Lagrangian and Legendrian Singularities

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These are notes of the mini-courses we lectured in Trieste in 2003 and Luminy in 2004. The courses were based on the books [1, 3, 2].

1 Symplectic and contact geometry

1.1 Symplectic geometry

A symplectic form ω on a manifold M is a closed 2-form, non-degenerate as a skew-symmetric bilinear form on the tangent space at each point. So $d\omega = 0$ and ω^n is a volume form, $\dim M = 2n$.

Manifold M equipped with a symplectic form is called symplectic. It is necessarily even-dimensional.

If the form is exact, $\omega = d\lambda$, the *symplectic area* of a 2-chain S is $\int_{\partial S} \lambda$. When λ exists and is fixed M is called *exact symplectic*.

Examples.
1. Let $K = M = \mathbf{R}^{2n} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ be a vector space, and

$$\lambda = pdq = \sum_{i=1}^{n} p_i dq_i \,, \qquad \omega = d\lambda = dp \wedge dq \,.$$

In these coordinates the form ω is constant. The corresponding bilinear form on the tangent space at a point is given by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Notice: for any non-degenerate skew-symmetric bilinear form on a linear space, there exists a basis (called Darboux basis) in which the form has this matrix.

2. $M = T^*N$. Take for λ the *Liouville form* defined in an invariant (coordinate-free) way as

$$\lambda(\alpha) = \pi(\alpha)\Big(\rho_*(\alpha)\Big)\,,$$

where

$$\alpha \in T(T^*N)\,, \quad \pi : T(T^*N) \to T^*N \quad \text{and} \quad \rho : T^*N \to N\,.$$

This is an exact symplectic manifold. If $q_1, \ldots, q_n$ are local coordinates on the base N, the dual coordinates $p_1, \ldots, p_n$ are the coefficients of the decomposition of a covector into linear combination of the differentials dq_i:

$$\lambda = \sum_{i=1}^{n} p_i dq_i \,.$$

3. On a Kähler manifold M, the imaginary part of its Hermitian structure $\omega(\alpha,\beta) = Im(\alpha,\beta)$ is a skew-symmetric 2-form which is closed.

4. Product of two symplectic manifolds. Given two symplectic manifolds (M_i,ω_i), $i = 1,2$, their product $M_1 \times M_2$ equipped with the 2-form $(\pi_1)_*\omega_1 - (\pi_2)_*\omega_2$, where the π_i are the projections to the corresponding factors, is a symplectic manifold.

A diffeomorphism $\varphi : M_1 \to M_2$ which sends the symplectic structure ω_2 on M_2 to the symplectic structure ω_1 on M_1,

$$\varphi^*\omega_2 = \omega_1 \,,$$

is called a symplectomorphism between (M_1,ω_1) and (M_2,ω_2). When the (M_i,ω_i) are the same, a symplectomorphism preserves the symplectic structure. In particular, it preserves the volume form ω^n.

Symplectic group.
For $K = (\mathbf{R}^{2n}, dp\wedge dq)$ of our first example, the group $Sp(2n)$ of linear symplectomorphisms is isomorphic to the group of matrices S such that

$$S^{-1} = -JS^tJ \,.$$

Here t is for transpose. The characteristic polynomial of such an S is reciprocal: if α is an eigenvalue, then α^{-1} also is. The Jordan blocks for α and α^{-1} are the same.

Introduce an auxiliary scalar product $(\cdot,\cdot)$ on K, with the matrix I_{2n} in our Darboux basis. Then

$$\omega(a,b) = (a, \widetilde{J}b) \,,$$

where $\widetilde{J}$ is the operator on K with the matrix J. Setting $q = \mathrm{Re}\, z$ and $p = \mathrm{Im}\, z$ makes K a complex Hermitian space, with the multiplication by $i = \sqrt{-1}$ being the application of $\widetilde{J}$. The Hermitian structure is

$$(a,b) + i\omega(a,b) \,.$$

From this,

$$Gl(n,\mathbf{C}) \bigcap O(2n) = Gl(n,\mathbf{C}) \bigcap Sp(2n) = O(2n) \bigcap Sp(2n) = U(n) \,.$$

Remark. The image of the unit sphere $S_1^{2n-1} : q^2+p^2 = 1$ under a linear symplectomorphism can belong to a cylinder $q_1^2 + p_1^2 \le r$ only if $r \ge 1$.

The non-linear analog of this result is rather non-trivial: $S_1^{2n-1} \in T^*\mathbf{R}^n$ (in the standard Euclidean structure) cannot be symplectically embedded into the cylinder $\{q_1^2 + p_1^2 < 1\} \times T^*\mathbf{R}^{n-1}$. This is Gromov's theorem on symplectic camel.

Thus, for $n > 1$, symplectomorphisms form a thin subset in the set of diffeomorphisms preserving the volume ω^n.

The dimension k of a linear subspace $L^k \subset K$ and the rank r of the restriction of the bilinear form ω on it are the complete set of $Sp(2n)$-invariants of L.

Define the skew-orthogonal complement $L^\angle$ of L as

$$L^\angle = \{v \in K | \omega(v, u) = 0 \quad \forall u \in L\} .$$

So $\dim L^\angle = 2n - k$. The kernel subspace of the restriction of ω to L is $L \bigcap L^\angle$. Its dimension is $k - r$.

A subspace is called <u>isotropic</u> if $L \subset L^\angle$ (hence $\dim L \leq n$).
Any line is isotropic.

A subspace is called <u>co-isotropic</u> if $L^\angle \subset L$ (hence $\dim L \geq n$).
Any hyperplane H is co-isotropic. The line $H^\angle$ is called the <u>characteristic direction</u> on H.

A subspace is called <u>Lagrangian</u> if $L^\angle = L$ (hence $\dim L = n$).

Lemma. Each Lagrangian subspace $L \subset K$ has a regular projection to at least one of the 2^n coordinate Lagrangian planes (p_I, q_J), along the complementary Lagrangian plane (p_J, q_I). Here $I \bigcup J = \{1, \ldots, n\}$ and $I \bigcap J = \emptyset$.

Proof. Let L_q be the intersection of L with the q-space and $\dim L_q = k$. Assume $k > 0$, otherwise L projects regularly onto the p-space. The plane L_q has a regular projection onto some q_I-plane (along q_J) with $|I| = k$. If L does not project regularly to the p_J-plane (along (q, p_I)) then L contains a vector $v \in (q, p_I)$ with a non-trivial p_I-component. Due to this non-triviality, the intersection of the skew-orthogonal complement $v^\angle$ with the q-space has a $(k-1)$-dimensional projection to q_I (along q_J) and so does not contain L_q. This contradicts to L being Lagrangian. $\square$

A Lagrangian subspace L which projects regularly onto the q-plane is the graph of a self-adjoint operator S from the q-space to the p-space with its matrix symmetric in the Darboux basis.

Splitting $K = L_1 \bigoplus L_2$ with the summands Lagrangian is called a <u>polarisation</u>. Any two polarisations are symplectomorphic.

The Lagrangian Grassmanian $Gr_L(2n)$ is diffeomorphic to $U(n)/O(n)$. Its fundamental group is $\mathbf{Z}$.

The Grassmanian $Gr_k(2n)$ of isotropic k-spaces is isomorphic to $U(n)/(O(k) + U(n-k))$.

Even in a non-linear setting symplectic structure has no local invariants (unlike Riemannian structure) according to the following

Darboux Theorem. Any two symplectic manifolds of the same dimension are locally symplectomorphic.

Proof. We use the homotopy method. Let ω_t, $t \in [0,1]$, be a family of germs of symplectic forms on a manifold coinciding at the distinguished point A. We are looking for a family $\{g_t\}$ of diffeomorphisms such that $g_t^* \omega_t = \omega_0$ for all t. Differentiate this by t:

$$\mathcal{L}_{v_t} \omega_t = -\gamma_t$$

where $\gamma_t = \partial \omega_t / \partial t$ is a known closed 2-form and $\mathcal{L}_{v_t}$ is the Lie derivative along the vector field to find. Since $\mathcal{L}_v = i_v d + d i_v$, we get

$$d i_{v_t} \omega_t = -\gamma_t \, .$$

Choose a 1-form α_t vanishing at A and such that $d\alpha_t = -\gamma_t$. Due to the non-degeneracy of ω_t, the equation $i_{v_t}\omega_t = \omega(\cdot, v_t) = \alpha_t$ has a unique solution v_t vanishing at A. □

Weinstein's Theorem. A submanifold of a symplectic manifold is defined, up to a symplectomorphism of its neighbourhood, by the restriction of the symplectic form to the tangent vectors to the ambient manifold at the points of the submanifold.

In a similar local setting, the inner geometry of a submanifold defines its outer geometry:

Givental's Theorem. A germ of a submanifold in a symplectic manifold is defined, up to a symplectomorphism, by the restriction of the symplectic structure to the tangent bundle of the submanifold.

Proof of Givental's Theorem. It is sufficient to prove that if the restrictions of two symplectic forms, ω_0 and ω_1, to the tangent bundle of a submanifold $G \subset M$ at point A coincide, then there exits a local diffeomorphism of M fixing G point-wise and sending one form to the other. We may assume that the forms coincide on $T_A M$.

We again use the homotopy method, aiming to find a family of diffeomorphism-germs g_t, $t \in [0,1]$, such that

$$g_t|_G = id_G \, , \quad g_0 = id_M \, , \qquad g_t^*(\omega_t) = \omega_0 \quad (*) \qquad \text{where} \quad \omega_t = \omega_0 + (\omega_1 - \omega_0)t \, .$$

Differentiating $(*)$ by t, we again get

$$\mathcal{L}_{v_t}(\omega_t) = d(i_{v_t}\omega_t) = \omega_0 - \omega_1$$

where v_t is the vector field of the flow g_t. Using the "relative Poincare lemma", it is possible to find a 1-form α so that $d\alpha = \omega_0 - \omega_1$ and α vanishes on G. Then the required vector field v_t exists since ω_t is non-degenerate. □

Darboux theorem is a particular case of Givental's theorem: take a point as a submanifold.

If at each point x of a submanifold L of a symplectic manifold M the subspace T_xL is Lagrangian in the symplectic space T_xM, then L is called Lagrangian.

Examples.
1. In T^*N, the following are Lagrangian submanifolds: the zero section of the bundle, fibres of the bundle, graph of the differential of a function on N.

2. The graph of a symplectomorphism is a Lagrangian submanifold of the product space (it has regular projections onto the factors). An arbitrary Lagrangian submanifold of the product space defines a so-called Lagrangian relation.

3. Weinstein's theorem implies that a tubular neighbourhood of a Lagrangian submanifold L in any symplectic space is symplectomorphic to a tubular neighbourhood of the zero section in T^*N.

A fibration with Lagrangian fibres is called Lagrangian.

Locally all Lagrangian fibrations are symplectomorphic (the proof is similar to that of Darboux theorem).

A cotangent bundle is a Lagrangian fibration.

Let $\psi : L \to T^*N$ be a Lagrangian embedding and $\rho : T^*N \to N$ the fibration. The product $\rho \circ \psi : L \to N$ is called a Lagrangian mapping. It critical values

$$\Sigma_L = \{q \in N | \exists p : (p, q) \in L,\ \operatorname{rank} d(\rho \circ \psi) < n\}$$

form the caustic of the Lagrangian mapping. The equivalence of Lagrangian mappings is that up to fibre-preserving symplectomorphisms of the ambient symplectic space. Caustics of equivalent Lagrangian mappings are diffeomorphic.

Hamiltonian vector fields.
Given a real function $h : M \to \mathbf{R}$ on a symplectic manifold, define a Hamiltonian vector field v_h on M by the formula

$$\omega(\cdot, v_h) = dh\,.$$

This field is tangent to the level hypersurfaces $H_c = h^{-1}(c)$:

$$\forall a \in H_c \quad dh(T_aH_c) = 0 \qquad \Longrightarrow \qquad T_aH_c = v_h^{\angle}\,, \quad \text{but} \quad v_h \in v_h^{\angle}\,.$$

The directions of v_h on the level hypersurfaces H_c of h are the characteristic directions of the tangent spaces of the hypersurfaces.

Associating v_h to h, we obtain a Lie algebra structure on the space of functions:

$$[v_h, v_f] = v_{\{h,f\}} \qquad \text{where} \quad \{h, f\} = v_h(f)\,,$$

the latter being the Poisson bracket of the Hamiltonians h and f.

A Hamiltonian flow (even if h depends on time) consists of symplectomorphisms. Locally (or in $\mathbf{R}^{2n}$), any time-dependent family of symplectomorphisms that starts from the identity is a phase flow of a time-dependent Hamiltonian. However, for example, on a torus $\mathbf{R}^2/(\mathbf{Z}^2)$ (which is the quotient of the plane by an integer lattice) the family of constant velocity displacements are symplectomorphisms but they cannot be Hamiltonian since a Hamiltonian function on a torus must have critical points.

Given a time-dependent Hamiltonian $\widetilde{h} = \widetilde{h}(t, p, q)$, consider the extended space $M \times T^*\mathbf{R}$ with auxiliary coordinates (s, t) and the form $pdq - sdt$. An auxiliary (extended) Hamiltonian $\widehat{h} = -s + \widetilde{h}$ determines a flow in the extended space generated by the vector field

$$\dot{p} = -\frac{\partial \widehat{h}}{\partial q} \qquad\qquad \dot{q} = -\frac{\partial \widehat{h}}{\partial p}$$

$$\dot{t} = -\frac{\partial \widehat{h}}{\partial s} = 1 \qquad\qquad \dot{s} = \frac{\partial \widehat{h}}{\partial t}$$

The restrictions of this flow to the $t = const$ sections are essentially the flow mappings of $\widetilde{h}$.

The integral of the extended form over a closed chain in $M \times \{t_o\}$ is preserved by the $\hat{h}$-Hamiltonian flow. Hypersurfaces $-s + \widetilde{h} = const$ are invariant. When $\widetilde{h}$ is autonomous, the form pdq is also a relative integral invariant.

A (transversal) intersection of a Lagrangian submanifold $L \subset M$ with a Hamiltonian level set $H_c = h^{-1}(c)$ is an isotropic submanifold L_c. All Hamiltonian trajectories emanating from L_c form a Lagrangian submanifold $exp_H(L_c) \subset M$. The space Ξ_{H_c} of the Hamiltonian trajectories on H_c inherits, at least locally, an induced symplectic structure. The image of the projection of $exp_H(L_c)$ to Ξ_{H_c} is a Lagrangian submanifold there. This is a particular case of a symplectic reduction which will be discussed later.

Example. The set of all oriented straight lines in $\mathbf{R}^n_q$ is T^*S^{n-1} as a space of characteristics of the Hamiltonian $h = p^2$ on its level $p^2 = 1$ in $K = \mathbf{R}^{2n}$.

1.2 Contact geometry

An odd-dimensional manifold M^{2n+1} equipped with a maximally non-integrable distribution of hyperplanes (contact elements) in the tangent spaces of its points is called a contact manifold.

The maximal non-integrability means that if locally the distribution is determined by zeros of a 1-form α on M then $\alpha \wedge (d\alpha)^n \neq 0$ (cf. the Frobenius condition of complete integrability being $\alpha \wedge d\alpha = 0$.)

Examples.
1. A projectivised cotangent bundle PT^*N^{n+1} with the projectivisation of the Liouville form $\alpha = pdq$. This is also called a space of contact elements on N. The spherisation of PT^*N^{n+1} is a 2-fold covering of PT^*N^{n+1} and its points are co-oriented contact elements.

2. The space J^1N of 1-jets of functions on N^n. (Two functions have the same m-jet at a point x if their Taylor polynomials of degree k at x coincide). The space of all 1-jets at all points of N has local co-ordinates $q \in N$, $p = df(q)$ which are the partial derivatives of a function at q, and $z = f(q)$. The contact form is $pdq - dz$.

Contactomorphisms are diffeomorphisms preserving the distribution of contact elements.

Contact Darboux theorem. All equidimensional contact manifolds are locally contactomorphic.

An analog of Givental's theorem also holds.

Symplectisation.
Let $\widetilde{M}^{2n+2}$ be the space of all linear forms vanishing on contact elements of M. The space $\widetilde{M}^{2n+2}$ is a "line" bundle over M (fibres do not contain the zero forms). Let

$$\widetilde{\pi} : \widetilde{M} \to M$$

be the projection. On $\widetilde{M}$, the symplectic structure (which is homogeneous of degree 1 with respect to fibres) is the differential of the canonical 1-form $\widetilde{\alpha}$ on $\widetilde{M}$ defined as

$$\widetilde{\alpha}(\xi) = p(\widetilde{\pi}_*\xi)\,, \qquad \xi \in T_p\widetilde{M}\,.$$

A contactomorphism F of M lifts to a symplectomorphism of $\widetilde{M}$:

$$\widetilde{F}(p) := (F^*_{F(x)})^{-1}p\,.$$

This commutes with the multiplication by constants in the fibres and preserves $\widetilde{\alpha}$. The symplectisation of contact vector fields (= infinitesimal contactomorphisms) yields Hamiltonian vector fields with homogeneous (of degree 1) Hamiltonian functions $h(rx) = rh(x)$.

Assume the contact structure on M is defined by zeros of a fixed 1-form β. Then M has a natural embedding $x \mapsto \beta_x$ into $\widetilde{M}$.

Using the local model $J^1\mathbf{R}^n$, $\beta = pdq - dz$, of a contact space we get the following formulas for components of the contact vector field with a homogeneous Hamiltonian function $K(x) = h(x_\beta)$ (notice that $K = \beta(X)$ where X is the corresponding contact vector field):

$$\dot{z} = pK_p - K, \quad \dot{p} = -K_q - pK_z, \quad \dot{q} = K_p.$$

where the subscripts mean the partial derivations.

Various homogeneous analogs of symplectic properties hold in contact geometry (the analogy is similar to that between affine and projective geometries).

In particular, a hypersurface (transversal to the contact distribution) in a contact space inherits a field of characteristics.

Contactisation.
To an exact symplectic space M^{2n} associate $\widehat{M} = \mathbf{R} \times M$ with an extra coordinate z and take the 1-form $\alpha = \lambda - dz$. This gives a contact space.

Here the vector field $\chi = -\frac{\partial}{\partial z}$ is such that $i_\chi \alpha = 1$ and $i_\chi d\alpha = 0$. Such a field is called a Reeb vector field. Its direction is uniquely defined by a contact structure. It is transversal to the contact distribution. Locally, projection along χ produces a symplectic manifold.

A Legendrian submanifold $\widehat{L}$ of M^{2n+1} is an n-dimensional integral submanifold of the contact distribution. This dimension is maximal possible for integral submanifolds.

Examples.
1. To a Lagrangian $L \subset T^*M$ associate $\widehat{L} \subset J^1M$:

$$\widehat{L} = \left\{(z, p, q) \mid z = \int pdq, \ (p, q) \in L\right\}.$$

Here the integral is taken along a path on L joining a distinguished point on L with the point (p, q). Such an $\widehat{L}$ is Legendrian.

2. The set of all covectors annihilating tangent spaces to a given submanifold (or variety) $W_0 \subset N$ form a Legendrian submanifold (variety) in PT^*N.

3. If the intersection I of a Legendrian submanifold $\widehat{L}$ with a hypersurface Γ in a contact space is transversal, then I is transversal to the characteristic vector field on Γ. The set of characteristics emanating from I form a Legendrian submanifold.

A Legendrian fibration of a contact space is a fibration with Legendrian fibres. For example, $PT^*N \to N$ and $J^1N \to J^0N$ are Legendrian. Any two Legendrian fibrations of the same dimension are locally contactomorphic.

The projection of an embedded Legendrian submanifold $\widehat{L}$ to the base of a Legendrian fibration is called a Legendrian mapping. Its image is called the wave front of $\widehat{L}$.

Examples.

1. Embed a Legendrian submanifold $\widehat{L}$ into J^1N. Its projection to J^0N, wave front $W(\widehat{L})$, is a graph of a multivalued action function $\int pdq + c$ (again we integrate along paths on the Lagrangian submanifold $L = \pi_1(\widehat{L})$, where $\pi_1 : J^1N \to T^*N$ is the projection dropping the z co-ordinate). If $q \in N$ is not in the caustic Σ_L of L, then over q the wave front $W(\widehat{L})$ is a collection of smooth sheets.

If at two distinct points $(p', q), (p'', q) \in L$ with a non-caustical value q, the values z of the action function are equal, then at (z, q) the wave front is a transversal intersection of graphs of two regular functions on N.

The images under the projection $(z, q) \mapsto q$ of the singular and transversal self-intersection loci of $W(\widehat{L})$ are respectively the caustic Σ_L and so-called Maxwell (conflict) set.

2. To a function $f = f(q)$, $q \in \mathbf{R}^n$, associate its Legendrian lifting $\widehat{L} = j^1(f)$ (also called the 1-jet extension of f) to $J^1\mathbf{R}^n$. Project $\widehat{L}$ along the fibres parallel to the q-space of another Legendrian fibration

$$\pi_1^\wedge(z, p, q) \mapsto (z - pq, p)$$

of the same contact structure $pdq - dz = -qdp - d(z - pq)$. The image $\pi_1^\wedge(\widehat{L})$ is called the Legendre transform of the function f. It has singularities if f is not convex.

This is an affine version of the projective duality (which is also related to Legendrian mappings). The space PT^*P^n (P^n is the projective space) is isomorphic to the projectivised cotangent bundle $PT^*P^{n\wedge}$ of the dual space $P^{n\wedge}$. Elements of both are pairs consisting of a point and a hyperplane, containing the point. The natural contact structures coincide. The set of all hyperplanes in P^n tangent to a submanifold $S \subset P^n$ is the front of the dual projection of the Legendrian lifting of S.

Wave front propagation.

Fix a submanifold $W_0 \subset N$. It defines the (homogeneous) Lagrangian submanifold $L_0 \subset T^*N$ formed by all covectors annihilating tangent spaces to W_0.

Consider now a Hamiltonian function $h : T^*N \to \mathbf{R}$. Let I be the intersection of L_0 with a fixed level hypersurface $H = h^{-1}(c)$. Consider the Lagrangian submanifold $L = exp_H(I) \subset H$ which consists of all the characteristics emanating from I. It is invariant under the flow of H.

The intersections of the Legendrian lifting $\widehat{L}$ of L into J^1N ($z = \int pdq$) with coordinate hypersurfaces $z = const$ project to Legendrian submanifolds (varieties) $\widehat{L}_z \subset PT^*N$. In fact, the form pdq vanishes on each tangent vector to $\widehat{L}_z$. In general, the dimension of $\widehat{L}_z$ is $n-1$.

The wave front of $\widehat{L}$ in J^0N is called the big wave front. It is swept out by the family of fronts W_z of the $\widehat{L}_z$ shifted to the corresponding levels of the z-coordinate. Notice that, up to a constant, the value of z at a point over a point (p, q) is equal to $z = \int p\frac{\partial h}{\partial p}dt$ along a segment of the Hamiltonian trajectory going from the initial I to (p, q).

When h is homogeneous of degree k with respect to p in each fibre, then $z_t = kct$. Let $I_t \subset L$ be the image of I under the flow transformation g_t for time t. The projectivised I_t are Legendrian in PT^*N. The family of their fronts in N is $\{W_{kct}\}$. So the W_t are momentary wavefronts propagating from the initial W_0. Their singular loci sweep out the caustic Σ_L.

The case of a time-depending Hamiltonian $h = h(t, p, q)$ reduces to the above by considering the extended phase space $J^1(N \times \mathbf{R})$, $\alpha = pdq - rdt - dz$. The image of the initial Legendrian subvariety $\widehat{L}_0 \subset J^1(N \times \{0\})$ under g_t is a Legendrian $L_t \subset J^1(N \times \{t\})$.

When z can be written locally as a regular function in q, t it satisfies the Hamilton-Jacobi equation $-\frac{\partial z}{\partial t} + h(t, \frac{\partial z}{\partial q}, q) = 0$.

2 Generating families

2.1 Lagrangian case

Consider a co-isotropic submanifold $C^{n+k} \subset M^{2n}$. The skew-orthogonal complements $T_c^{\angle}C$, $c \in C$, of tangent spaces to C define an integrable distribution on C. Indeed, take two regular functions whose common zero level set contains C. At each point $c \in C$, the vectors of their Hamiltonian fields belong to $T_c^{\angle}C$. So the corresponding flows commute. Trajectories of all such fields emanating from $c \in C$ form a smooth submanifold I_c integral for the distribution.

By Givental's theorem, any co-isotropic submanifold is locally symplectomorphic to a coordinate subspace $p_I = 0$, $I = \{1, \ldots, n-k\}$, in $K = \mathbf{R}^{2n}$. The fibres are the sets $q_J = const$.

Proposition. Let L^n and C^{n+k} be respectively Lagrangian and co-isotropic submanifolds of a symplectic manifold M^{2n}. Assume L meets C transversally at a point a. Then the intersection $X_0 = L \bigcap C$ is transversal to the isotropic fibres I_c near a.

The proof is immediate. If $T_a X_0$ contains a vector $v \in T_a I_c$, then v is skew-orthogonal to $T_a L$ and also to $T_a C$, that is to any vector in $T_a M$. Hence $v = 0$.

Isotropic fibres define the fibration $\xi : C \to B$ over a certain manifold B of dimension $2k$ (defined at least locally). We can say that B is the manifold of isotropic fibres.

It has a well-defined induced symplectic structure ω_B. Given any two vectors u, v tangent to B at a point b take their liftings, that is vectors $\tilde{u}, \tilde{v}$ tangent to C at some point of $\xi^{-1}(b)$ such that their projections to B are u and v. The value $\omega(\tilde{u}, \tilde{v})$ depends only on the vectors u, v. For any other choice of liftings the result will be the same. This value is taken for the value of the two-form ω_B on B.

Thus, the base B gets a symplectic structure which is called a symplectic reduction of the co-isotropic submanifold C.

Example. Consider a Lagrangian section L of the (trivial) Lagrangian fibration $T^*(\mathbf{R}^k \times \mathbf{R}^n)$. The submanifold L is the graph of the differential of a function $f = f(x, q)$, $x \in \mathbf{R}^k$,

$q \in \mathbf{R}^n$. The dual coordinates y, p are given on L by $y = \frac{\partial f}{\partial x}$, $p = \frac{\partial f}{\partial q}$. Therefore, the intersection $\widetilde{L}$ of L with the co-isotropic subspace $y = 0$ is given by the equations $\frac{\partial f}{\partial x} = 0$. The intersection is transversal iff the rank of the matrix of the derivatives of these equations, with respect to x and q, is k. If so, the symplectic reduction of $\widetilde{L}$ is a Lagrangian submanifold L_r in $T^*\mathbf{R}^n$ (it may be not a section of $T^*\mathbf{R}^n \to \mathbf{R}^n$).

This example leads to the following definition of a generating function (the idea is due to Hörmander).

Definition. A generating family of the Lagrangian mapping of a submanifold $L \subset T^*N$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p, q) \quad \mid \quad \exists x : \frac{\partial F(x, q)}{\partial x} = 0, \quad p = \frac{\partial F(x, q)}{\partial q} \quad \right\}.$$

Here $q \in N$, and x is in the fibre over q. We also assume that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } \quad (x, q) \mapsto \frac{\partial F}{\partial x}.$$

The latter guarantees L being a smooth manifold.

Remark. The points of the intersection of L with the zero section of T^*N are in one-to-one correspondence with the critical points of the function F. In symplectic topology, when interested in such points, it is desirable to avoid a possibility of having no critical points at all (as it may happen on a non-compact manifold E).

So dealing with global generating families defining Lagrangian submanifolds globally, generating families with good behaviour at infinity should be considered.

A generating family F is said to be quadratic at infinity (QI) if it coincides with a fibre-wise quadratic non-degenerate form $Q(x, q)$ outside a compact.

On the topological properties of such families and on their role in symplectic topology see the papers by C.Viterbo, for example [4].

Existence and uniqueness (up to a certain equivalence relation) of QI generating families for Lagrangian submanifolds which are Hamiltonian isotopic to the zero section in T^*N of a compact N was proved by Viterbo, Laundeback and Sikorav in the 80s:

Given any two QI generating families for L, there is a unique integer m and a real ℓ such that $H^k(F_b, F_a) = H^{k-m}(F_{b-\ell}, F_{a-\ell})$ for any pair of $a < b$.

Here F_a is the inverse image under F of the ray $\{t \leq a\}$.

However, we shall need a local result which is older and easier.

Existence.
Any germ L of a Lagrangian submanifold in $T^*\mathbf{R}^n$ has a regular projection to some (p_J, q_I) co-ordinate space. In this case there exists a function $f = f(p_J, q_I)$ (defined up to a constant) such that

$$L = \left\{ \quad (p, q) \quad \mid \quad q_J = -\frac{\partial f}{\partial p_J}, \quad p_I = \frac{\partial f}{\partial q_I} \quad \right\}.$$

Then the family $F_J = xq_J + f(x, q_I)$, $x \in \mathbf{R}^{|J|}$, is generating for L. If $|J|$ is minimal possible, then $\mathrm{Hess}_{xx} F_J = \mathrm{Hess}_{p_J p_J} f$ vanishes at the distinguished point.

Uniqueness.
Two family-germs $F_i(x, q)$, $x \in \mathbf{R}^k$, $q \in \mathbf{R}^n$, $i = 1, 2$, at the origin are called $\mathcal{R}_0$-equivalent if there exists a diffeomorphism $\mathcal{T} : (x, q) \mapsto (X(x, q), q)$ (i.e. preserving the fibration $\mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$) such that $F_2 = F_1 \circ \mathcal{T}$.

The family $\Phi(x, y, q) = F(x, q) \pm y_1^2 \pm \ldots, \pm y_m^2$ is called a stabilisation of F.

Two family-germs are called stably $\mathcal{R}_0$-equivalent if they are $\mathcal{R}_0$-equivalent to appropriate stabilisations of the same family (in a lower number of variables).

Lemma. Up to addition of a constant, any two generating families of the same germ L of a Lagrangian submanifold are stably $\mathcal{R}_0$-equivalent.

Proof. Morse Lemma with parameters implies that any function-germ $F(x, q)$ (with zero value at the origin which is taken as the distinguished point) is stably $\mathcal{R}_0$-equivalent to $\widetilde{F}(y, q) \pm z^2$ where $x = (y, z)$ and the matrix $\mathrm{Hess}_{yy} \widetilde{F}|_0$ vanishes. Clearly $\widetilde{F}(y, q)$ is a generating family for L if we assume that $F(x, q)$ is.

Since the matrix $\partial^2 \widetilde{F}/\partial y^2$ vanishes at the origin, the Morse condition for $\widetilde{F}$ implies that there exists a subset J of indices such that the minor $\partial^2 \widetilde{F}/\partial y \partial q_J$ is not zero at the origin. Hence the mapping

$$\Theta : (y, q) \mapsto (p_J, q) = (\partial \widetilde{F}/\partial q_J, q)$$

is a local diffeomorphism. The family $G = \widetilde{F} \circ \Theta^{-1}$, $G = G(p_J, q)$, is also a generating family for L.

The variety $\partial \widetilde{F}/\partial y = 0$ in the domain of Θ is mapped to the Lagrangian submanifold L in the (p, q)-space by setting $p = \partial \widetilde{F}/\partial q$ and forgetting y. Therefore, the variety $X = \{\partial G/\partial p_J = 0\}$ in the (p_J, q)-space is the image of L under its (regular) projection $(p, q) \mapsto (p_J, q)$.

Compare now G and the standard generating family F_J defined above (with p_J in the role of x). We may assume their values at the origin coinciding. Then the difference $G - F_J$ has vanishing 1-jet along X. Since X is a regular submanifold, $G - F_J$ is in the square of the ideal $\mathcal{I}$ generated by the equations of X, that is by $\partial F_J/\partial p_J$.

The homotopy method applied to the family $A_t = F_J + t(G - F_J)$, $0 \le t \le 1$, shows that G and F_J are $\mathcal{R}_0$-equivalent. Indeed, it is clear that the homological equation

$$-\frac{\partial A_t}{\partial t} = F_J - G = \frac{\partial A_t}{\partial p_J} \dot{p}_J$$

has a smooth solution $\dot{p}_J$ since $F_J - G \in \mathcal{I}^2$ while the $\partial A_t/\partial p_J$ generate $\mathcal{I}$ for any fixed t. $\square$

2.2 Legendrian case

Definition. A generating family of the Legendrian mapping $\pi|_L$ of a Legendrian submanifold $L \subset PT^*(N)$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ \quad (p,q) \quad | \quad \exists x : \quad F(x,q) = 0\,, \quad \frac{\partial F(x,q)}{\partial x} = 0\,, \quad p = \frac{\partial F(x,q)}{\partial q} \quad \right\},$$

where $q \in N$ and x is in the fibre over q, provided that the following Morse condition is satisfied:

$$0 \text{ is a regular value of the mapping } (x,q) \mapsto \{F, \frac{\partial F}{\partial x}\}\,.$$

Definition. Two function family-germs $F_i(x,q)$, $i = 1, 2$, are called V-equivalent if there exists a fibre-preserving diffeomorphism $\Theta : (x,q) \mapsto (X(x,q), q)$ and a function $\Psi(x,q)$ not vanishing at the distinguished point such that $F_2 \circ \Theta = \Psi F_1$.

Two function families are called stably V-equivalent if they are stabilisations of a pair of V-equivalent functions (may be in a lower number of variables x).

Theorem. Any germ $\pi|_L$ of a Legendrian mapping has a generating family. All generating families of a fixed germ are stably V-equivalent.

Proof. For an n-dimensional N, we use the local model $\pi_0 : J^1N' \to J^0N'$, $N' = \mathbf{R}^{n-1}$, for the Legendrian fibration.

Consider the projection $\pi_1 : J^1N' \to T^*N'$ restricted to L. Its image is a Lagrangian germ $L_0 \subset T^*N$. If $F(x,q)$ is a generating family for L_0, then $F(x,q) - z$ considered as a family of functions in x with parameters $(q,z) \in J^0N' = N$ is a generating family for L and vice versa. Now the theorem follows from the Lagrangian result and an obvious property: multiplication of a Legendrian generating family by a function-germ not vanishing at the distinguished point gives a generating family. After multiplication by an appropriate function Ψ, a generating family (satisfying the regularity condition) takes the form $F(x,q) - z$ where (q,z) are local coordinates in N. □

Remarks.
A symplectomorphism φ preserving the bundle structure of the standard Lagrangian fibration $\pi : T^*\mathbf{R}^n \to \mathbf{R}^n$, $(q,p) \mapsto q$ has a very simple form

$$\varphi : (q,p) \mapsto \Big(Q(q), DQ^{-1*}(q)(p + df(q))\Big)\,,$$

where $DQ^{-1*}(q)$ is the dual of the derivative of the inverse mapping of the base of the fibration, $Q \circ \pi = \pi \circ \varphi$, and f is a function on the base.

To see this, it is sufficient to write in the coordinates the equation $\varphi_*\lambda - \lambda = df$.

The above formula shows that fibres of any Lagrangian fibration posses a well-defined affine structure.

Consequently, a contactomorphism ψ of the standard Legendrian fibration $PT^*\mathbf{R}^n \to \mathbf{R}^n$ acts by projective transformations in the fibres:

$$\psi : (q,p) \mapsto (Q(q), DQ^{-1*}(q)p)\,.$$

Hence, there is a well-defined projective structure on the fibres of any Legendrian fibration.

We also see that Lagrangian equivalences act on generating families as $\mathcal{R}$-equivalences $(x, q) \mapsto (X(x, q), Q(q))$ and additions of function in parameters q.

Legendrian equivalences act on Legendrian generating families just as $\mathcal{R}$-equivalences.

We see that the results of this section relate local singularities of caustics and wavefronts to those of discriminants and bifurcation diagrams of families of functions depending on parameters. In particular, this explains the famous results of Arnold and Thom on the classification of stable singularities of low-dimensional wavefronts by the discriminants of the Weyl groups.

The importance of the constructions introduced above for various applications is illustrated by the examples of the next section.

2.3 Examples of generating families

1. Consider a Hamiltonian $h : T^*\mathbf{R}^n \to \mathbf{R}$ which is homogeneous of degree k with respect to the impulses p: $h(\tau p, q) = \tau^k h(p, q)$, $\tau \in \mathbf{R}$.

An initial submanifold $W_0 \subset \mathbf{R}^n$ (initial wavefront) defines an exact isotropic $I \subset H_c = h^{-1}(c)$. Assume I is a manifold transversal to v_h. Put $c = 1$.

The exact Lagrangian flow-invariant submanifold $L = exp_h(I)$ is a cylinder over I with local coordinates $\alpha \in I$ and time t from a real segment (on which the flow is defined).

Assume that in a domain $U \subset T^*R^n \times \mathbf{R}$ the restriction to L of the phase flow g_t of v_h is given by the mapping $(\alpha, t) \mapsto (Q(\alpha, t), P(\alpha, t))$ with $\frac{\partial P}{\partial \alpha, t} \neq 0$. Then the following holds.

Proposition. a) The family $F = P(\alpha, t)(q - Q(\alpha, t)) + kt$ of functions in α, t with parameters $q \in \mathbf{R}^n$ is a generating family of L in the domain U.

b) For any fixed t, the family $\widetilde{F}_t = P(\alpha, t)(q - Q(\alpha, t))$ is a Legendrian generating family of the momentary wavefront W_t.

The proof is an immediate verification of the Hörmander definition using the fact that value of the form pdq on each vector tangent to $g_t(I)$ vanishes and on the vector v_h it is equal to $p\frac{\partial h}{\partial p} = kh = k$.

2. Let $\varphi : T^*\mathbf{R}^n \to T^*\mathbf{R}^n$, $(q, p) \mapsto (Q, P)$ be a symplectomorphism close to the identity. Thus the system of equations $q' = Q(q, p)$ is solvable for q. Write its solution as $q = \widetilde{q}(q', p)$.

Assume the Lagrangian mapping of a Lagrangian submanifold L has a generating family $F(x, q)$. Then the following family G of functions in x, q, p with parameters q' is a generating family of $\varphi(L)$:

$$G(x, p, q; q') = F(x, \widetilde{q}) + p(\widetilde{q} - q) + S(p, q') .$$

Here $S(q', p)$ is the "generating function" in the sense of Hamiltonian mechanics of the canonical transformation φ, that is

$$dS = PdQ - pdq\,.$$

Notice that, if φ coincides with the identity mapping outside a compact, then G is a quadratic form at infinity with respect to the variables (q, p).

The expression $p(\tilde{q} - q) + S(p, q')$ from the formula above is the generating family of the symplectomorphism φ.

3. Represent a symplectomorphism φ of $T^*\mathbf{R}^n$ into itself homotopic to the identity as a product of a sequence symplectomorphisms each of which is close to the identity. Iterating the previous construction, we obtain a generating family of $\varphi(L)$ as a sum of the initial generating family with the generating families of each of these transformations. The number of the variables becomes very large, $\dim(x) + 2mn$, where m is the number of the iterations. Namely, consider a partition of the time interval $[0, T]$ into m small segments $[t_i, t_{i+1}]$, $i = 0, \ldots, m-1$. Let $\varphi = \varphi_m \circ \varphi_{m-1} \circ \ldots \varphi_1$ where $\varphi_i : (Q_i, P_i) \mapsto (Q_{i+1}, P_{i+1})$ is the flow map on the interval $[t_i, t_{i+1}]$. Then the generating family is

$$G(x, Q, P, q) = F(x, Q_0) + \sum_{i=0}^{m-1} \left(P_i(U({}_iQ_{i+1}, P_i) - Q_i) + S_i(P_i, Q_{i+1})\right)\,,$$

where:

$Q = Q_0, \ldots,\ Q_{m-1},\ q = Q_m, \quad Q_i \in \mathbf{R}^n, \quad q \in \mathbf{R}^n,$

S_i is a generating function of φ_i,

$U_i(Q_{i+1}, P_i)$ are the solutions of the system of equations $Q_{i+1} = Q_{i+1}(Q_i, P_1)$ defined by φ_i.

One can show that if φ is a flow map for time $t = 1$ of a Hamiltonian function which is convex with respect to the impulses then the generating family G is also convex with respect to the P_i and these variables can be removed by the stabilisation procedure. This provides a generating family of $\varphi(L)$ depending just on x, Q, q which are usually taken from a compact domain. Therefore, the function attains minimal and maximal values on the fibre over point q, this property being important in applications.

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