two choices correspond to two rank 2 representations of G_{10} each: the eigenvalue of a can be either $\omega = e^{2\pi i/3}$ or $\bar{\omega}$, and that of b either i or $-i$.

For G_{20}, another choice also exists: with $\cos\frac{4\pi}{5}$ instead of $\cos\frac{2\pi}{5}$ in (1). Again, the choices provide two representations each: the operators a and b must have the same eigenvalue which can be either ω or $\bar{\omega}$.

The four exact representations of each of the two groups described here will be referred to as the *standard* representations.

1.3. GROUPS G_{12} AND G_{22}

These are also rank 2 groups. Each is generated by three order 2 reflections. The degrees of the basic invariants are respectively 6 and 8, and 12 and 20. The discriminants are 4/3 and 5/3 parabolas (that is, E_6 and E_8 curves).

Figure 3 shows that the two groups form a mini-series of four with A_2 and $G(4,2,2)$ (cf. Figure 1 for the latter). The sum of the three roots in each case is zero. The diagrams of the figure can easily be obtained from those in [11].

The change of the sign of i in Figure 3 gives one more exact representation of each of the groups except for A_2. Two further exact representations of G_{22} correspond to the angle $2\pi/5$ used instead of $\pi/5$.

These two representations of G_{12} and four of G_{22} will be also referred to as their *standard* representations.

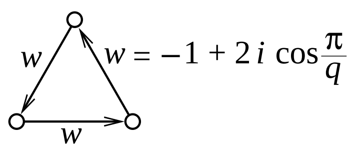

Figure 3. The diagrams of A_2, $G(4,2,2)$, G_{12} and G_{22}, provided q is 2, 3, 4 or 5 respectively.

2. Automorphisms of simple singularities

2.1. MILNOR REGULAR AUTOMORPHISMS

Let f be a holomorphic function-germ on $(\mathbf{C}^{r+1},0)$. Consider a diffeomorphism-germ g of $(\mathbf{C}^{r+1},0)$ sending the hypersurface $f=0$ into itself. It multiplies the function f by a certain function c not vanishing at the origin. In what follows we assume g has finite order, hence c is just a constant, a root of unity.

Consider now the space $\mathcal{O}(g,c)$ of all holomorphic function-germs on $(\mathbf{C}^{r+1},0)$ multiplied by c under the action of g. The group $\mathcal{R}_g$ of biholomorphism-germs of $(\mathbf{C}^{r+1},0)$ commuting with g acts on $\mathcal{O}(g,c)$. The equivalence relation induced on $\mathcal{O}(g,c)$ is a nice geometric equivalence in the sense of Damon [6]. Therefore, the standard theorem on versal deformations holds for it. In particular, the base of an $\mathcal{R}_g$-miniversal deformation of a function in $\mathcal{O}(g,c)$ is smooth.

Definition 2.1 (cf. [14]) An automorphism g of a hypersurface $f = 0$ is *Milnor regular* if an $\mathcal{R}_g$-versal deformation of function f contains members with smooth zero sets.

This implies that either f is g-invariant (hence $c = 1$ and an $\mathcal{R}_g$-miniversal deformation contains the free term) or c is an eigenvalue of the operator dual to g (hence an $\mathcal{R}_g$-miniversal deformation contains a linear form). The latter condition is not sufficient for regularity.

Notice that an $\mathcal{R}_g$-miniversal deformation F_g of a function $f \in \mathcal{O}(g, c)$ can be extended to an ordinary $\mathcal{R}$-miniversal deformation F of f. To get F_g back from F one has to take only those members of F which are in $\mathcal{O}(g, c)$. Similarly, the *equivariant discriminant* $\Sigma_g(f)$ of f, that is, the set of those members of the family F_g whose zero-sets are singular, can be viewed as a section of the ordinary discriminant $\Sigma(f)$ of the family F.

Milnor regularity implies that the zero set V of a generic member of an $\mathcal{R}_g$-versal deformation is a Milnor fibre of f and allows one to consider the intersection form on the middle homology $H_r(V)$ without any complications. The automorphism g induces an automorphism of $H_r(V)$, which we also denote by g, slightly abusing the notations. The space $H_r(V; \mathbf{C})$ splits into a direct sum of the eigenspaces H_χ of g (the χ are the corresponding eigenvalues).

2.2. THE LIST OF AUTOMORPHISMS

Now assume f is a simple function-germ on $(\mathbf{C}^{r+1}, 0)$. We shall be more interested in the splitting $H_r = \oplus_\chi H_\chi$ of its vanishing homology by an automorphism g than in the element g itself. Therefore, it is sufficient for our purposes to consider only the case $r + 1 = \operatorname{corank} f$ and then extend the automorphisms to the stably equivalent functions in any consistent way. Recall that, in this minimal dimension and with f in its quasi-homogeneous normal form, the reductive automorphism group $Aut(f)$ of $f = 0$ is just the group $\mathbf{C} \setminus \{0\}$ of quasi-homogeneous dilations of $\mathbf{C}^{r+1}$ if f is of the type A_n, D_{odd} or E_n. For the other singularities, the group $Aut(f)$ is a direct product of the group of the quasi-homogeneous dilations and either S_3 (for D_4) or $\mathbf{Z}_2$ (for all the other D_{even}).

A case-by-case analysis (which is too straightforward, lengthy and boring to write about) provides the following result.

Theorem 2.2 *The complete list of Milnor regular automorphisms of simple hypersurface singularities in $\mathbf{C}^3$ is given in the two tables below.*

The completeness here is in the following sense. One starts with $\mathbf{C}^{\operatorname{corank} f}$ and the automorphisms considered up to the groups they generate. In this setting, a function and an automorphism are brought to a simultaneous normal form (f', g') by a choice of coordinates. After this, the function is stabilised to three variables and the automorphism extended to provide a table pair (f, g). Any other, stabilisation and extension of (f', g') results in absolutely the same decomposition of the homology of a Milnor fibre into the sum of character subspaces as does (f, g). A priori, the character assignments to individual subspaces may change at this point. However, this does not affect the monodromy on the subspaces, which will be our main concern.

In the table, in each of the cases we write out monomials $e_1, \ldots, e_\tau$ spanning an equivariant miniversal deformation (here τ is the *equivariant Tjurina number*). We list all the characters for which $\dim H_\chi = \tau$. A kth root of unity is denoted by ϵ_k if it is completely arbitrary, and by ε_k if it is arbitrary primitive.

For the characters listed, the table gives the monodromy representations on the H_χ (the order of χ is denoted o_χ). If $\tau = 1$, we give the eigenvalue of the only Picard-Lefschetz operator on H_χ. The equivariant monodromy is studied in more detail in the next section.

We keep the notation of [13] for the unitary reflection groups. The notation used for $\tau > 1$ invariant functions is that from [7]. The notation for the equivariant singularities studied in [8] is slightly altered.

The singularity D_4 occurs in the table in two normal forms. One of them is more convenient for expressing automorphisms involving order 3 elements of S_3 and the other for those involving the transpositions.

In all the series, $\tau = k \geq 1$.

Remark 2.3 One of the immediate observations the table provides, as an extension of properties of Arnold's simple boundary singularities, is as follows.

Let $D(f)$ be the set of weights of parameters of a quasi-homogeneous $\mathcal{R}$-miniversal deformation F of a simple singularity f. Assume that the weights are normalised so that the largest of them is the Coxeter number of the related Weyl group X (then, of course, $D(f)$ coincides with the set $D(X)$ of degrees of basic invariants of X).

Now, for an automorphism g of $f = 0$, denote by $D(f, g) \subset D(f)$ the set of weights of the parameters of the g-equivariant miniversal deformation F_g of f. Let X_g be the unitary reflection group which is the monodromy group of a character subspace H_χ, where χ is of the maximal order, $X_g \subset X$. In all the table cases, $D(f, g) = D(X_g)$ (for all other χ, such that $\mathrm{rk}\, H_\chi = \tau$, the set $D(f, g)$ is a multiple of the set of the degrees of the monodromy group) and hence $D(X_g) \subset D(X)$. Moreover, the discriminants of the function and of the group, $\Sigma_g(f)$ and $\Sigma(X_g)$, are diffeomorphic.

See [14] for an explanation of these facts.

function	$g: x, y, z \mapsto$	$\vert g\vert$	$\{e_i\}$	χ	monodromy on H_χ	notation
A_n : x^{n+1} $+yz$	$e^{2\pi i/m}x,\ y,\ z$ $n+1=km$	m	$1, x^m, x^{2m}, \dots, x^{(k-1)m}$	$\epsilon_m \neq 1$	$G(o_\chi, 1, k)$; $\langle\chi\rangle$ if $k=1$	$B_k^{(m)}$
	$e^{2\pi i/m}x$, $e^{2\pi i/m}y,\ z$ $n=km$	m	$x, x^{m+1}, x^{2m+1}, \dots, x^{(k-1)m+1}$	ϵ_m	$G(o_\chi, 1, k)$; $\langle\chi\rangle$ if $k=1$	$A_{km}/\mathbf{Z}_m$
D_4 : x^3+y^3 $+z^2$	$\omega x,\ y,\ z$	3	$1, y$	ε_3	G_4	$A_2^{(3)}$
	$\omega x,\ \omega^2 y,\ z$	3	$1, xy$	1	G_2	G_2
	$\omega x,\ \omega y,\ z$	3	1	ε_3	$\mathbf{Z}_6 = \langle -\chi\rangle$	$D_4\vert\mathbf{Z}_3$
	$-x,\ -\omega y,\ iz$	12	x	ε_{12}	$\mathbf{Z}_4 = \langle\chi^3\rangle$	$D_4/\mathbf{Z}_{12}$
D_n, $n \geq 4$: x^2y $+y^{n-1}$ $+z^2$	$e^{-2\pi i/2m}x$, $e^{2\pi i/m}y,\ z$, $n-1=km$	$2m$	$1, y^m, y^{2m}, \dots, y^{(k-1)m}$	$\chi^m=-1$, rk $H_1 = 1$	$G(o_\chi, 1, k)$; $\langle -\chi^{1-m}\rangle$ if $k=1$	$D_{km+1}\vert\mathbf{Z}_{2m}$; C_k if $m=1$
	$e^{2\pi i/m}x$, $e^{-2\pi i/m}y$, $e^{2\pi i/2m}z$, $n=km$	$2m$	$x, y^{m-1}, y^{2m-1}, \dots, y^{(k-1)m-1}$	$\chi^m=-1$	$G(2o_{\chi^2}, 2, k)$; $\langle\chi^{-2}\rangle$ if $k=1$	$D_{km}/\mathbf{Z}_{2m}$

3. Monodromy on the character subspaces

Now we shall demonstrate that the monodromy on the character subspaces for the automorphisms listed is that given in the table. We proceed case-by-case for the automorphisms not considered in [7, 8].

3.1. CODIMENSION ONE SINGULARITIES

The characters for the table singularities with $\tau = 1$ can easily be calculated from the cohomological point of view, using the residue-forms. The monodromy eigenvalues are obtained from weighted-homogeneous considerations. We do this in detail for one of the automorphisms, for illustration.

function	$g: x, y, z \ \mapsto$	$\lvert g\rvert$	$\{e_i\}$	χ	monodromy on H_χ	notation
E_6 : x^3+y^4 $+z^2$	$x,\ -y,\ z$	2	$1, x, y^2, xy^2$	-1	$F_4 = G_{28}$	F_4
	$\omega x,\ y,\ z$	3	$1, y, y^2$	ε_3	G_{25}	$A_3^{(3)}$
	$x,\ iy,\ z$	4	$1, x$	$\pm i, -1$	G_8 on $H_{\pm i}$; A_2 on H_{-1}	$A_2^{(4)}$
	$\omega x,\ -y,\ z$	6	$1, y^2$	ε_6	G_5	$B_2^{(3,3)}$
	$\omega x,\ iy,\ z$	12	1	$\varepsilon_{12}, -\varepsilon_3$	$\langle -\chi \rangle$	$E_6 \lvert \mathbf{Z}_{12}$
	$-x,\ e^{2\pi i/8}y,\ iz$	8	x	$\varepsilon_8, \pm 1$	$\langle \chi^3 \rangle$	$E_6/\mathbf{Z}_8$
	$e^{2\pi i/9}x,\ \omega y,$ $-\omega^2 z$	18	y	ε_{18}	$\mathbf{Z}_9 = \langle \chi^4 \rangle$	$E_6/\mathbf{Z}_{18}$
E_7 : x^3+xy^3 $+z^2$	$x,\ \omega y,\ z$	3	$1, x, y^3$	ε_3	G_{26}	$C_3^{(3)}$
	$\omega x,\ e^{4\pi i/9}y,\ z$	9	1	$\varepsilon_9, 1$	$\langle -\chi \rangle$	$E_7 \lvert \mathbf{Z}_9$
	$e^{6\pi i/7}x,\ e^{4\pi i/7}y,$ $e^{2\pi i/7}z$	7	y	ϵ_7	$\langle -\chi \rangle$	$E_7/\mathbf{Z}_7$
E_8 : x^3+y^5 $+z^2$	$\omega x,\ y,\ z$	3	$1, y, y^2, y^3$	ε_3	G_{32}	$A_4^{(3)}$
	$x,\ e^{2\pi i/5}y,\ z$	5	$1, x$	ε_5	G_{16}	$A_2^{(5)}$
	$\omega x,\ e^{2\pi i/5}y,\ z$	15	1	ε_{15}	$\mathbf{Z}_{30} = \langle -\chi \rangle$	$E_8 \lvert \mathbf{Z}_{15}$
	$-x,\ -y,\ iz$	4	y, x, y^3, xy^2	$\pm i$	G_{31}	$E_8/\mathbf{Z}_4$
	$-ix,\ iy,\ e^{2\pi i/8}z$	8	y, xy^2	ε_8	G_9	$E_8/\mathbf{Z}_8$
	$-\omega x,\ -y,\ iz$	12	y, y^3	ε_{12}	G_{10}	$B_2^{(4,3)}$
	$-x,\ e^{3\pi i/5}y,\ -iz$	20	x	ε_{20}	$\mathbf{Z}_{20} = \langle \chi \rangle$	$E_8/\mathbf{Z}_{20}$

Example 3.1 $E_6/\mathbf{Z}_8$. The equivariant miniversal deformation is

$$F(x,y,z;\alpha) = x^3 + y^4 + z^2 + \alpha x\,.$$

Set $f_\alpha = F|_{\alpha=\text{const}}$ and $V_\alpha = \{f_\alpha = 0\} \subset \mathbf{C}^3$. Assume $\alpha \neq 0$. Consider the monomial residue-forms generating $H_2(V_\alpha;\mathbf{C})$:

$$\varphi\, dxdydz/df_\alpha\,, \quad \varphi = 1, y, y^2, x, xy, xy^2\,.$$

Put $\delta = e^{2\pi i/8}$. Our automorphism $g : (x,y,z) \mapsto (\delta^4 x, \delta y, \delta^2 z)$ multiplies the above forms respectively by $\delta^3, \delta^4, \delta^5, \delta^7, 1, \delta$. These are the characters.

The monodromy is defined by the homotopy $\alpha \cdot e^{2\pi i t}$, $t \in [0,1]$, raised in the weighted-homogeneous way to $\mathbf{C}^4_{x,y,z,\alpha}$. This ends up with the transformation

$$h : (x,y,z) \mapsto (\delta^4 x, \delta^3 y, \delta^6 z) = g^3(x,y,z)$$

of 3-space. Hence h has eigenvalue χ^3 on H_χ.

3.2. SINGULARITIES OF HIGHER CODIMENSION

For each automorphism we shall construct distinguished sets of generators of the H_χ. After this we shall show that the corresponding Picard-Lefschetz operators do generate the unitary reflection group claimed.

We start the construction in the traditional way, fixing a generic point $*$ in the base $\mathbf{C}^\tau$ of an equivariant miniversal deformation of the function f. Let $V \subset \mathbf{C}^{r+1}$ be the smoothing corresponding to $*$.

Denote by m the order of our Milnor regular automorphism g. Let $V' \subset V$ be the subset of all points with non-trivial stationary subgroups under the action of $\mathbf{Z}_m = \langle g \rangle$. In all our cases, V' has positive codimension in V, not necessarily 1. Consider the quotient varieties, $W = V/\mathbf{Z}_m$ and $W' = V'/\mathbf{Z}_m$, and the *integer* relative homology group $H_r(W,W';\mathbf{Z})$.

Let $\Sigma_g \subset \mathbf{C}^\tau$ be the discriminant of our equivariant singularity. Consider a generic line $\ell \subset \mathbf{C}^\tau$ passing through the point $*$, and a set of paths $\{\gamma_j\}$ in ℓ from $*$ to all the points of $\ell \cap \Sigma_g$. As usual, we assume that the paths have no mutual- or self-intersections, except for the common starting point.

In all the table cases, except for $D_4|\mathbf{Z}_3$, it turns out that exactly one relative cycle is contracted by the homotopy of (W,W') corresponding to motion along the path γ_j to its end-point. Moreover, the set of cycles vanishing along all the paths γ_j generates the torsion-free $H_r(W,W';\mathbf{Z})$. This is what we call a *distinguished set of generators* of this homology group. Notice that, for all our table singularities with $\tau > 1$, the rank of $H_r(W,W';\mathbf{Z})$ is τ.

Let e be one of the vanishing cycles on (W,W') just obtained. Since g acts freely on $V \setminus V'$, it has m independent preimages $e_0,\ldots,e_{m-1}$ in

$H_r(V,V';\mathbf{Z})$. Assume that they are ordered so that the automorphism g permutes them cyclically:

$$g : e_0 \mapsto e_1 \mapsto e_2 \mapsto \ldots \mapsto e_{m-1} \mapsto e_0 \,.$$

For each mth root χ of unity, the cycle

$$e_\chi = \sum_{s=0}^{m-1} \chi^{-s} e_s \tag{2}$$

is in the χ-eigenspace $H_{r,\chi}(V,V',\mathbf{C})$ of the relative action of g.

The action of g splits the exact sequence of the pair into the exact sequences of the character subspaces:

$$0 \to H_\chi = H_{r,\chi}(V;\mathbf{C}) \to H_{r,\chi}(V,V';\mathbf{C}) \to H_{r-1,\chi}(V';\mathbf{C}) \to \ldots \,.$$

The rank of the relative homology group here does not depend on the choice of χ, this is just the rank of $H_r(W,W';\mathbf{Z})$. On the other hand, $H_{r-1,\chi}(V',\mathbf{C}) = 0$ if χ is a primitive mth root of unity (more generally, if $\chi^d \neq 1$, where $\mathbf{Z}_{m/d}$ is a stationary subgroup of a codimension 1 subset of points in V, $d \neq m$). Therefore, if $\chi = \varepsilon_m$ then the rank of H_χ is maximal possible, τ, for all our table singularities (except for $D_4|\mathbf{Z}_3$ when it is maximal, $2 = \tau + 1$, for $\chi = 1$). In fact, the primitive character subspaces provide the most interesting cases since the monodromy on them projects to those on the other character subspaces.

Definition 3.2 The element e_χ in (2) is called a *vanishing χ-cycle on V* if it belongs to the kernel of the boundary operator $H_{r,\chi}(V,V';\mathbf{C}) \to H_{r-1,\chi}(V';\mathbf{C})$. The set of vanishing χ-cycles corresponding to the distinguished set of generators of $H_r(W,W';\mathbf{Z})$ is called *distinguished.*

Of course, χ-cycles in a distinguished set are naturally ordered by the counter-clockwise order in which the corresponding paths in ℓ leave the base point. However, in all our cases the equivariant Dynkin diagrams are just straight chains and therefore an arbitrary order can be achieved (see [10]).

Remark 3.3 The above is in fact a generalisation of Arnold's construction used for boundary function singularities. The first step towards this generalisation was taken in [7]. The distinguished generating sets of vanishing χ-cycles obtained in [8] can also be produced by the algorithm described here.

The $\tau > 1$ singularities which did not appear in [7, 8] are $A_{km}/\mathbf{Z}_m$, $D_{km+1}|\mathbf{Z}_m$ and $B_2^{(4,3)}$. We deal with them in turn. Each of them is considered in two particular dimensions, one odd and one even. The intersection forms are Hermitian and skew-Hermitian respectively. Stabilisation by adding a generic quadratic form in two new variables to a function (with the automorphism action appropriately extended) changes the sign of the intersection form [9] but does not affect the monodromy group.

Diagrams of the equivariant functions will be obtained from Dynkin diagrams of the ordinary singularities in $\mathbf{C}^{r+1}$. The conventions to draw the latter are standard:

- the square of each root is $(-1)^{(r+1)(r+2)/2}\Big((-1)^r+1\Big)$;
- in the skew-symmetric case, an edge $a \to b$ means $(a,b)=1$;
- in the symmetric case, a dashed edge is drawn for the intersection number $(-1)^{r/2}$ and a solid edge for the negative of this.

3.2.1. $A_{km}/\mathbf{Z}_m$

For $m=3$, this singularity was studied in [8]. Now we consider arbitrary m.

Hermitian case. Take the singularity in just one variable: $f(x)=x^{km+1}$, with $g(x)=e^{2\pi i/m}x$. The family

$$x^{km+1}+\lambda_k x^{(k-1)m+1}+\ldots+\lambda_{k-1}x^{m+1}+\lambda_k x = xp_\lambda(x^m) \qquad (3)$$

is its equivariant miniversal deformation. The discriminant of the family is that of the Weyl group B_k corresponding to the polynomial p_λ having either zero or multiple roots.

We take a function $f_*(x)=xp_*(x^m)$, such that p_* has all its roots real positive and simple, for a marked generic member of (3). Consider the zero-set of f_*:

$$V=\{0, e^{2\pi i s/m}a_j,\ s=0,\ldots,m-1,\ j=1,\ldots,k\}\subset \mathbf{C}\,.$$

We order the real numbers a_j so that $0<a_1<a_2<\ldots<a_k$.

A distinguished basis of vanishing χ-cycles on V can be obtained by varying the free term of p_*. For example, such a basis can be taken to consist of

(i) a *short* χ*-cycle* $\sum_{s=0}^{m-1}\chi^{-s}[e^{2\pi i s/m}a_1]$ vanishing at the origin (for $\chi=1$, the term $m[0]$ must be subtracted from this sum);

(ii) $k-1$ *long* χ*-cycles* $\sum_{s=0}^{m-1}\chi^{-s}([e^{2\pi i s/m}a_j]-[e^{2\pi i s/m}a_{j-1}])$, $j=2,\ldots,k$.

The self-intersection number of a long χ-cycle is $2m$, and of a short χ-cycle either m^2+m if $\chi=1$ or m otherwise.

The Dynkin diagram of the chosen basis is drawn top left in Figure 4. The upper half of the figure also shows how this diagram can be obtained by m-folding of the symmetric Dynkin diagram of the ordinary singularity A_{km}. According to the expressions in (i) and (ii), each of the k character cycles is a linear combination of m ordinary vanishing cycles situated on one of the concentric circles of the A_{km} diagram (these are orbits of the Z_m-action). In the linear combination, the crossed cycle is taken with the coefficient 1, the cycle counter-clockwise next to it with the coefficient χ^{-1}, the next one with the coefficient χ^{-2}, and so on.

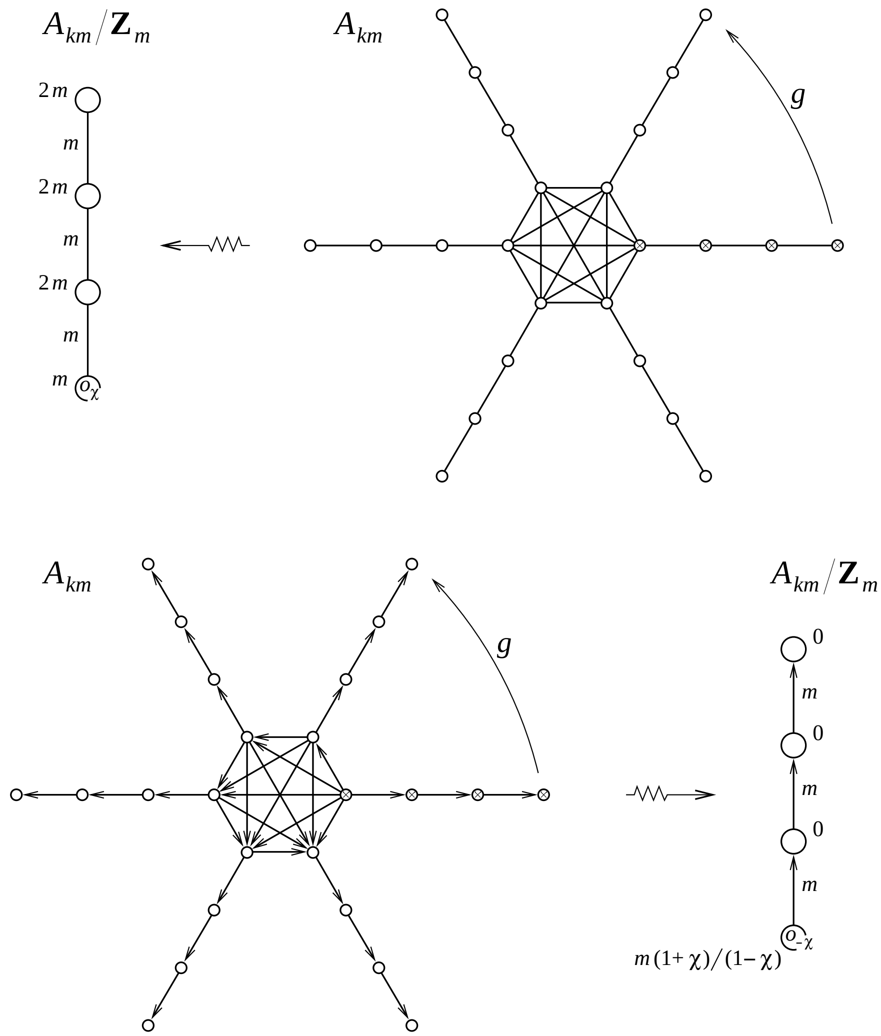

Figure 4. Folding symmetric Dynkin diagrams of A_{km} to the character diagrams of $A_{km}/\mathbf{Z}_m$, $\chi \neq 1$, $k = 4$ and $m = 6$. The Hermitian diagrams, of the one-variable singularities, are in the upper half of the figure. The skew-Hermitian diagrams, of the two-variable functions, are in the lower half.

Skew-Hermitian case. Addition of t^2 to the family (3) provides a miniversal equivariant deformation of the function $x^{km+1} + t^2$ associated with the

Milnor regular automorphism $g : (x,t) \mapsto (e^{2\pi i/m}x, e^{2\pi i/2m}t)$ of order $2m$. Now $\chi^m = -1$. Similar to the Hermitian version, one can obtain a Dynkin diagram of the equivariant function by m-folding of that of the ordinary singularity as shown in the lower half of Figure 4.

Picard-Lefschetz operators. One of the possible versions of the $A_{km}/\mathbf{Z}_m$ singularity in $\mathbf{C}^{r+1}$ is provided by the addition of the quadratic form $\sum_{j=1}^{[r/2]} y_j z_j$ to the above one- or two-variable function, with the automorphism g multiplying all the y_j by $e^{2\pi i/m}$ and acting trivially on the z_j. The character subspaces in the homology are H_χ, $\chi^m = (-1)^r$. The intersection form on H_χ is the one we had in the one- or two-dimensional case multiplied by $(-1)^{[r/2]}$.

The Picard-Lefschetz operator on H_χ corresponding to the short χ-cycle e has the eigenvalue $(-1)^r\chi$ and acts by the formula

$$h_e : \quad c \mapsto c + (-1)^{(r+1)(r+2)/2}(1-\chi)(c,e)e/m\,. \tag{4}$$

The monodromy operator on H_χ, associated with a long χ-cycle e, is induced by the product of the m commuting Picard-Lefschetz operators of the ordinary singularity corresponding to the vertices on the related circle of the symmetric A_{km} diagram. The operator is

$$h_e : \quad c \mapsto c + (-1)^{(r+1)(r+2)/2}(c,e)e/m\,. \tag{5}$$

Clearly, for $r = 0$, the unitary reflections (4) and (5), associated with the vertices of the top left diagram of Figure 4, do generate the Shephard-Todd group $G(o_\chi, 1, k)$ (see Remark 1.1 for the $\chi = 1$ and $k = 1$ cases).

3.2.2. $D_{km+1}|\mathbf{Z}_{2m}$

Skew-Hermitian case. We start with the two-variable singularity. It has the invariant miniversal deformation

$$x^2y + y^{km} + \lambda_k y^{(k-1)m} + \ldots + \lambda_2 y^m + \lambda_1 = x^2 y + p_\lambda(y^m)\,, \tag{6}$$

with respect to the automorphism $g(x,y) = (e^{-2\pi i/2m}x, e^{2\pi i/m}y)$ of order $2m$.

The quotient W of an invariant Milnor fibre V under the action of $\mathbf{Z}_{2m}$ is diffeomorphic to a coordinate line $\mathbf{C}_v$, $v = y^m$, with a puncture at the origin. The subset $W' \subset W$ of irregular orbits consists of m points, roots of the equation $p_\lambda(v) = 0$. Hence, the discriminant of (6) is also isomorphic to the discriminant of B_k.

Similar to what was done for the previous singularity, we mark a point $*$ in the base of deformation (6), so that the polynomial $p_*(v)$ has all its roots real positive and simple. This choice easily provides the Dynkin diagram for a character subspace $H_\chi \subset H_1(V;\mathbf{C})$, $\chi^m = -1$, which is the top left diagram of Figure 5. This diagram can also be obtained by m-folding of a symmetric Dynkin diagram of the ordinary D_{km+1} singularity as shown in the upper half of the figure. For a better symmetry, we introduced there an auxiliary cycle (drawn fine, not participating in the actual D_{km+1} diagram) which is the negative of the sum of all the cycles in the bold D_{m+1}-subdiagram. Each vanishing cycle in the $D_{km+1}|\mathbf{Z}_{2m}$ diagram is a linear combination of the m cycles in the D_{km+1} diagram situated on one of the concentric circles: they are taken with the coefficients $1, \chi^{-1}, \chi^{-2}, \ldots$, starting from the crossed cycle and going counter-clockwise.

The lowest vertex in the character diagram corresponds to the polynomial $p_\lambda(v)$ of (6) having a zero root. Any other vertex corresponds to $p_\lambda(v)$ getting a double root.

Hermitian case. Similar diagrams for the one-variable stabilisation of the above, with the automorphism g acting trivially on the added coordinate, are drawn in the lower half of Figure 5.

Picard-Lefschetz operators. In $\mathbf{C}^{r+1}$, with g acting trivially on the 2-jet, the cycle in H_χ, $\chi^m = -1$, corresponding to the lowest vertex of the character diagram has self-intersection number

$$(e,e) = (-1)^{(r+1)(r+2)/2}\Big((-1)^r\chi - 1\Big)(1-\bar\chi)m\,.$$

Its Picard-Lefschetz operator has eigenvalue $(-1)^r\chi$ and, therefore, acts by the formula

$$h_e: \quad c \mapsto c + (-1)^{(r+1)(r+2)/2}\frac{(c,e)}{m(1-\bar\chi)}e\,. \tag{7}$$

This operator is induced on H_χ by the classical monodromy defined by the bold D_{m+1}-subdiagrams in Figure 5.

The vanishing χ-cycle corresponding to any other vertex of the character diagram has self-intersection number

$$(e,e) = (-1)^{(r+1)(r+2)/2}\Big((-1)^r + 1\Big)m\,.$$

Its Picard-Lefschetz operator on H_χ comes from the product of m commuting reflections on $H_r(V;\mathbf{C})$ corresponding to the vertices on the appropriate circle in the D_{km+1} diagram. It acts on H_χ by the formula (5).

In fact, division of the lowest root in the $D_{km+1}/\mathbf{Z}_{2m}$ diagrams by $\bar\chi - 1$ reduces them to the character diagrams of Figure 4. The operator (7) transforms into (4). Therefore, the monodromy groups on the character subspaces of the equivariant A- and D-singularities are the same. In particular, in the Hermitian D-case, we do obtain the groups $G(o_\chi, 1, k)$.

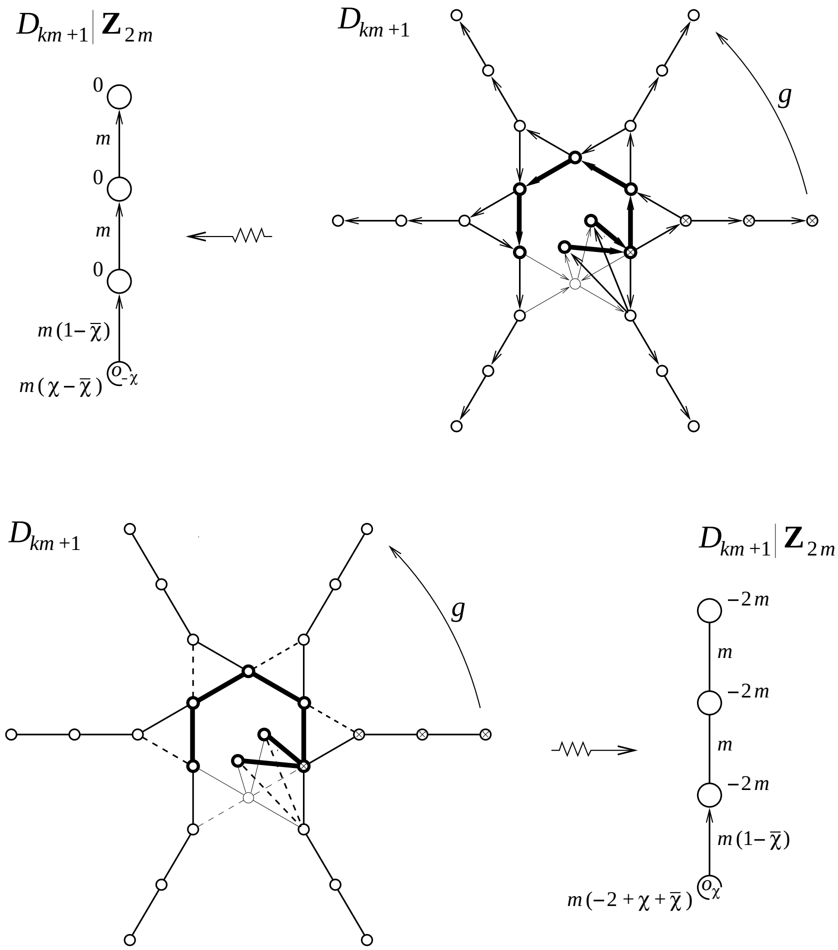

Figure 5. Folding symmetric Dynkin diagrams of the D_{km+1} singularity to $D_{km+1}|\mathbf{Z}_{2m}$ diagrams, $k=4$ and $m=6$. The two-variable case is in the upper half of the figure, and the three-variable is in the lower.

Remark 3.4 To be more precise, the correspondence observed here is between all the character subspaces of $D_{km+1}/\mathbf{Z}_{2m}$ and $A_{km}/\mathbf{Z}_m$ in the skew-Hermitian case, and between all the character subspaces of $D_{km+1}/\mathbf{Z}_{2m}$ and just half of those of $A_{2km}/\mathbf{Z}_{2m}$ in the Hermitian. This follows just from the comparison of the character sets.

Remark 3.5 The above folding of the D_{km+1} diagrams could be done in two steps: via factorisation by the action of $\mathbf{Z}_2 = \langle g^m \rangle$ followed by the further $\mathbf{Z}_m$-factorisation. The intermediate output here is Arnold's C_{km} singularity, with its $\mathbf{Z}_2$-anti-invariant homology. This, for example, immediately eliminates the generator of the rank 1 character subspace $H_{\chi=1}$ of $D_{km+1}|\mathbf{Z}_{2m}$, which is the difference of the two cycles at the whiskers of the central D_{m+1}-subdiagram in our figures.

3.2.3. $B_2^{(4,3)}$

Hermitian case. The singularity has a two-parameter equivariant miniversal deformation

$$x^3 + y^5 + \beta y^3 + \alpha y + z^2\,. \tag{8}$$

This corresponds to the Milnor regular automorphism $g(x, y, z) = (-\omega x, -y, iz)$ of the E_8 surface. The bifurcation diagram of the family is isomorphic to the discriminant of B_2 and consists of two strata:

$$2A_2 = \{b^2 = 4\alpha\} \qquad \text{and} \qquad D_4|\mathbf{Z}_{12} = \{\alpha = 0\}\,.$$

The rectangular Dynkin diagram of the ordinary singularity E_8 folds to a diagram of the equivariant function in two steps (see Figure 6):

(i) fusion to the $A_4^{(3)}$ diagram corresponding to the 4-subspace in the vanishing homology of E_8 on which g^4 is the multiplication by ε_3 (the cycles in it have self-intersection numbers -3, and an edge $a \to b$ has weight $(a, b) = 3/(1-\varepsilon_3)$);

(ii) further 2-folding of $A_4^{(3)}$ (this picks up the character subspace $H_{\chi=-\varepsilon_3\varepsilon_4}$ of the $B_2^{(4,3)}$ singularity; the vertices of the diagram corresponding to the reflections of orders 3 and 4 are $v_1 = e_1 + \varepsilon_4 e_4$ and $v_2 = e_2 + \chi e_3$ respectively).

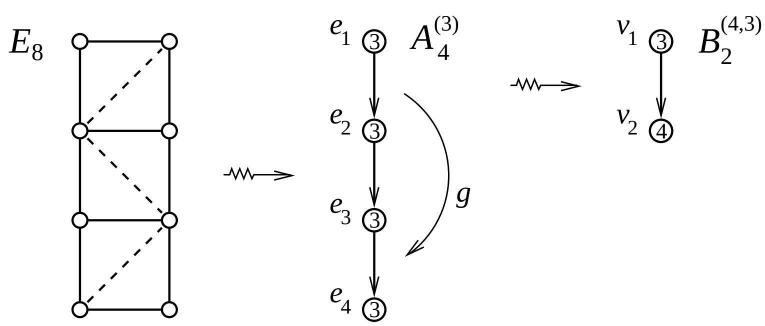

Figure 6. The two-step folding of the E_8 diagram to the diagram of the $B_2^{(4,3)}$ singularity. The weights of the vertices and edges are omitted.

The intersection form thus obtained is

$$-6\begin{pmatrix} 1 & \frac{1}{\varepsilon_3-1} \\ \frac{1}{\bar\varepsilon_3-1} & 1-\frac{\chi\varepsilon_3}{\varepsilon_3-1} \end{pmatrix}.$$

From this,

$$\frac{(v_1,v_2)(v_2,v_1)}{(v_1,v_1)(v_2,v_2)} = \frac{\cos\frac{\pi}{12}}{2\sin\frac{\pi}{3}\sin\frac{\pi}{4}}\,, \quad \frac{\cos\frac{5\pi}{12}}{2\sin\frac{\pi}{3}\sin\frac{\pi}{4}}$$

depending on the choice of ε_3 and ε_4. Therefore, the four options to choose ε_3 and ε_4 correspond to the four standard representations of G_{10} (see Section 1.2) acting on the subspaces $H_{\chi=-\varepsilon_3\varepsilon_4}$.

In terms of the Picard-Lefschetz operators h_{e_j} of $A_4^{(3)}$ (each with the eigenvalue ε_3), the Picard-Lefschetz operators h_{v_s} generating the representation of G_{10} on $H_{\chi=-\varepsilon_3\varepsilon_4}$ are

$$h_{v_1} = h_{e_1}h_{e_4} \qquad \text{and} \qquad h_{v_2} = h_{e_2}h_{e_3}h_{e_2} = h_{e_3}h_{e_2}h_{e_3}\,, \tag{9}$$

with matrices

$$\begin{pmatrix} \varepsilon_3 & -\varepsilon_3 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 1-\chi\varepsilon_3 & \varepsilon_4 \end{pmatrix}.$$

Skew-Hermitian case. For the two-variable singularity (no z in (8)), the characters χ of g on the first homology of the curve are $\pm\varepsilon_3$. Constructing the diagram of $B_2^{(4,3)}$, we again first obtain the $A_4^{(3)}$ diagram for the 4-dimensional χ^2-eigenspace of g^2 in the first homology. This diagram looks absolutely the same as in Figure 6, but with a modified interpretation: now the self-intersection of each cycle is $3\frac{\chi^2+1}{\chi^2-1}$ and an edge $a \to b$ has weight $(a,b) = -3/(1-\bar{\chi}^2)$. For a distinguished basis in the two-dimensional eigenspace H_χ of g one can take $v_1 = e_1 - \chi^3 e_4$ and $v_2 = e_2 - \chi e_3$. The generators (9) of the equivariant monodromy group now have the matrices

$$\begin{pmatrix} -\bar{\chi}^2 & \bar{\chi}^2 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 1-\bar{\chi} & \chi^3 \end{pmatrix}.$$

In particular, we get a finite group, $G(6,1,2)$, if $\chi = -\varepsilon_3$.

4. Three rank 2 groups

Among various relations between the sets of degrees of basic invariants of Shephard-Todd groups, similar to those observed in Remark 2.3, there are relations giving rise to singularity theory interpretations of three further rank 2 unitary reflection groups:

(i) $D(G_{20}) = D(H_4) \cap D(G_{32})$, corresponding to $G_{20} = H_4 \cap G_{32}$ in E_8;
(ii) $D(G_{22}) = D(H_4) \cap D(G_{31})$, related to $G_{22} = H_4 \cap G_{31}$ in E_8;
(iii) $D(G_{12}) \subset D(F_4) \subset D(E_6)$, corresponding to $G_{12} \subset F_4 \subset E_6$.

We write out appropriate two-parameter subfamilies of the $\mathcal{R}$-versal deformations of E_8 and E_6, and show that their monodromy groups are exactly the expected G_{20}, G_{22} and G_{12}. This time symmetry will be combined with special distribution of singular points on critical levels.

4.1. GROUP G_{20}

O.Shcherbak showed [12] that there exist two 4-parameter subfamilies in an $\mathcal{R}$-miniversal deformation of E_8 whose discriminant is that of the Coxeter group H_4. Those are either functions with all critical points of type A_2 or functions having four critical levels with two Morse points on each of them. Consider the latter subfamily:

$$x^3 + \alpha x + \beta + y^5/5 + y^3(\gamma x + \delta) + y(\gamma x + \delta)^2 + z^2 \,. \qquad (10)$$

Under the monodromy action, the vanishing homology of E_8 breaks up into the sum of the standard representation of H_4 and its conjugate.

Intersect the family (10) with the invariant miniversal deformation of $A_4^{(3)}$ (whose monodromy group is G_{32}, see the table). This is the same as taking the invariant part of the family with respect to the $\mathbf{Z}_3$-action, $g(x, y, z) = (\omega x, y, z)$:

$$x^3 + \beta + y^5/5 + \delta y^3 + \delta^2 y + z^2 \,. \qquad (11)$$

These are functions having two critical levels, with two A_2 points on each of them.

The family obtained has discriminant $25\beta^2 + 4\delta^5 = 0$ isomorphic to the discriminant of the group G_{20}. Therefore, the monodromy group has two generators, a and b, satisfying the relation $ababa = babab$. Moreover, $a^3 = b^3 = id$ since an elementary degeneration in (11) involves only A_2 points on the zero-level and the Coxeter number of the Weyl group A_2 is 3.

The relations obtained are those defining G_{20}. Hence the monodromy group of the family (11) is an 8-dimensional representation of G_{20}.

In fact we obtain the sum of all four standard rank 2 representations of G_{20} mentioned in Section 1.3. Indeed, consider the 4-dimensional character space $H_{\chi=\varepsilon_3}$ of g in the vanishing homology of E_8. On H_{ε_3}, the restriction of the whole $A_4^{(3)}$ monodromy (with four A_2 points situated on independent levels) to the subfamily (11) corresponds to the zig-zag folding of the diagram in the middle of Figure 6. The order 3 operators $h_{e_1}h_{e_3}$ and $h_{e_2}h_{e_4}$ split H_{ε_3} into two invariant subspaces, generated by

$$u_1 = e_1 + \bar{\varepsilon}_3(\varepsilon_5^2 + \bar{\varepsilon}_5^2)e_3 \qquad \text{and} \qquad u_2 = e_2 - \bar{\varepsilon}_3(\varepsilon_5 + \bar{\varepsilon}_5)e_4$$

(the two subspaces differ by the choice of the values of the expressions in the brackets). For these vectors,

$$\frac{(u_1,u_2)(u_2,u_1)}{(u_1,u_1)(u_2,u_2)} = \frac{1+\cos\frac{2\pi}{5}}{2\sin^2\frac{\pi}{3}}\,, \quad \frac{1+\cos\frac{4\pi}{5}}{2\sin^2\frac{\pi}{3}}$$

depending on the value of $\varepsilon_5 + \bar{\varepsilon}_5$. Now, comparison with Section 1.3 yields the desired identification of the representations of G_{20}.

Remark 4.1 The sum of the spaces of two of these representations of G_{20}, with the same value of $\varepsilon_5 + \bar{\varepsilon}_5$ and different ε_3, is the space of one of the two representations of H_4 on the vanishing homology of the E_8 singularity.

4.2. GROUPS WITH EQUILATERAL TRIANGULAR DIAGRAMS

4.2.1. G_{22}

The part of the H_4 family (10) contained in the equivariant versal deformation of the $E_8/\mathbf{Z}_4$ singularity (the monodromy group of the latter is G_{31}) is

$$x^3 + \alpha x + y^5/5 + \delta y^3 + \delta^2 y + z^2\,. \tag{12}$$

Its discriminant, $25\alpha^3 = 27\delta^5$, is the discriminant of the group G_{22}. A generic degeneration is 4 Morse points on the zero level $V_{\alpha,\delta} \subset \mathbf{C}^3_{x,y,z}$.

Consider the line $\ell = \{\delta = \delta_0\}$, where δ_0 is a real negative constant, in the base of deformation (12). Take V_{0,δ_0} for the marked Milnor fibre. The line ℓ intersects the discriminant at three points, with coordinates α_0, $\omega\alpha_0$ and $\omega^2\alpha_0$, where α_0 is real. We take the straight paths on ℓ from $(0,\delta_0)$ to the discriminantal points to define vanishing cycles on V_{0,δ_0}.

The real part of the plane curve $V_{\alpha_0,\delta_0} \cap \{z=0\}$ is the sabirification of E_8 that yields the rectangular Dynkin diagram of E_8. The product of the four commuting Picard-Lefschetz operators, corresponding to the cycles on V_{0,δ_0} vanishing at the nodes of this curve, is an elementary Picard-Lefschetz operator in the family (12).

The two quadruples of cycles on V_{0,δ_0} vanishing at $(\omega\alpha_0,\delta_0)$ and $(\omega^2\alpha_0, \delta_0)$ are obtained from those vanishing at (α_0,δ_0) by multiplication of x by ω^2 and ω.

The intersection diagram of the twelve vanishing cycles on V_{0,δ_0} is given in Figure 7. The sum of three cycles on each of the four horizontal levels of the prism is zero. It is easy to spot an embedding of the rectangular E_8 diagram into our diagram.

The three generators of the monodromy group of the family (12) are the products of the Euclidean reflections in the cycles in one vertical quadruple in Figure 7. The vanishing homology of E_8 splits into the sum of four rank 2 invariant subspaces. The elements spanning these subspaces are the linear

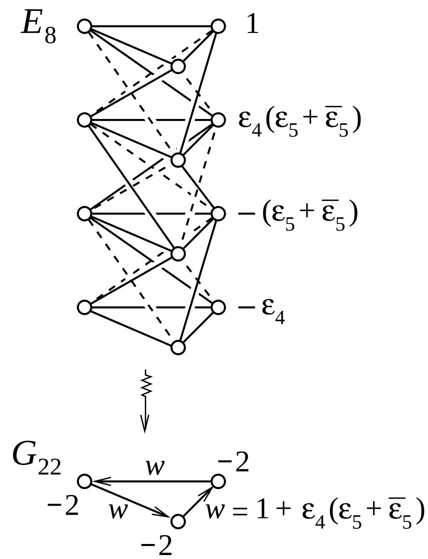

Figure 7. Folding E_8 diagram to G_{22} diagram.

combinations of the cycles in each quadruple taken with the coefficients shown in Figure 7 (the coefficients are defined by a horizontal level of the diagram). Different choices of the numbers ε_4 and $\varepsilon_5+\bar{\varepsilon}_5$ give four different rank 2 subspaces.

Rescaling the spanning vectors to make their squares -2, we get the triangular intersection diagram shown at the bottom of Figure 7. Comparing it with the G_{22} diagram of Figure 3, we see that the four rank 2 subspaces obtained provide all four standard representations of G_{22}.

4.2.2. G_{12}
Consider now the subfamily

$$x^3+\alpha x+y^4+\beta y^2+\beta^2/8+z^2 \qquad (13)$$

of the miniversal deformation of E_6. Its discriminant, $(\alpha/3)^3+(\beta/4)^4=0$, is the discriminant of the group G_{12}.

This time the elementary degeneration is three Morse points on the zero-level situated symmetrically with respect to the change of sign of y (thus, one of them is on $y=0$). For a real point of the discriminant having $\beta>0$, the real part of the curve $V_{\alpha,\beta}\cap\{z=0\}$ is the open trefoil yielding the rectangular Dynkin diagram of E_6. The product of the Picard-Lefschetz operators on the vanishing homology of E_6 corresponding to the nodes of the trefoil is an elementary monodromy operator in the present situation.

Considerations similar to the G_{22} case provide the prismatic diagram of Figure 8(a). Again the sum of the cycles on each of the three horizontal levels here is zero. The generators of the monodromy group Γ of the family (13) are the products of the Euclidean reflections corresponding to the vertices in the vertical triplets.

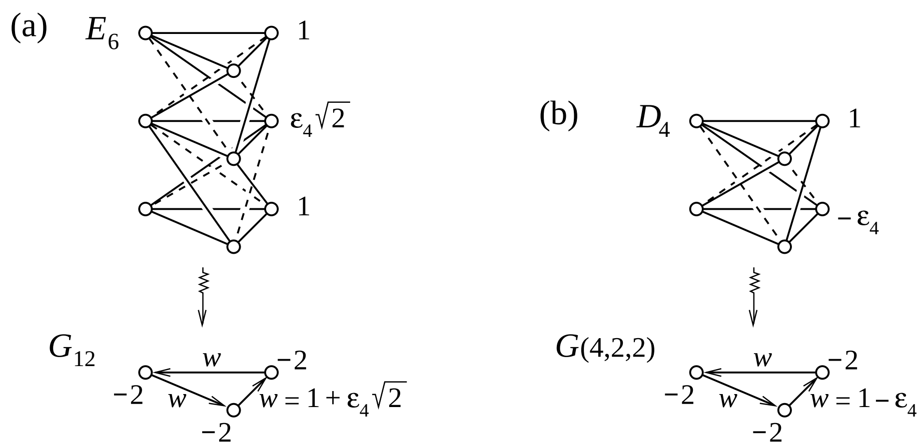

Figure 8. Obtaining G_{12} diagram from E_6, and $G(4,2,2)$ from D_4.

The linear combinations of the cycles in each of the vertical triplets, with the coefficients shown in the figure, span one of the three rank 2 representations of Γ. Scaling the spanning vectors to make their squares -2, we obtain the equilateral triangular intersection diagram of Figure 8(a). Comparison with Section 1.3 shows that two possible values of ε_4 provide the two standard representations of G_{12}.

The sum of the two G_{12} representations described is the anti-invariant part of the homology of E_6 with respect to the change of sign of y (that is, the F_4 vanishing homology). The sign change acts on the prismatic diagram as the reflection in the medial triangle followed by the change of orientation of all the cycles. Therefore, the third rank 2 representation of Γ in the homology of E_6 is just A_2 acting on the span of the differences of the corresponding vertices of the top and bottom faces of the prism.

4.2.3. *The mini-series*

The E_6 prism of Figure 8(a) is one store shorter than that of Figure 7. Cutting off one level more (Figure 8(b)), we get the diagram corresponding to the equivariant mini-versal deformation of $D_4/\mathbf{Z}_4$ (cf. the table where it is given in a different normal form):

$$x^3 + \alpha x + y^3 + \beta y + z^2\,. \tag{14}$$

Here $g(x, y, z) = (-x, -y, iz)$. We can assume that g sends the vertices of the top face down to the underlying vertices of the bottom, and those from the bottom up to the negatives of those on the top. The monodromy group of (14) is $G(4, 2, 2)$, and the discriminant, $\alpha^3 = \beta^3$, of the family is that of the reflection group. The elementary degeneration is $2A_1$.

The linear combinations of the vertical pairs of the cycles with the coefficients shown in Figure 8(b) span the rank 2 subspace $H_{\chi=\varepsilon_4}$ in the homology of D_4. The result of the folding operation is the equilateral diagram of $G(4, 2, 2)$ (cf. Figure 3).

The degenerate prism, consisting of just one triangular level, corresponds to the A_2 $\mathcal{R}$-versal family $x^3 + \alpha x + \beta + y^2 + z^2$ and a non-generic line $\ell = \{\alpha = \text{const}\}$ in its base.

Remark 4.2 There is an interesting feature in all four examples of this subsection: the discriminant of the related two-dimensional unitary reflection group has the same singularity as the non-deformed singular curve $V_{0,0} \cap \{z = 0\}$. Does this have any explanation avoiding explicit calculations?

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References

1. V. I. Arnold and his school, *Arnold's Problems*, Phasis, Moscow, 2000, 452 pp. (in Russian).
2. E. Brieskorn, *Die Fundamentalgruppe des Raumes der regülaren Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.
3. M. Broué and G. Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189.
4. A. M. Cohen, *Finite complex reflection groups*, Ann. Sci. Ecole Norm. Sup. **9** (1976), 379–436.
5. H. S. M. Coxeter, Regular Complex Polytopes, Cambridge University Press, Cambridge, 1974, 195 pp.
6. J. N. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Mem. Amer. Math. Soc. **50** (1984), no.306, 100 pp.
7. V. V. Goryunov, Unitary reflection groups associated with singularities of functions with cyclic symmetry, Russian Mathematical Surveys **54** (1999), no.5, 873–893.
8. V. V. Goryunov and C. E. Baines, Cyclically equivariant function singularities and unitary reflection groups $G(2m, 2, n)$, G_9, G_{31}, St Petersburg Mathematical Journal **11** (2000), no.5, 000–000.
9. S. M. Gusein-Zade, *Monodromy groups of isolated singularities of hypersurfaces*, Russian Mathematical Surveys **32** (1977), no.2, 23–69.

Real pages required!

10. O. V. Lyashko, *Geometry of bifurcation diagrams* Journal of Soviet Mathematics **27** (1984), 2736–2759.
11. V. L. Popov, *Discrete complex reflection groups*, Commun. Math. Inst., Rijksuniversiteit Utrecht 15-1982 (1982), 89 pp.
12. O. P. Shcherbak, *Wavefronts and reflection groups*, Russian Math. Surveys **43** (1988), no.3, 125–160.
13. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.
14. P. Slodowy, *Simple Singularities and Complex Reflections*, these proceedings.
15. T. A. Springer, *Regular elements of finite reflection groups*, Inventiones Math. **25** (1975), 159–198.
16. T. Yano, *Free deformations of isolated singularities*, Sci. Rep. Saitama Univ. Ser A **9** (1980), 61–70.
17. T. Yano, *Invariants of finite reflection groups and flat coordinate systems of isolated singularities*, RIMS Kokyuroku **444** (1981), 209–235 (Lecture notes in Japanese).
18. T. Yano, *Deformations of rational double points associated with unitary reflection groups*, Sci. Rep. Saitama Univ. Ser A **10** (1981), 7–9.