Functions on space curves

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Abstract

We classify simple singularities of functions on space curves. We show that their bifurcation sets have properties very similar to those of functions on smooth manifolds and complete intersections [3, 4]: the $k(\pi, 1)$-theorem for the bifurcation diagram of functions is true, and both this diagram and the discriminant are Saito's free divisors.

More than 25 years ago Arnold started a tradition in singularity theory to ask a couple of standard questions every time when a new classification problem is considered: is the complement to a bifurcation variety of a simple singularity an Eilenberg-MacLane $k(\pi, 1)$-space? is a bifurcation variety of a singularity a free divisor? In almost all decent situations the answers to these questions are positive. For example, this is usually so when a function is involved: for discriminants and bifurcation diagrams of functions in the study of functions on smooth manifolds and manifolds with boundary, for bifurcation diagrams of functions on isolated complete intersection singularities [3, 4].

In the present paper we answer positively similar questions in the case of functions on space curves. Our initial interest in such functions was motivated by a search for singularity theory realisations of Hurwitz spaces, that is, of moduli spaces of functions on closed complex curves with given orders of poles. A classical example of such a space is the miniversal deformation of the A_k function singularity which provides a moduli space of rational functions with one pole (that is, polynomials in one variable), of order $k+1$. Similarly, miniversal deformations of simple functions on one-dimensional complete intersections [13] are convenient tools to study Laurent polynomials (see [2] for details) and elliptic functions with one pole. Naturally, the next in the line are functions on space curves, since a space curve singularity is sufficiently easy to handle: the base of its miniversal deformation is smooth.

The paper is organised as follows.

In Sections 1 and 2 we introduce an equivalence relation for functions on space curves and classify simple singularities.

In Section 3 we define analogs of Tjurina and Milnor numbers for our functions.

In Section 4 we list versal deformations of simple singularities as partial closures of certain Hurwitz spaces. We make use of this interpretation in Section 6.

Section 5 on the geometry of the bifurcation sets contains theorems on the free divisors and gives algorithms to construct basic vector fields tangent to the sets. It also contains Lyashko-Looijenga type theorem on bifurcation diagrams of functions of the simple singularities.

Calculation of the degree of the Lyashko-Looijenga mapping yields answers in some particular cases of the Hurwitz problem on enumeration of topologically distinct Morse meromorphic functions on closed curves with fixed orders of poles and fixed critical values (Section 6). In all our cases the answers were known earlier from combinatorial considerations [17, 22, 20]. Our approach looks promising for similar enumeration of the corresponding non-Morse functions: basically, one has to calculate the degree of the Lyashko-Looijenga mapping on the relevant stratum.

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1 Space curves

A germ of any reduced space curve is a determinantal variety: it is the zero set of all order n minors of a germ of some $n \times (n+1)$-matrix M on $\mathbf{C}^3$. In what follows we often treat a space curve as such a matrix.

Definition 1.1 *Two curve-germs at the origin, given by matrices M and M', are said to be* equivalent *if there exist two invertible matrix-germs A and B and a biholomorphism-germ h of $(\mathbf{C}^3, 0)$ such that*

$$AMB = M' \circ h\,.$$

We say that *corank* of a space curve-germ is c if the rank at the origin of its defining matrix is $n-c$. Obviously, such a curve can be given by a $c \times (c+1)$-matrix which is just the zero matrix at the origin.

Classification of simple space curves within this approach has been obtained in [10]. As a subset it contains simple plane A, D, E-curves and Giusti's list of simple 1-dimensional complete intersections in the 3-space [11, 12, 4]. Those are corank 1 curves. All the other simple space curves have corank 2. We are not reproducing the whole list since we will not need it.

2 Simple function singularities

Definition 2.1 A function on a space curve *is a pair of germs* (M, f) *on* $\mathbf{C}^3$*, where* M *is an* $n \times (n+1)$*-matrix and* f *a function.*

Definition 2.2 *Two germs of functions on curves at the origin,* (M, f) *and* (M', f')*, are* $\mathcal{R}_c$-equivalent *if there exist two invertible matrix-germs* A *and* B*, a biholomorphism-germ* h *of* $(\mathbf{C}^3, 0)$ *and a function* g *from the ideal generated by the maximal minors of* M *such that*

$$(AMB, f+g) = (M' \circ h, f' \circ h) .$$

Notation $\mathcal{R}_c$ is chosen to indicate that this is a sort of right equivalence of functions, this time restricted to curves.

The notion of $\mathcal{R}_c$-equivalence satisfies all the conditions of Damon's good geometrical equivalence [8, 9]. Thus all the standard theorems like those of versality and finite determinacy are valid in our case. We can apply traditional technique [5, 3] to classify singularities and construct their $\mathcal{R}_c$-versal deformations.

One of our aims is to classify $\mathcal{R}_c$-simple singularities. The $\mathcal{R}_c$-simplicity requires the participating curve to be simple as a determinantal variety. Thus we have to study functions either on simple plane curves, or on non-planar 1-dimensional complete intersections in $\mathbf{C}^3$, or on determinantal space curves of corank 2. The second case is easily shown to contain no $\mathcal{R}_c$-simple functions. The simple lists for the two others are given below.

2.1 Functions on plane curves

Theorem 2.3 [13] *The complete list of $\mathcal{R}_c$-simple functions on plane curves is as follows:*

notation	curve singularity	function	restrictions
A_k	$A_0 : \ y = 0$	x^{k+1}	$k \geq 0$
$C_{p,q}$	$A_1 : \ xy = 0$	$x^p + y^q$	$p \geq q \geq 1$
B_k	$A_{k-1} : \ x^2 + y^k = 0$	y	$k \geq 3$
F_k	$A_2 : \ x^2 + y^3 = 0$	$\begin{cases} y^r \\ xy^r \end{cases}$	$k = 2r + 1 \geq 5$ $k = 2r + 4 \geq 4$

In the exposition we shall take care of not making any confusion between curves A_k and our functions A_k.

All the adjacencies of the listed functions are compositions of the adjacencies within the series obtained by reducing the indices (we additionally set $B_2 = C_{1,1}$, $F_3 = B_3$) and

$$F_{p+q+1} \to C_{p,q} \to A_{p+q-1} \ .$$

There are just two bounding functions:

$$\begin{array}{llll} X_9^* : & x + \alpha y^2 & \text{on} & x^2 + y^4 = 0 \\ J_{10}^* : & x + \alpha y & \text{on} & x^3 + y^3 = 0 \ . \end{array}$$

Here α is a generic complex number (modulus). Both singularities are adjacent to B_4 and F_4. Also X_9^* is adjacent to $C_{3,1}$.

Remark 2.4 Singularity J_{10}^* is a generic function on a D_4 curve. Its non-simplicity guarantees absence of $\mathcal{R}_c$-simple functions on any curve singularity adjacent to D_4.

2.2 Functions on space curves

Theorem 2.5 *The list of simple functions on determinantal space curves of corank 2 is as follows:*

notation	curve singularity	function	restrictions
$C_{p,q,r}$	$A_1^L : \begin{vmatrix} x & y & 0 \\ 0 & y & z \end{vmatrix}$	$x^p + y^q + z^r$	$p \geq q \geq r \geq 1$
$\dot{F}_k$	$A_2^L : \begin{vmatrix} x & y & 0 \\ y^2 & x & z \end{vmatrix}$	$z + \begin{cases} y^r \\ xy^r \end{cases}$	$k = 2r + 3 \geq 5$ $k = 2r + 6 \geq 6$
$\check{E}_6$	$E_6' : \begin{vmatrix} x & y & z \\ z^2 & x & y \end{vmatrix}$	z	–

Here the notation of the curve singularities a bit differs from that in [10].

Since the classification technique is very similar to that used in numerous earlier classifications we do not give the proof of this theorem here. The only point we would like to briefly explain is the absence of other simple functions. There are two reasons for this:

(a) Figure 1 shows the hierarchy of space curves not adjacent to D_4 [10], that is, of those on which there can exist $\mathcal{R}_c$-simple functions due to Remark 2.4. In the figure, the curves with the same Tjurina number are in the same column.

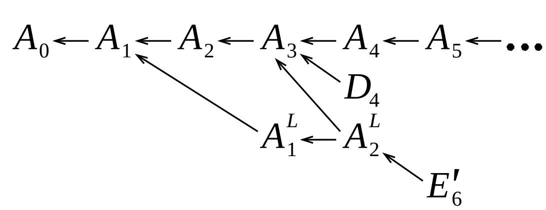

Figure 1: *The initial part of the hierarchy of space curves*

(b) The list of the theorem contains all finitely degenerate functions on the curve A_1^L. The list also contains all finitely degenerate functions on A_2^L and E_6' with z in the linear part. The latter condition is necessary for $\mathcal{R}_c$-simplicity: the evolution of the tangent cones to the curves in the deformations $E_6' \to A_2^L \to A_3$ (see [10]) shows that otherwise a function on A_2^L or E_6' is adjacent to the non-simple singularity X_9^* (which is a generic function on the space with an A_3 curve, such that the tangent cone to the curve is not transversal to the critical level of the function).

The above shows that the two planar bounding singularities, X_9^* and J_{10}^*, of the previous subsection serve as a complete bounding list in the corank ≤ 2 case as well.

One obtains obvious adjacencies of the singularities of Theorem 2.5 reducing the indices in the series. Some other adjacencies showing the relations between the series are:

$$C_{p,q,r} \to C_{p,q} \qquad \dot{F}_k \to F_{k-2} \qquad \dot{F}_{p+q+3} \to C_{p,q,1} \qquad \check{E}_6 \to \dot{F}_5$$

3 Tjurina number and Milnor number

Let (x_1, x_2, x_3) be coordinates in $(\mathbf{C}^3, 0)$ and $\mathcal{O}_3$ the space of holomorphic function-germs on it. Choosing an ordering of entries of pairs consisting of an $n \times (n+1)$-matrix and a function, we identify the space of their germs with the module $\mathcal{O}_3^{n(n+1)+1}$. The tangent space $T(M, f)$ to the (extended) $\mathcal{R}_c$-equivalence class of a germ (M, f) in this module is the $\mathcal{O}_3$-module generated by the elements

$$\begin{aligned}
(E_{ij}^n M, 0)\,, &\qquad i, j = 1, \ldots, n, \\
(M E_{kl}^{n+1}, 0)\,, &\qquad k, l = 1, \ldots, n+1, \\
(0, \varphi_r)\,, &\qquad r = 1, \ldots, n+1, \\
(\partial M/\partial x_s, \partial f/\partial x_s)\,, &\qquad s = 1, 2, 3
\end{aligned}$$

Here E_{ij}^n is the $n \times n$-matrix having 1 at the intersection of the ith row with the jth column and zeros everywhere else. The φ_r are the maximal minors of the matrix M.

We set $\tau(M,f)$ to be the dimension of the linear space $\mathcal{O}_3^{n(n+1)+1}/T(M,f)$ and call it the *Tjurina number* of the function singularity. This is the dimension of the base of an $\mathcal{R}_c$-miniversal deformation of (M,f). Such a deformation can be taken in the form

$$(M,f) + \sum_{i=0}^{\tau-1} \lambda_i e_i \ ,$$

where the λ_i are the parameters, and the e_i are elements of $\mathcal{O}_3^{n(n+1)+1}$ which project to a linear basis of the quotient $\mathcal{O}_3^{n(n+1)+1}/T(M,f)$.

For singularities $C_{p,q}$ and $C_{p,q,r}$ we have $\tau = p+q$ and $\tau = p+q+r+1$ respectively. For all the other simple singularities of our lists, τ is the subscript in the notation. Also $\tau(X_9^*) = \tau(J_{10}^*) = 6$.

Another characteristic of a function-germ on a curve is the number $\mu(M,f)$ of Morse critical points which a generic small perturbation of f has on a generic smoothing of curve M. We call $\mu(M,f)$ the *Milnor number* of the singularity.

Algebraically, $\mu(M,f)$ is the dimension of the linear space $\mathcal{O}_3/J$, where J is the ideal generated by the maximal minors $\varphi_1,\ldots,\varphi_{n+1}$ of M and by all maximal minors of the Jacobi matrix $\partial(\varphi_1,\ldots,\varphi_{n+1},f)/\partial(x_1,x_2,x_3)$. In fact, it is sufficient to take only those Jacobian minors which involve f: all the others are in the ideal generated by the φ_i.

Conjecture 3.1 $\qquad \tau(M,f) = \mu(M,f)$.

The conjecture is true for functions on complete intersections [13], for $\mathcal{R}_c$-simple singularities and in some other particular cases.

4 $\mathcal{R}_c$-versal deformations as moduli spaces of meromorhpic functions

For our further considerations we need explicit $\mathcal{R}_c$-miniversal deformations of the simple singularities. Instead of just straightforward writing out lengthy families of functions, we show in this section how to arrive at the same families from a geometrical point of view treating their *generic* members as meromorphic functions on smooth curves of low genus. As a result, the versal families come out as partial closures of moduli spaces of functions on curves with fixed orders of the poles. This will be helpful for the Hurwitz problem of enumeration of meromorphic functions considered in Section 6.

Below all the Greek letters and non-fixed coefficients in the polynomials are parameters of a deformation of a function on a curve. In correspondence with our further notation, we denote the free term of all the functions by λ_0. Each of the versal families obtained is quasihomogeneous, with all the arguments and parameters of positive weights. Therefore, each family is defined globally, on the whole of the complex linear space of the arguments and on the whole of the complex linear space of the parameters (not just on the germs of the two). Each of the deformations contains a versal deformation of the corresponding curve.

4.1 Functions on the complex line

$\mathbf{A_k}$. Any function on the complex line with just one pole, of order $k+1$, is equivalent to a polynomial

$$x^{k+1} + \lambda_{k-1}x^{k-1} + \ldots \lambda_1 x + \lambda_0 \ . \tag{1}$$

The family of all such polynomials is a versal deformation of function A_k.

$\mathbf{C_{p,q}}$. Any function on the complex line with two poles, of orders p and q, is equivalent to a member of the family of functions on the family of curves:

$$f_p(x) + g_q(y) + \lambda_0 \quad \text{on} \quad xy = \varepsilon \ , \tag{2}$$

where f_p and g_q are arbitrary monic polynomials with no free term, of degrees p and q respectively. This is a miniversal deformation of the $C_{p,q}$ singularity.

$\mathbf{C_{p,q,r}}$. Similarly (see [16]), any rational function with 3 poles, of orders p, q and r, is a member of the miniversal deformation of function $C_{p,q,r}$ which is

$$f_p(x) + g_q(y) + h_r(z) + \lambda_0 \quad \text{on} \quad \begin{vmatrix} x & y & \alpha \\ \beta & y+\gamma & z \end{vmatrix} \ . \tag{3}$$

Again f_p, g_q, h_r are monic polynomials of the corresponding degrees, with no free term. The discriminant of the curve A_1^L participating in this series of function singularities is $\alpha \cdot \beta \cdot \gamma = 0$.

4.2 Elliptic functions

$\mathbf{F_{n+1}}$. A genus 1 curve with a marked point is a plane curve

$$x^2 + y^3 + \alpha y + \beta = 0\ , \quad \alpha, \beta \in \mathbf{C}\,, \tag{4}$$

with its infinite point marked. Function y on (4) has a pole of order 2 at infinity, and x a pole of order 3. Therefore, any degree n elliptic function with only one pole is a member of a family of functions with the support of one of the two types shown in Figure 2. In the first case $n = 2r$, and in the second $n = 2r + 3$. The coefficient of the black monomial can be reduced to 1 by the quasihomogeneous rescaling allowed by a spare degree of freedom in (4).

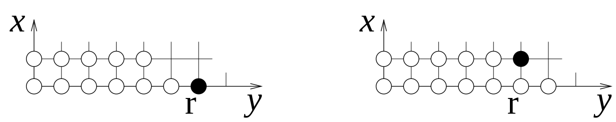

Figure 2: *Supports of elliptic functions with one pole.*

The family of functions of Figure 2 on the family of curves (4) is an $\mathcal{R}_c$-miniversal deformation of the F_{n+1} singularity.

$\dot{\mathbf{F}}_{\mathbf{n+3}}$. An elliptic curve with two distinct marked points is a curve

$$x(x+\gamma) - y(y^2 + \beta y + \delta) = 0\,, \tag{5}$$

with the infinity and origin marked. Up to an additive constant, a function on such curve with two simple poles at the marked points is $z = \alpha(x+\gamma)/y$ (see Figure 3). This time there is no freedom in (5) to normalise α to 1.

Function z lifts the planar curve (5) to a space curve

$$\left| \begin{array}{ccc} x & y & \alpha \\ y^2 + \beta y + \delta & x + \gamma & z \end{array} \right| . \tag{6}$$

The obtained family is a versal deformation of the space curve A_2^L.

Let $f_n(x, y)$ be a function with the support as in Figure 2 and n calculated as for the previous series of singularities. Now a function on an elliptic curve

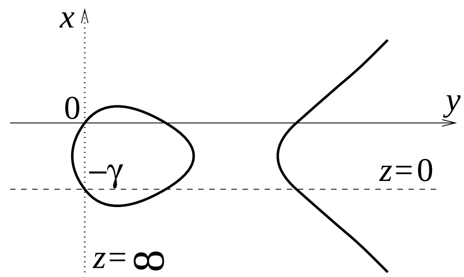

Figure 3: *Construction of an elliptic function with two simple poles.*

with two poles, of orders 1 and $n > 1$, is a function $z + f_n(x, y)$ on a curve (6). The family of all such functions on the family of all such curves is an $\mathcal{R}_c$-miniversal deformation of the singularity $\dot{F}_{n+3}$.

The case of two simple poles is covered by the next series (singularity B_3).

Remark 4.1 The discriminant of the family (6) is

$$\alpha \cdot (27\gamma^4 - 72\beta\gamma^2\delta - 16\beta^2\delta^2 + 16\beta^3\gamma^2 + 64\delta^3) = 0 \tag{7}$$

If $\alpha = 0$, the curve is a plane cubic wedged with a straight line transversal to the plane. The second factor in (7) is the discriminant of the plane family (5) which coincides with the discriminant of function A_3.

4.3 Degree 2 functions

$\mathbf{B_k}$. To carry a degree 2 function a curve must be hyperelliptic:

$$x^2 + y^k + \alpha_{k-2}y^{k-2} + \ldots \alpha_1 y + \alpha_0 = 0\,. \tag{8}$$

Function $y + \lambda_0$ on it has either two simple poles (for k even) or one order 2 pole (for k odd) at the infinity. Here again we can assume the coefficient of y in the function to be quasihomogeneously scaled down to 1.

The family of functions $y + \lambda_0$ on the family of curves (8) is an $\mathcal{R}_c$-miniversal deformation of the function B_k.

4.4 Degree 3 functions on genus 2 curves

$\check{\mathbf{E}}_6$. A genus 2 curve with a marked non-Weierstrass point is a hyperelliptic curve

$$Y^2 = X^6 + aX^4 + bx^3 + cX^2 + dX + e \ , \quad a, \ldots, e \in \mathbf{C} \ , \tag{9}$$

with one of the two points at the infinity marked.

The same curve can be written as a member of the miniversal deformation

$$\begin{vmatrix} x+\delta & y & z \\ z^2 + \alpha z + \beta + \gamma y & x & y + \varepsilon \end{vmatrix} \tag{10}$$

of the E_6' curve singularity with the marked point being the infinite one. The transformation from (9) to (10) is

$$\begin{array}{rcl} z & = & (\pm 2Y + 2X^3 + aX + b)/4 \\ y & = & zX + c/4 - a^2/16 \\ x & = & yX \ , \end{array} \tag{11}$$

with the parameter settings

$$a = -2\gamma \quad b = -2\alpha \quad c = -4\varepsilon + \gamma^2 \quad d = 4\delta + 2\alpha\gamma \quad e = -4\beta + 4\gamma\varepsilon + \alpha^2 \ . \tag{12}$$

The sign choice for z is the choice of one of the two infinite points of curve (9) as marked.

Now any degree 3 meromorphic function on a genus 2 curve with only one pole is a function $z + \lambda_0$ on a curve (10). We can normalise the coefficient of z in the function to be 1. The function family $z + \lambda_0$ on the curve family (10) is an $\mathcal{R}_c$-miniversal deformation of the function singularity $\check{E}_6$.

Remark 4.2 The relation (12) between the two families, (9) and (10), shows that the discriminant in the versal deformation of the curve E_6' is the bifurcation diagram of zeros of function A_5.

5 Geometry of bifurcation sets

5.1 Discriminant as a free divisor

Definition 5.1 *Consider the base $\mathbf{C}^\tau$ of an $\mathcal{R}_c$-miniversal deformation of singularity (M, f). The* discriminant *$\Delta(M, f) \subset \mathbf{C}^\tau$ of (M, f) is the set* *of those values of the deformation parameters for which the function on the curve has critical value 0.*

Consideration of the $C_{1,1}$ singularity shows that, within this definition and in what follows, a singular point of a curve must be treated as critical for a function on the curve.

We recall that a hypersurface H in N-dimensional complex linear space is called *a free divisor* if the algebra Θ_H of vector fields on $\mathbf{C}^N$ tangent to H (that is, preserving its ideal) is generated by N elements as a module over functions on $\mathbf{C}^N$.

Theorem 5.2 *Assume $\tau(M,f) = \mu(M,f)$. Then the discriminant $\Delta(M,f) \subset \mathbf{C}^\tau$ is a free divisor.*

Proof (cf. [24, 25, 26, 14, 15, 4]). Let $(\mathcal{M},F) = (\mathcal{M}(x,\lambda), F(x,\lambda))$ be an $\mathcal{R}_c$-miniversal deformation of (M,f), with $\lambda = (\lambda_0,\ldots,\lambda_{\tau-1}) \in \mathbf{C}^\tau$ being the parameters. For any $i = 0,\ldots,\tau-1$, due to the versality there exists a decomposition

$$F\frac{\partial}{\partial\lambda_i}(\mathcal{M},F) = (\mathcal{A}_i\mathcal{M}\mathcal{B}_i, \mathcal{G}_i) + \sum_{s=1}^{3} h_{is}\frac{\partial}{\partial x_s}(\mathcal{M},F) + \sum_{j=0}^{\tau-1} v_{ij}\frac{\partial}{\partial\lambda_j}(\mathcal{M},F)\ ,$$

where $\mathcal{A}_i(x,\lambda)$ and $\mathcal{B}_i(x,\lambda)$ are matrix-germs, $h_{is}(x,\lambda)$ and $v_{ij}(\lambda)$ function-germs, and $\mathcal{G}_i(x,\lambda)$ is an element of the ideal generated by the maximal minors of $\mathcal{M}$ in the ring of functions in x,λ. A functional factor or differentiation in front of a pair (matrix, function) means multiplication by the function or the differentiation of both items.

The vector fields $\nu_i = \sum_{j=0}^{\tau-1} v_{ij}(\lambda)\partial_{\lambda_j}$ are tangent to Δ. Assume that the deformation $(\mathcal{M},F)$ is monomial and λ_0 is the free term of F. Then it is easily verified that $\det(v_{ij}(\lambda_0,0,\ldots,0)) = \lambda_0^\tau$. On the other hand, $\det(v_{ij})$ has to vanish on Δ, and hence is proportional to its defining equation. Since the latter is a polynomial of degree $\mu = \tau$ in λ_0, $\det(v_{ij}) = 0$ may be taken to be such an equation itself. This implies that the vector fields $\nu_0,\ldots,\nu_{\tau-1}$ generate Θ_Γ as a free module over $\mathcal{O}_\tau$. □

Thus, for example, the discriminant of any $\mathcal{R}_c$-simple singularity is a free divisor. Modulo Conjecture 3.1 this is true for any singularity of finite $\mathcal{R}_c$-codimension.

5.2 Bifurcation diagram of functions as a free divisor

Consider a *truncated* $\mathcal{R}_c$-miniversal deformation $(\mathcal{M}, F')$ of a singularity (M, f), that is, one allowing just functions vanishing at $0 \in \mathbf{C}^3$. Its base is of dimension $\tau(M, f) - 1$. Note that in this case the deformation $(\mathcal{M}, F' + \lambda_0)$, where λ_0 is an additional parameter, is $\mathcal{R}_c$-miniversal for (M, f).

Definition 5.3 The bifurcation diagram of functions $\Sigma(M, f) \subset \mathbf{C}^{\tau-1}$ *is the set of those values of parameters of the truncated deformation for which either the corresponding curve is not smooth or the function on it has either a degenerate critical point or at least two critical points on the same level.*

In general, Σ has three irreducible components responsible for the three degenerations mentioned in the definition.

Theorem 5.4 *Assume* $\tau(M, f) = \mu(M, f)$*. Then the bifurcation diagram of functions* $\Sigma(M, f) \subset \mathbf{C}^{\tau-1}$ *is a free divisor.*

A proof of this statement is absolutely similar to those for functions on smooth manifolds [7, 23] and for functions on complete intersections [14, 15]. The generators of Θ_Σ

$$\omega_i = \sum_{j=1}^{\tau-1} w_{ij}(\lambda')\partial_{\lambda_j}\,, \qquad i = 1, \ldots, \tau - 1, \qquad \lambda' = (\lambda_1, \ldots, \lambda_{\tau-1}) \in \mathbf{C}^{\tau-1}\,,$$

are obtained from the decompositions

$$(0, F'^i) = (\mathcal{A}_i' \mathcal{M} \mathcal{B}_i', \mathcal{G}_i') + \sum_{s=1}^{3} h_{is}' \frac{\partial}{\partial x_s}(\mathcal{M}, F') + (0, w_{i0}) + \sum_{j=1}^{\tau-1} w_{ij} \frac{\partial}{\partial \lambda_j}(\mathcal{M}, F')\,,$$

where the $\mathcal{A}_i'(x, \lambda')$ and $\mathcal{B}_i'(x, \lambda')$ are matrix-germs, $h_{is}'(x, \lambda')$ and $w_{ij}(\lambda')$ function-germs, and $\mathcal{G}_i'(x, \lambda')$ is an element of the ideal generated by the maximal minors of $\mathcal{M}$.

Example 5.5 As Section 4 implies, the singularity $C_{1,1,1}$ has a truncated $\mathcal{R}_c$-miniversal deformation

$$\left(\left| \begin{array}{ccc} x & y & \alpha \\ \beta & y+\gamma & z \end{array} \right|, \quad x + y + z \right).$$

The algebra of vector fields on $\mathbf{C}^3$ tangent to the bifurcation diagram $\Sigma(C_{1,1,1})$ is a free module over $\mathcal{O}_3$ generated by the fields of degree 1 (the Euler field), 2 and 3.

The projectivisation of $\Sigma(C_{1,1,1})$ in $\mathbf{C}P^2$ is a nodal cubic with its three tangent lines at the inflection points (thus, as expected, its degree is the sum of the degrees of the basic fields). The cubic corresponds to functions with degenerate critical points on smooth curves and the three lines $\alpha \cdot \beta \cdot \gamma = 0$ to non-smooth curves. On the left-hand side of Figure 4 we show this projectivisation (the isolated point in the centre is the node of the cubic). On the right-hand side of the same figure there is given the bifurcation diagram in $\mathbf{R}^3$ of the other obvious real version of the complex singularity $C_{1,1,1}$: the isolated straight real line inside the cone bounded by the cubic is the intersection of the two conjugate planes tangent to the cubic along its complex parabolic lines.

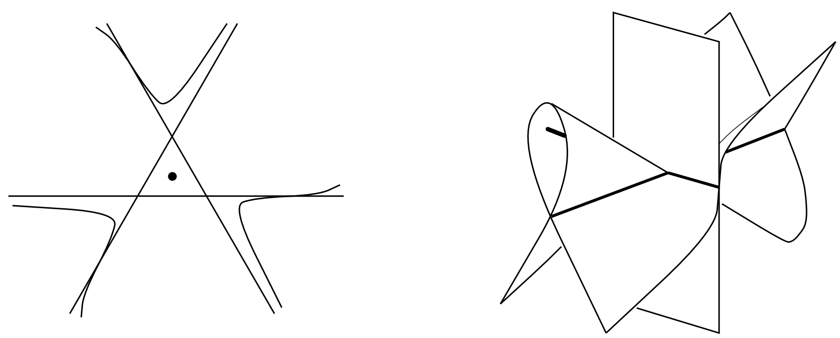

Figure 4: *Two real versions of the bifurcation diagram of complex singularity* $C_{1,1,1}$.

5.3 Lyashko-Looijenga mapping

Let $\mathbf{C}^{\mu-1}$ be the space of all monic polynomials in one variable of degree μ with vanishing sum of the roots, and $\Xi \subset \mathbf{C}^{\mu-1}$ the set of polynomials with multiple roots.

For a singularity (M, f), with its Tjurina number equal to its Milnor number μ, consider the mapping

$$\mathbf{C}^{\mu-1} \setminus \Sigma(M,f) \to \mathbf{C}^{\mu-1} \setminus \Xi$$

from the complement of the bifurcation diagram of functions which sends a Morse function on a smooth curve to the unordered set of its critical values shifted by their arithmetic mean, that is, to the monic polynomial whose roots are the shifted critical values. This mapping is easily verified to be extendible to that between the ambient complex linear spaces [13]. Again, the extension counts every singular point of a curve to be critical for a function on the curve. We call the extension *the Lyashko-Looijenga mapping* and denote it by LL.

Theorem 5.6 *For an $\mathcal{R}_c$-simple function on a space curve, the Lyashko-Looijenga mapping is a finite covering. As a mapping from $\mathbf{C}^{\mu-1} \setminus \Sigma$ to $\mathbf{C}^{\mu-1} \setminus \Xi$ it has no branching.*

Since the space $\mathbf{C}^{\mu-1} \setminus \Xi$ is a classification space of the Artin braid group $B(\mu)$ on μ threads, we get

Corollary 5.7 *For an $\mathcal{R}_c$-simple singularity (M,f), the complement to its bifurcation diagram of functions in the base of its trunkated $\mathcal{R}_c$-miniversal deformation is a $k(\pi,1)$-space, where π is a subgroup of finite index in the group of braids on $\mu(M,f)$ threads.*

Both statements may be understood not on the germ level only, but also globally, as related to the truncations of the quasihomogeneous $\mathcal{R}_c$-miniversal deformations of Section 4.

The index $[B(\mu):\pi]$ is the degree of the mapping $LL : \mathbf{C}^{\mu-1} \to \mathbf{C}^{\mu-1}$. The latter is finite quasihomogeneous for a quasihomogeneous deformation, and hence its degree is the ratio of the products of the weights of the coordinate functions and of the arguments. Using the truncations of the deformations of Section 4, we obtain the following indices:

A_k	$C_{p,q}$	F_k	B_k
$(k+1)^{k-1}$	$\frac{(p+q-1)!\,p^p\,q^q}{(p-1)!\,(q-1)!}$	$\frac{(k-2)(k-1)^k k}{24}$	1

$C_{p,q,r}$	$\dot{F}_k$	$\check{E}_6$
$\frac{(p+q+r+1)!\,p^p\,q^q\,r^r}{(p-1)!\,(q-1)!\,(r-1)!}$	$\frac{(k-3)^k(k-2)(k-1)k}{24}$	3^5

Remark 5.8 The theorem and its corollary are analogous to the classical theorem on simple functions on smooth manifolds [1, 19, 6, 3] and generalise similar assertions about simple functions on plane curves [13, 4] (the latter provide the upper half of the index table). The assertion for the $C_{p,q,r}$ series is a particular case of the Lyashko-Looijenga type theorem for rational functions proved in [16]. Therefore, only $\dot{F}_r$ and $\check{E}_6$ are really new cases here.

Proof of Theorem 5.6. Consider the version $\widetilde{LL} : \mathbf{C}^\mu \to \mathbf{C}^\mu$ of the Lyashko-Looijenga mapping defined on the base of a non-truncated miniversal deformation of an $\mathcal{R}_c$-simple function singularity (M, f) and sending a member of the family to the monic degree μ polynomial whose roots are non-shifted critical values of the function. The base $\mathbf{C}^\mu$ contains extended bifurcation diagram of functions $\widetilde{\Sigma}(M, f)$ defined in the obvious way and isomorphic to $\Sigma(M, f) \times \mathbf{C}$. The dicriminant $\widetilde{\Xi}$ in the target polynomial space $\mathbf{C}^\mu$ is similarly cyllindrical: $\widetilde{\Xi} \cong \Xi \times \mathbf{C}$.

The theorem is equivalent to the analogous claim for the mapping $\widetilde{LL}$ and we shall prove the latter. The proof will be carried out for the deformations of Section 4 defined globally.

As in the versions of the theorem mentioned in the last remark, we have to prove two facts:

1) mapping $\widetilde{LL}$ is proper,

2) mapping $\widetilde{LL}$ is a local diffeomorphism out of $\widetilde{\Sigma}$.

1) Since the weights of all the variables in the miniversal deformations of Section 4 are positive, it is sufficient to show that $\widetilde{LL}^{-1}(0) = 0$.

Consider a member (M_λ, f_λ) of the miniversal deformation of an $\mathcal{R}_c$-simple singularity. Denote by $\Gamma_\lambda \subset \mathbf{C}^3$ the curve defined by M_λ. Assume f_λ has only one critical value $0 \in \mathbf{C}$ on Γ_λ.

Take an arbitrary regular value $w \in \mathbf{C}$ of f_λ. Join it with 0 by the straight path I. The inverse image $f_\lambda^{-1}(I) \subset \Gamma_\lambda$ is a disjoint union of wedges of intervals, with the wedge points being the critical ponts of f_λ. On the other hand, $f_\lambda^{-1}(I)$ is a retract of the connected curve Γ_λ. Hence, $f_\lambda^{-1}(I)$ is just one wedge. Therefore, Γ_λ is contractible and has at most one singular point. For the curves participating in the $\mathcal{R}_c$-simple singularities this means that Γ_λ is just the undeformed curve. Now a quick check of the miniversal deformations of Section 4 shows that a function on the undeformed curve with a single critical value 0 is the undeformed function.

2) Consider the critical values $c_0, \ldots, c_{\mu-1}$ of the members (M_λ, f_λ) of a miniversal family $(\mathcal{M}, F)$. Out of the diagram $\widetilde{\Sigma}$, each of the c_i is locally a holomorphic function of the parameters $\lambda = (\lambda_0, \ldots, \lambda_{\mu-1})$ of the deformation. Local diffeomorphness of $\widetilde{LL}$ at some point $\widehat{\lambda} \in \mathbf{C}^\mu \setminus \widetilde{\Sigma}$ is equivalent to the non-degeneracy of the $\mu \times \mu$-matrix $(\partial c_i/\partial \lambda_j)$ at $\widehat{\lambda}$.

For the sake of uniformity we shall assume that all the curves in the $\mathcal{R}_c$-simple singularities are space curves defined by 2×3-matrices.

Denote by Φ_r the order 2 minor of the 2×3-matrix $\mathcal{M}$ obtained by omitting its rth column, $r = 1, 2, 3$. Those are the equations of the family of the curves participating in the function deformation.

An exercise in calculus shows that the velocity $(\partial c_i/\partial \lambda_j)(\widehat{\lambda})$ is the value of the ratio of the determinants

$$\rho_j = \frac{||\partial(\Phi_r, \Phi_{r'}, F)/\partial(x_s, x_{s'}, \lambda_j)||}{||\partial(\Phi_r, \Phi_{r'})/\partial(x_s, x_{s'})||} \tag{13}$$

at the critical point of function $f_{\widehat{\lambda}}$ at which it attains the value c_i. Here x_1, x_2, x_3 are the coordinates in the space.

We have an easy

Lemma 5.9 *The ratio (13) of the evaluated determinants does not depend on the choice of r, r', s, s' for which the denominator does not vanish.*

On the other hand, the family $\mathcal{C} \subset \mathbf{C}^{3+\mu}$ of the critical loci of functions f_λ is defined by the ideal J generated by the Φ_r and three maximal minors of the Jacobi matrix $\partial(\Phi_1, \Phi_2, \Phi_3, F)/\partial(x_1, x_2, x_3)$ involving F. The quotient $\mathcal{O}_{3+\mu}/J$ is a free rank μ $\mathcal{O}_\mu$-module (this is easy to straightforwardly check for the simple singularities). Let functions $\eta_0, \ldots, \eta_{\mu-1} \in \mathcal{O}_{3+\mu}$ represent its generators.

Non-degeneracy of the matrix $(\partial c_i/\partial \lambda_j)$ out of $\widetilde{\Sigma}$ is equivalent to the existence of a representation on $\mathcal{C}$

$$\eta_\ell = \sum_{j=0}^{\mu-1} a_{\ell j} \rho_j \ , \qquad \ell = 0, \ldots, \mu - 1, \tag{14}$$

such that

(i) the functions $a_{\ell j} = a_{\ell j}(\lambda)$ are holomorphic on $\mathbf{C}^\mu$,

(ii) the $\mu \times \mu$-matrix $(a_{\ell j})$ is invertible on $\mathbf{C}^\mu \setminus \widetilde{\Sigma}$.

It is not so difficult to verify that for the simple singularities such representations do exist. In fact in each case $\det(a_{\ell j}) = 0$ turns out to be an equation of $\widetilde{\Sigma}$. □

Remark 5.10 Condition (ii) reflects the fact that vanishing of the evaluations of all possible denominators in (13) is equivalent to the corresponding curve being singular.

The sense of the decompositions (14) is discussed in the next section.

5.4 Discriminants of space curves

Consider a miniversal deformation $\mathcal{M}$ of a space curve M. Let $\mathbf{C}^{\tau_1}$ be its base. Let $\Delta(M) \subset \mathbf{C}^{\tau_1}$ be the *discriminant* of M, that is, the set of those values of the deformation parameters for which the corresponding curve is singular.

Theorem 5.11 [21] *The discriminant of a space curve is a free divisor.*

Apparently there exists the following algorithm to construct generators of $\Theta_{\Delta(M)}$ based on the decompositions (14).

Take an arbitrary function f on $\mathbf{C}^3$ so that the pair (M, f) is finitely $\mathcal{R}_c$-degenerate. Extend deformation $\mathcal{M}$ to an $\mathcal{R}_c$-miniversal deformation $(\mathcal{M}, F)$ of (M, f). Let $\mathbf{C}^{\tau_1} \times \mathbf{C}^{\tau_2}$ be the base of the extension.

Assume that $\mu(M, f) = \tau(M, f)$, that is, $\mu(M, f) = \tau_1 + \tau_2$. Assume also that the representations (14) for $(\mathcal{M}, F)$ exist (with the obvious modification for curves of arbitrary corank). Each of them defines on $\mathbf{C}^{\tau_1+\tau_2}$ a vector field

$$a_\ell = \sum_{j=0}^{\tau_1+\tau_2-1} a_{\ell j}(\lambda)\partial_{\lambda_j} .$$

Let $\bar{a}_\ell$ be the restriction of a_ℓ to $\mathbf{C}^{\tau_1} \times 0$.

Modulo Conjecture 3.1 we have

Conjecture 5.12 *Decompositions (14) satisfying conditions (i) and (ii) exist for any finitely degenerate function on a space curve. Within the above construction, one can choose elements $\eta_0, \dots, \eta_{\tau_1+\tau_2-1} \in \mathcal{O}_{3+\tau_1+\tau_2}$ so that the vector fields $\bar{a}_0, \dots, \bar{a}_{\tau_1-1}$ generate the $\mathcal{O}_{\tau_1}$-module $\Theta_{\Delta(M)}$.*

Example 5.13 Consider function $\dot{F}_{2r+3}$ and its miniversal deformation of Section 4.2. Take $1, y, \dots, y^{r+1}, x, xy, \dots, xy^{r-1}, z$ for the elements η_ℓ. All of these, except for z, y^r, xy^{r-1} and y^{r+1}, are among the velocities ρ_j. The complement of the ρ-set to its intersection with the η-set consists of the velocities involving the derivatives with respect to the four parameters $\alpha, \beta, \gamma, \delta$ of the versal deformation (6) of the curve A_2^L. Decompositions (14) of the elements $z, y^r, xy^{r-1}, y^{r+1}$ (in this order) provide the following matrix of the components of basic vector fields in $\mathbf{C}^4$ tangent to the discriminant of A_2^L:

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 2\beta & 3\gamma & 4\delta \\ 0 & 3\gamma & 2\delta & 2\beta\gamma \\ 0 & 4\delta & 2\beta\gamma & 2\beta\delta + 3\gamma^2/2 \end{pmatrix} \begin{pmatrix} \partial_\alpha \\ \partial_\beta \\ \partial_\gamma \\ \partial_\delta \end{pmatrix} \qquad (15)$$

To be precise, to obtain the above matrix one has to correct basic vector fields of higher quasihomogeneous degree given by (14) for our choice of the basic elements by the vector fields of lower degree. Also the fields must be multiplied by some non-zero constants. The first two vector fields are Euler. Notice the symmetry of the matrix. The determinant of the matrix is the discriminant (7) of the curve A_2^L.

6 Enumeration of meromorphic functions

Corollary 5.7 provides a singularity theory approach to the problem on enumeration of certain holomorphic coverings of the 2-sphere. The problem goes back to Hurwitz who stated it for rational functions [18]. The setting is as follows.

Consider two holomorphic mappings, f and f', from closed connected complex curves Γ and Γ' to $\mathbf{C}P^1$. We say that they are of *the same topological type* if there exists a commutative diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma' \\ f \searrow & & \swarrow f' \\ & \mathbf{C}P^1 & \end{array}$$

in which the horizontal arrow is a homeomorphism.

Mark a point (infinity) on $\mathbf{C}P^1$ and call its inverse images *poles*. Consider the space of holomorphic coverings of $\mathbf{C}P^1$ by curves of fixed genus having fixed orders of poles. We shall be interested in the case when all finite critical points of the coverings are Morse and situated on different levels. Such coverings will be called *Morse meromorphic functions*.

The Riemann-Hurwitz theorem provides a relation on the genus g of a curve, the degree d of a Morse function, and the numbers μ and s of the Morse points and poles:

$$2 - 2g - s = d - \mu \ .$$

Theorem 6.1 *The number N of topologically different Morse meromorphic functions on genus g curves having fixed critical values and s poles of fixed orders $d_1, \ldots, d_s$ (and thus the degree $d = d_1 + \ldots + d_s$) is given, in the following cases, by the table:*

g	$\{d_i\}$	N	singularity	restrictions
0	n	n^{n-3}	A_{n-1}	$n > 2$
0	p, q	$\frac{(p+q-1)!p^p q^q}{\epsilon!\,p!\,q!}$	$C_{p,q}$	$p \geq q \geq 1, p > 1$
0	p, q, r	$\frac{(p+q+r+1)!p^p q^q r^r}{\epsilon!\,p!\,q!\,r!}$	$C_{p,q,r}$	$p \geq q \geq r \geq 1$
1	n	$\frac{n^n(n^2-1)}{24}$	F_{n+1}	$n \geq 3$
1	$1, n$	$\frac{n^{n+2}(n+1)(n+2)(n+3)}{24}$	$\dot{F}_{n+3}$	$n \geq 2$
2	3	3^4	$\check{E}_6$	–
g	2	1	B_{2g+1}	$g \geq 1$
g	$1, 1$	1	B_{2g+2}	$g \geq 1$

In both the C-series, ϵ is the number of poles of coinciding order.

The space $\mathcal{T}$ of various topological types of all the Morse meromorphic functions in each of these cases is a smooth complex μ-dimensional variety. The variety $\mathcal{T}$ is a $k(\pi, 1)$-space, where π is a subgroup of index N in the Artin group of braids on μ threads.

Proof. All Morse functions of the listed cases enter the $\mathcal{R}_c$-miniversal deformations of Section 4. Representatives of all topological types of Morse functions with fixed critical values are contained in one fibre of the corresponding Lyashko-Looijenga mapping $\widetilde{LL}$. On each fibre there acts the group Aut of homeomorphisms of the curve which preserves the topological type of a function. In each case this is a subgroup of quasihomogeneous automorphisms of the miniversal family (see Example 6.2 below). The subgroups are easily seen to be as follows:

A_{n-1}	$C_{p,q}$	$C_{p,q,r}$	F_{n+1}	$\dot{F}_{n+3}$	$\check{E}_6$	B_k
$\mathbf{Z}_n$	$S_\epsilon \times \mathbf{Z}_p \times \mathbf{Z}_q$	$S_\epsilon \times \mathbf{Z}_p \times \mathbf{Z}_q \times \mathbf{Z}_r$	$\mathbf{Z}_n$	$\mathbf{Z}_n$	$\mathbf{Z}_3$	1

To prove the theorem one just needs to show that the action of each of these groups on the complement to the diagram $\tilde{\Sigma}$ is free. This is easy to check in all the cases and we are not going to demonstrate this.

The space $\mathcal{T}$ of topological types of Morse meromorphic functions is the quotient-space $(\mathbf{C}^\mu \setminus \tilde{\Sigma})/Aut$. Let φ be the factorisation map. The Lyashko-Looijenga mapping $\widetilde{LL}$ factors through it: there exists a mapping $\widetilde{\ell\ell} : \mathcal{T} \to \mathbf{C}^\mu \setminus \tilde{\Xi}$ such that $\widetilde{LL} = \widetilde{\ell\ell} \circ \varphi$. This implies the $k(\pi, 1)$ claim of the theorem. This also calculates the index of $\pi_1(\mathcal{T})$ in the braid group $B(\mu)$ as the ratio $[B(\mu) : \pi_1(\mathbf{C}^\mu \setminus \tilde{\Sigma})]/|Aut|$. □

Example 6.2 Consider the singularity $\check{E}_6$. Set $\omega = \sqrt[3]{1}$. The action of the group $Aut(\check{E}_6) = \mathbf{Z}_3$ on the miniversal deformation of Section 4.4 multiplies each of the variables v of the deformation by $\omega^{wt(v)}$, where $wt(v)$ is the weight of v, with the normalisation $wt(z) = 3$.

Remark 6.3 The Hurwitz problem can be considered as a problem on enumeration of certain connected graphs with d vertices and μ ordered edges (see [2] for details). For example, the entry $N = n^{n-3}$ for the A_{n-1} singularity in the table of the theorem is the number of trees with n vertices and ordered edges which is a famous theorem by Cayley.

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