Vassiliev type invariants in Arnold's J^+-theory of plane curves without direct self-tangencies

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Abstract

We show that the spaces of complex-valued Vassiliev type invariants for oriented regular plane curves without direct self-tangencies and for oriented framed knots in a solid torus [8] coincide. The isomorphism is provided by the Legendrian lift of plane curves to the solid torus $ST^*\mathbf{R}^2$.

1 Introduction

A generic plane curve is an immersed curve with a finite number of double points of transversal self-intersection. In the space of all C^∞-immersions $S^1 \to \mathbf{R}^2$, the complement to the set of all generic curves consists of three hypersurfaces [1, 2]. They correspond to the three possible degenerations in generic 1-parameter families of immersed curves. In such families there can appear either a curve with a triple point or a curve with one of two types of self-tangencies. A self-tangency can be either *direct* (when the two velocity

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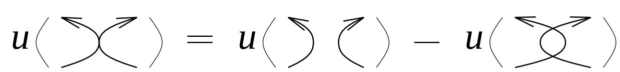
vectors at the self-tangency point have the same direction) or *inverse* (when they are opposite).

In this paper we study invariants of oriented immersed plane curves without direct self-tangencies. The values of such invariants on isotopy classes of curves do not change during inverse-self-tangency and triple-point transformations. The first invariant of this kind was defined by Arnold [1, 2] and called J^+. That is why the whole theory of invariants that we consider is called J^+-theory.

Our main Theorems 7.3 and 7.4 give the description of finite order complex-valued Vassiliev type invariants in the J^+-theory. They turn out to be the same as in the theory of oriented framed knots in a solid torus (ST) [8]. To show this we go through the following steps.

Invariants of regular plane curves without direct self-tangencies have a natural extension to curves with finitely many simple (quadratic) direct self-tangencies. The extension is done in the spirit of Vassiliev theory for knots by taking the difference of the values of an invariant u on the two resolutions of a self-tangency point (cf. [13, 3, 4, 5, 9]):



It is not very difficult to see that the extensions are subject to the 2- and 4-term relations:

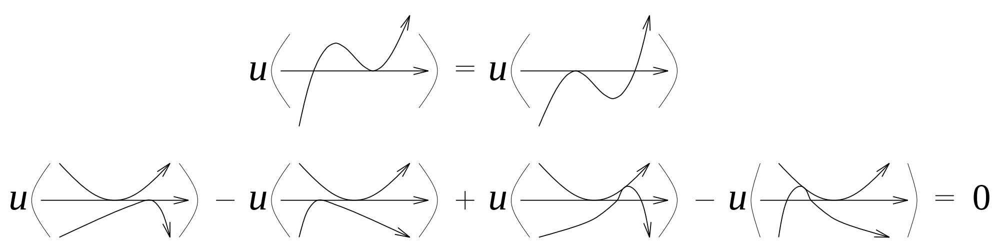


Figure 1: *Relations for extended invariants.*

When we raise plane curves to Legendrian curves in the solid torus $ST^*\mathbf{R}^2$, the 4-term relations become the 4-term relations of the theory of knots in ST [8].

We have shown in [8] that the graded space of complex finite order Vassiliev type invariants of oriented framed knots in ST is in a sense dual to the graded $\mathbf{C}$-linear space $\mathcal{M}$ spanned by all marked chord diagrams on a circle modulo all marked 4-term relations. $\mathcal{M}$ is graded by the number of chords of a diagram.

The nature of the marking is as follows. Consider an immersion of an oriented circle into ST with a finite number of double points. As usual, on the source circle, we connect the two preimages of a double point by the chord. The two-side marking of each chord is done by the classes of the images of the subtended halves of the oriented circle in $\pi_1(\mathrm{ST})$. We mark the whole circle with the fundamental class of its image. The sum of the two markings on a chord is the marking of the circle. With all the markings omitted, the marked 4-term relations are ordinary 4-term relations of the Vassiliev theory for knots in $\mathbf{R}^3$ [13, 4, 5, 9].

In the J^+-theory we get the marked-chord-diagram interpretation completely parallel to that of framed knots in ST. Now, instead of drawing a chord that joins the two preimages of a double point of a singular knot, we join the two preimages of a point of direct self-tangency. The markings on a chord are the Whitney winding numbers of the corresponding halves of a plane curve. The marking of the circle is the winding number of the whole curve.

Every marked chord diagram is the diagram of a plane curve. The singular knot in $ST^*\mathbf{R}^2$ which is the Legendrian lift of a plane curve has the marked chord diagram which is exactly that of the underlying plane curve. The Legendrian lift of a plane curve has the canonical framing.

The Whitney-Graustein theorem [14] and the "2-term-relation" move imply connectedness of the set of immersed plane curves, whose direct self-tangencies are subject to a given marked chord diagram (Theorem 4.6). Thus the restriction of an extended invariant of order n to the set of plane curves with n quadratic direct self-tangencies is actually a function on the set of marked diagrams with n chords considered modulo the marked 4-term relation.

So, the chord-diagram theories that appear in the J^+-theory and in the theory of oriented framed knots in ST are identical. In order to prove our main result, it remains to show that the order n graded part of the space of our extended complex invariants is isomorphic to the space of linear functions on the order n graded part of the space $\mathcal{M}$. The required isomorphism is provided by the introduction of the universal Vassiliev-Kontsevich invariant for the J^+-theory. The universal invariant is defined rather straightforwardly as induced, by the Legendrian lifting, from the universal invariant of framed knots in ST [8].

Remark 1.1 In the similar way one can show the coincidence of the spaces of complex-valued Vassiliev type invariants of finite order for two other settings. On one side of the equality there is the theory of oriented regular plane curves without self-tangencies (both direct and inverse). This is Arnold's $J^{\pm}$-theory [1, 2]. On the other side we put oriented framed knots in the solid torus $PT^*\mathbf{R}^2$ which define even classes in the fundamental group.

I am thankful to Sergei Chmutov for very useful discussions.

2 Extended invariants

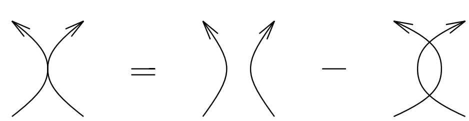
Let $\mathcal{P}$ be the space of all C^∞-immersions of an oriented circle S^1 to $\mathbf{R}^2$. In this paper, for the *discriminant* $\Sigma \subset \mathcal{P}$ we take the closure of the set of all the immersions which have one point of direct self-tangency. This is a hypersurface in $\mathcal{P}$.

Definition 2.1 A self-tangency of a plane curve is called *quadratic* or *simple* if, by a local diffeomorphism of the plane, a local equation of the curve can be brought to the form $y^2 = x^4$.

A regular point of Σ is an immersion with one simple direct self-tangency. A regular point of the n-tuple self-intersection of Σ is an immersion with n simple direct self-tangencies.

Definition 2.2 *An invariant of oriented immersed plane curves with no direct self-tangencies* is an element of $H^0(\mathcal{P} \setminus \Sigma)$ (with any coefficients).

Definition 2.3 *An extended invariant* is the extension of an element of $H^0(\mathcal{P} \setminus \Sigma)$ to plane curves with a finite number of quadratic direct self-tangencies via the recurrence setting



Here and below, all the equalities involving curves are actually relations between the values of some invariant on these curves. All the curves in such equalities coincide modulo the shown fragments.

Our definition of an extended invariant corresponds to the following coorientation of the hypersurface Σ in $\mathcal{P}$ at its regular points. We call a small perturbation of the immersion which transforms the self-tangency into two transverse double points *negative*. Respectively, a perturbation which makes the two local branches disjoint is *positive*. So, the bifurcation of Fig.2 is done in the positive direction. This coorientation is opposite to that used by Arnold [1, 2]. But it is exactly the coorientation which is induced on $\Sigma \subset \mathcal{P}$, via the Legendrian lifting (see Section 7 below), from the usual coorientation of the subset of singular knots in the space of all mappings of S^1 to $\mathbf{R}^3$ or $ST^*\mathbf{R}^2$ [13, 3, 4, 5, 9, 8].

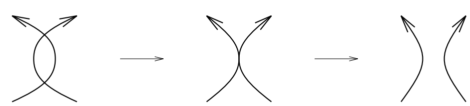
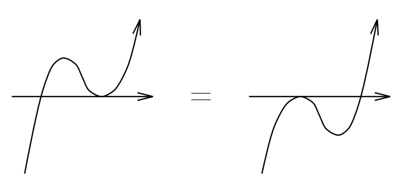


Figure 2: *Positive direct self-tangency bifurcation.*

3 Relations

3.1 Two-term relation

Proposition 3.1 *Any extended invariant satisfies the two-term relation:*



This relation shows that for an extended invariant it does not matter which of the two tangent branches is to, say, the left of the other.

Proof. The existence of the 2-term relation is due to the shape of a transversal section of $\Sigma \subset \mathcal{P}$ at a point which corresponds to an immersion that has a cubic tangency of two branches. The events in this two-dimensional transversal are shown in Fig.3. The two-term relation is easily read from that figure by the use of the definition of an extended invariant. □

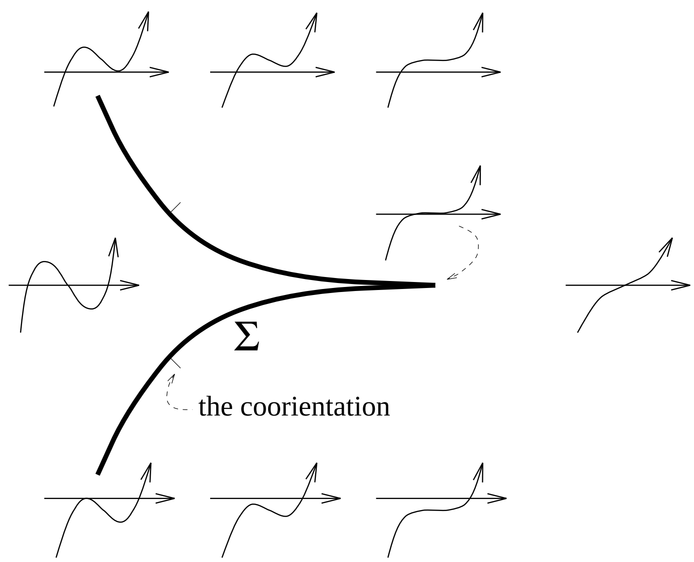
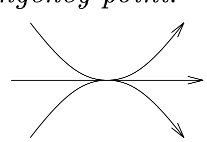


Figure 3: *The cubic self-tangency bifurcation implies the two-term relation.*

3.2 Four-term relations for plane curves

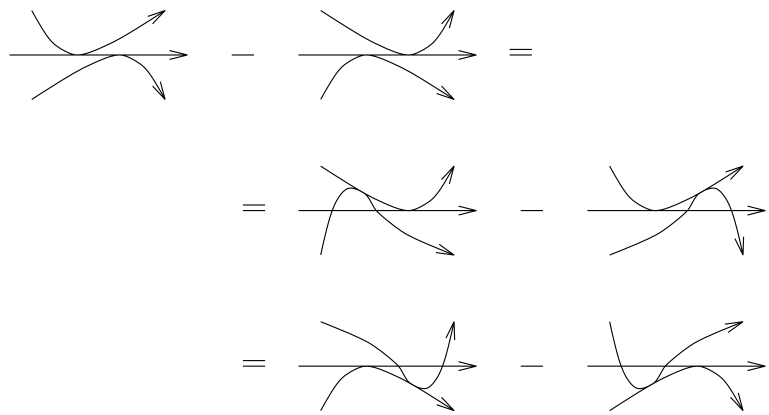
Consider *a triple direct self-tangency point*:



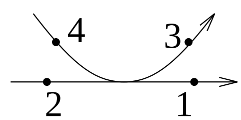
Each pair of the branches here has a quadratic tangency.

Let us perturb the above curve fragment generating two simple self-tangency points in various ways. We compare the values of extended invariants on these perturbations.

Proposition 3.2 *Any extended invariant satisfies the four-term relations:*



Another way to express the 4-term relations for plane curves is as follows (cf. [3, 9]). Let us fix two of the three branches at a triple self-tangency point. We slightly move the third branch making it tangent to those fixed at one of the four points 1,2,3,4:



Let $C_1, \ldots, C_4$ be the corresponding curves (recall Fig.1). Then, for any extended invariant u,

$$\sum_{j=1}^{4} (-1)^j u(C_j) = 0.$$

Proof (cf. [3, 6]). Let us consider, for example, the difference of the first two lines of the claim. In Fig.4, using the definition of an extended invariant, we express it in 8 terms. There we get 3 pairs of isotopic curves: 1st and 3rd, 2nd and 5th, 4th and 7th. The values of an invariant on the two remaining curves, 6th and 8th, are rewritten, after the second equality sign, using the definition once again. Now we have two isotopic curves and two other curves which can be transformed to one another by triple-point moves only. Thus the whole expression vanishes. □

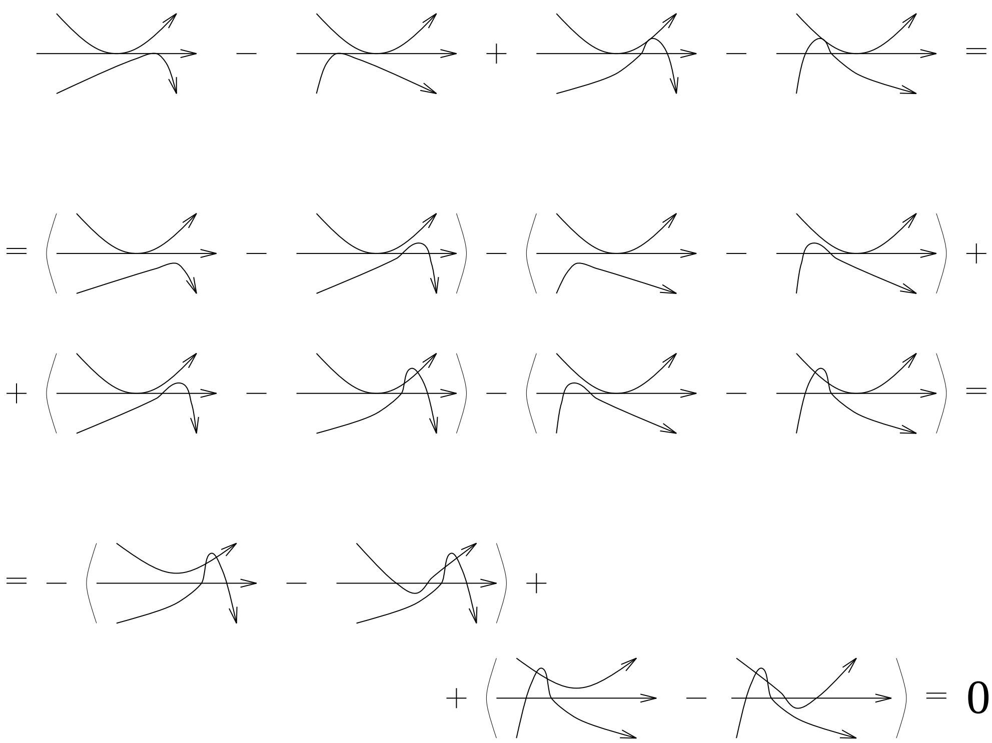


Figure 4: *Proof of the 4-term relation for plane curves.*

Remark 3.3 Similar 2- and 4-term relations exist in the theory of invariants of immersed plane curves without any self-tangencies, both direct and inverse. They are valid for curves on any surface as well.

4 Chord diagrams and the stratification of the discriminant

4.1 Marked chord diagrams of direct self-tangencies

Definition 4.1 Consider an oriented circle with $2n$ distinct points on it that are split into n non-ordered pairs. *A chord diagram* is an equivalence class of such objects with respect to orientation preserving diffeomorphisms of the circle.

We represent a chord diagram as a standard counter-clockwise oriented circle on a plane with the chords connecting the pairs of points.

Definition 4.2 Mark the circle of a chord diagram with an integer number w. Mark each side of each chord with integers so that the sum of the two markings on a chord is w. *A marked chord diagram* is an equivalence class of such objects with respect to diffeomorphisms of the circle that preserve its orientation and all the markings.

Now consider an oriented curve $S^1 \to \mathbf{R}^2$ with a finite number of simple direct self-tangencies. We take, on the source circle, the pairs of the preimages of all the points of direct self-tangency.

Definition 4.3 The chord diagram of this pairing is called *the chord diagram* of the plane curve.

Such a diagram possesses a natural marking. Let us fix an orientation of the target plane. Any chord of the diagram cuts the circle into two arcs. Each of the arcs is mapped onto a closed C^1-curve. Each of the two curves has its Whitney winding number [14, 1, 2]. We show this information on the chord diagram of the plane curve putting the two winding numbers on the sides of the corresponding chord (Fig.5). The winding number of the image of an arc is put on the side facing this arc.

We also mark the whole circle with the winding number of the whole curve.

Definition 4.4 The obtained marked chord diagram is called *the marked chord diagram* of a plane curve.

4.2 Irreducibility of the strata of the discriminant

Below we consider an analogue of the following obvious fact of knot theory [13]. Let Δ_n be the set of all oriented spatial curves that have exactly n non-self-tangent double points and no other singularities. We say that two curves

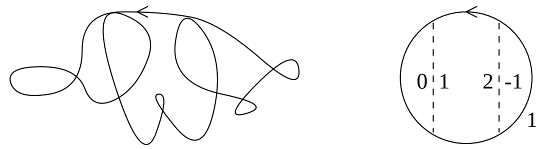


Figure 5: *A plane curve and its marked chord diagram.*

from this set are *related* if it is possible to join them by a smooth homotopy that stays, almost all the time, in Δ_n and, at the remaining finitely many moments, passes transversally through Δ_{n+1}. The classes of related spatial curves are enumerated by the types of the chord diagrams of their double points.

Now, in J^+-theory, let Σ_n be the set of all immersed plane curves that have exactly n simple direct self-tangencies and all the other local singularities not very complicated. Namely, the complete list of local singularities allowed for curves from this set consists of: transverse double points, triple points with pairwise transversal branches, inverse self-tangencies, third branches passing through any of the n direct self-tangencies transversally to the two tangent branches.

Let Σ'_n be the set of all immersed plane curves with exactly one cubic and $n-1$ quadratic direct self-tangencies, and with the other local singularities being only transverse double points.

Definition 4.5 *Two plane curves* from the set Σ_n *are related* if they can be connected by a smooth homotopy that stays, almost all the time, in Σ_n and, at the remaining finitely many moments, passes through $\Sigma_{n+1} \cup \Sigma'_n$ in a generic way.

A generic way to pass through Σ_{n+1} is to be transversal to this set.

A generic way to pass through Σ'_n is to follow the discriminant in Fig.3. Since the value of any extended invariant does not change during such a two-term-relation move, it is natural to allow this move to relate curves in J^+-theory.

Theorem 4.6 *Two plane curves are related if and only if they have the same marked chord diagrams.*

Proof. The "only if" part is obvious. So we are proving only the "if" one.

We will homotop a plane curve $\mathcal{C}$ with quadratic direct self-tangencies to some sort of a normal form represented in Fig.6.

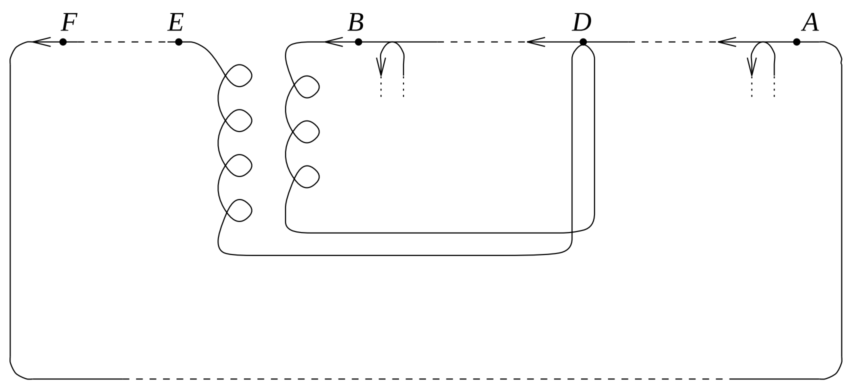


Figure 6: *Normal curve with direct self-tangencies.*

Let us fix an arbitrary generic point A of the curve $\mathcal{C}$ as the initial point. We follow the curve in the direction of its orientation. Instantaneously generating extra self-tangencies and using the two-term-relation move (Fig.7), we arrange all the direct self-tangencies so that the branch of the second-time visit is to the left of the branch of the first-time visit.

Then we start pulling the self-tangencies back to A along the curve. We do this in succession following our curve from the point A. We come to the first point of direct self-tangency and move it back close to A, by sliding the second-time visit along the unperturbed first-time-visit branch, so that there would be no self-intersection of the part of the curve between A and the first visit to the first self-tangency point.

Then we go for the second direct self-tangency. If it is not the second visit to the point we dealt with on the previous step, we slide it back along the curve arranging no self-intersections on the part of the curve between the point A and the second direct self-tangency.

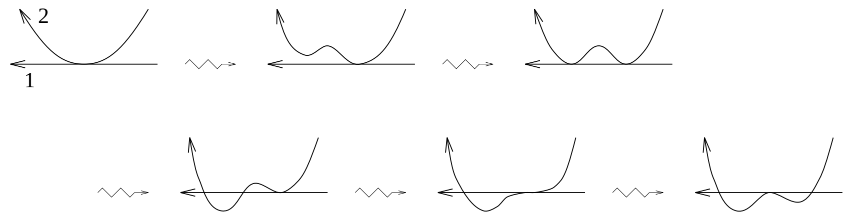


Figure 7: *Switching of a self-tangency from one side of the first-time-visit branch to the other.*

We do this with all the first-time visits until we see that the next visit will be a second-time one. We make the part of the curve traced by now horizontal. This gives the interval AB of Fig.6.

Let $D \in AB$ be the self-tangency point to which we have to go from B with no intermediate visit to any other direct self-tangency. On the marked chord diagram of the initial curve $\mathcal{C}$, the points that are mapped now to D are connected by a chord. The marking on the side of this chord that faces the preimage of the loop DBD says that DBD should have a certain winding number α. So, going from B to D, we have to make α rotations. By the Whitney-Graustein theorem [14], keeping small neighbourhoods of B and D fixed, we can make a generic regular homotopy of the path from B to D to the U-shaped curly path (Fig.6) which has the proper (defined by α) number of small curls, either all positive or all negative (Fig.8).

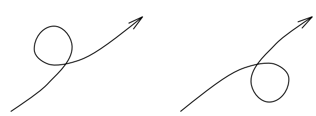


Figure 8: *Positive and negative curls.*

Now we would like to return from D to a point E close to B (Fig.6) so that the winding number of AE is zero. In order to do this, we introduce, shortly after leaving D for the second time, a series of negative-positive (or a series of positive-negative) pairs of curls (Fig.9). We take a point G in the middle of this series and pull it to the position E of Fig.6. Taking the curl series of the proper (again, defined by α) length, we guarantee the winding number of BE to be zero.

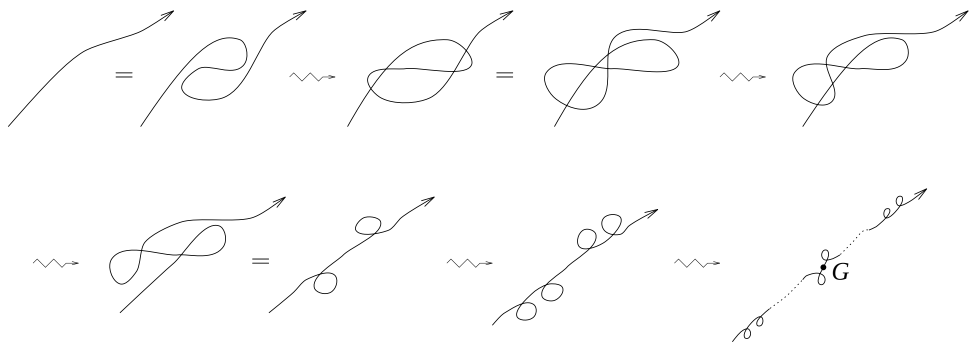


Figure 9: *Insertion of a series of negative-positive pairs of curls.*

We continue with our pulling-out and walking-along-curly-paths normalization procedure until all the visits to all the self-tangecies are exhausted. Now we are at the point F of Fig.6. By the Whitney-Graustein theorem, there exists a generic regular homotopy that brings the remaining part of our curve, from F to A, to the path of Fig.6 with small curls (either all positive or all negative) along the dashed interval. The curls make the winding number of the normalized curve (that is the winding number of its part from F to A) equal to the winding number of the original curve $\mathcal{C}$ (that is the marking of the circle of the diagram).

All the normalization homotopy can be done in a generic way, so that it relates $\mathcal{C}$ and the normal curve.

A normalized curve depends on the choice of the initial point A. To get rid of this ambiguity and obtain a common normal form for all the curves with a given marked chord diagram, we fix the preimage of A on the marked diagram. The relative disposition of the curly parts like BDE does not matter up to generic homotopies which we are considering. $\square$

Corollary 4.7 *Assignment of the marked chord diagram of its direct self-tangencies to a plane curve is a one-to-one correspondence between all the* *classes of related curves with n simple direct self-tangencies and all marked n-chord diagrams.*

Proof. Indeed, every marked chord diagram is the marked chord diagram of an appropriate normal curve. □

5 Invariants of finite order

5.1 Order of an invariant

Definition 5.1 *An invariant* of plane curves with no direct self-tangencies *is of order not greater than n* if its extension vanishes on all the curves with more than n points of direct self-tangency.

The linear space of invariants of finite order has an increasing filtration by the subspaces U_n of all the invariants of order not greater than n.

Unlike the case of knots in 3-space, the quotients U_n/U_{n-1} are infinite-dimensional linear spaces.

Example 5.2 Elements of U_0 are locally constant functions on connected components of the space of immersed plane curves. The components are enumerated by the Whitney winding number $w \in \mathbf{Z}$.

Definition 5.3 *An invariant of order n* is an element of $U_n \setminus U_{n-1}$.

Example 5.4 Arnold's invariant J^+ [1, 2], which takes the value -2 on any plane curve with one simple direct self-tangency, has order 1.

5.2 Symbols

Definition 5.5 *The symbol of an invariant of order n* is its restriction to the set of curves with n points of quadratic direct self-tangency.

Proposition 5.6 *The value of the symbol of an invariant of order n on a curve with n simple direct self-tangencies depends only on the marked chord diagram of the curve.*

Proof. This follows from Theorem 4.6.

Indeed, consider a homotopy that relates two curves which have n simple direct self-tangencies each and no other singularities except transverse double points. Then:

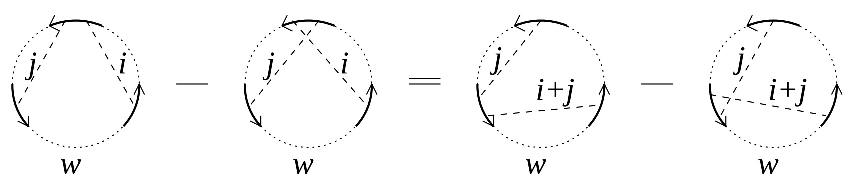
a) The value of our symbol does not change when we pass through the set Σ_{n+1}, since the invariant is of order less than $n+1$.

b) The value also does not change when we pass through the set Σ'_n. This is due to the two-term relation.

c) Finally, the value of the symbol does not change when we pull a third branch through a point of direct self-tangency. To show this we resolve the self-tangency using the definition of the extension of an invariant. The resolutions of the before-the-bifurcation curve are homotopic to the corresponding resolutions of the after-the-bifurcation curve via triple-point moves only (cf. Fig.4). □

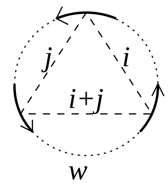
Thus the value of a symbol on a marked chord diagram is well-defined.

Proposition 5.7 *The values of symbols on marked chord diagrams are subject to the marked four-term relation:*



Here we indicate only partial markings that determine the complete ones. All the chords based on the solid arcs are shown. We do not show any chords based on the dotted arcs (those parts are assumed to be the same in all 4 diagrams).

Proof. This is an immediate implication of the four-term relations for extended invariants (subsection 3.2) and of the observation that the sum of the winding numbers of the three subcurves into which a triple self-tangency point cuts a curve is the winding number of the whole curve, so that the partial marking on the corresponding chord triangle should be



6 Vassiliev type invariants of oriented framed knots in a solid torus

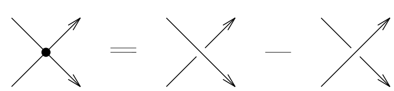
The relation of Proposition 5.7 is exactly the relation for symbols of Vassiliev type invariants for knots in a solid torus (ST) obtained in [8]. Since we are going to trace further analogy between J^+-theory and framed knots in ST, we recall the main constructions of [8].

6.1 The invariants

An oriented non-singular framed knot K in ST is a smoothly embedded oriented circle equipped with a smooth section v of the bundle $T_K(\mathrm{ST})$ induced from the tangent bundle of ST. The field v is assumed to be nowhere tangent to K.

By a singular knot we mean an immersed circle that has a finite number of double points with non-tangent branches and no other singularities. Its framing is, in general, double-valued at the double points.

As in the theory of knots in $\mathbf{R}^3$, an isotopy invariant of non-singular (framed) knots in solid torus possesses the recursive extension to singular (framed) knots:



An invariant is said to be *of order not greater than* n if it vanishes on any knot with more than n double points.

The definition of the symbol of an invariant of finite order is obvious.

The definition of the marked chord diagram of a singular knot in ST was mentioned in the Introduction. Every marked chord diagram is the marked chord diagram of a singular knot.

The value of the symbol of an order n invariant on a framed knot with n double points depends only on the marked chord diagram of the knot.

The values of a symbol on singular knots in ST are subject to the marked 4-term relation of Proposition 5.7. For **C**-valued invariants of oriented framed knots, this relation turns out to be the only restriction on the symbols.

Namely, consider the set of all marked n-chord diagrams. Let $\mathcal{M}_n$ be the space of all formal complex linear combinations of finitely many elements of this set modulo the marked 4-term relation (that is the relation of Proposition 5.7 for diagrams themselves rather than functions on them). Let $\mathcal{M}_n^*$ be the space of **C**-linear functions on $\mathcal{M}_n$.

We denote by W_n the linear space of order less than $n+1$ complex-valued invariants of oriented framed knots in ST.

The main result of [8] is

Theorem 6.1 $\quad W_n/W_{n-1} = \mathcal{M}_n^*.$

The marked 4-term relation for symbols identifies W_n/W_{n-1} as a subspace of $\mathcal{M}_n^*$. As in [9, 4], the equality is provided by the introduction of the universal Vassiliev-Kontsevich invariant.

6.2 The universal Vassiliev-Kontsevich invariant for framed knots in a solid torus

The approach of [8] is distinct from that by Lê and Murakami [10, 11] and covers more general situation.

We represent ST as the direct product $\mathbf{C}\times S^1$ with the complex coordinate z and the angular coordinate $\theta \bmod 2\pi$.

We say that a knot in ST is *a Morse knot* if θ is a Morse function on it.

Let K be an oriented non-singular framed Morse knot in ST.

A. For small $\varepsilon > 0$, we shift K in the direction of its framing v:

$$(z, \theta) \mapsto (z, \theta) + \varepsilon v(z, \theta).$$

We denote by K_ε the result of the shift. For all sufficiently small ε, K_ε is a knot that does not intersect K.

B. In order to have a good definition of a chord diagram later on, we make an adjustment of the link $K \cup K_\varepsilon$. Near a local maximum of the function θ on K, θ has the local maximum on K_ε as well. We take the lowest of the two critical levels and remove the small arc of $K \cup K_\varepsilon$ that is locally above this level. In the similar way, we remove the small arc that is locally below the highest of the two critical levels near a local minimum of θ on K. After the surgery at all the local extrema, we remain with the subsets $\widehat{K} \subset K$ and $\widehat{K}_\varepsilon \subset K_\varepsilon$.

The shift along the framing provides the one-to-one correspondence between the sets of intervals of monotonicity of the function θ on K and K_ε. For each non-critical point $(z', \theta) \in \widehat{K}_\varepsilon$ this correspondence uniquely defines its *neighbour* $(z'', \theta) \in \widehat{K}$ on the same θ-level.

C. Now we take n different non-critical levels $0 < \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$. In each section $\theta = \theta_j$ of $\widehat{K} \cup \widehat{K}_\varepsilon$, we choose an *ordered* pair of points $(z_j, z'_j) = (z_j, z'_j)(\theta_j) \in \widehat{K} \times \widehat{K}_\varepsilon$. Let P be a set of n such pairs, one pair per level.

The set P defines the marked n-chord diagram as follows (see Fig.10).

In each pair we substitute $z'_j \in \widehat{K}_\varepsilon$ by its neighbour $z''_j \in \widehat{K}$. The knot K is the image of an immersion of an oriented circle that we take to be a standard counter-clockwise oriented circle on the plane. If $z_j \neq z''_j$, we join the preimages of the points z_j and z''_j on the source circle by the chord. The chord has the two-side marking by the fundamental classes of the two loops obtained by a homotopy of K in ST that glues together z_j and z''_j and is the identity outside a small neighbourhood of the section $\theta = \theta_j$. We assume here that a generator of $\pi_1(\mathrm{ST}) = \mathbf{Z}$ is fixed. Say, it goes once around the torus in the direction of increase of θ.

If $z_j = z''_j$, we draw a small chord between two arbitrary points on the circle that are very close to the preimage of z_j. We mark the side of the chord facing the small arc with 0 and its other side with the class of K in $\pi_1(\mathrm{ST})$.

The whole circle is marked with the class of K in $\pi_1(\mathrm{ST})$ as well.

We denote by $D(P)$ the equivalence class of the obtained marked chord diagram in the $\mathbf{C}$-linear space $\mathcal{M}_n$.

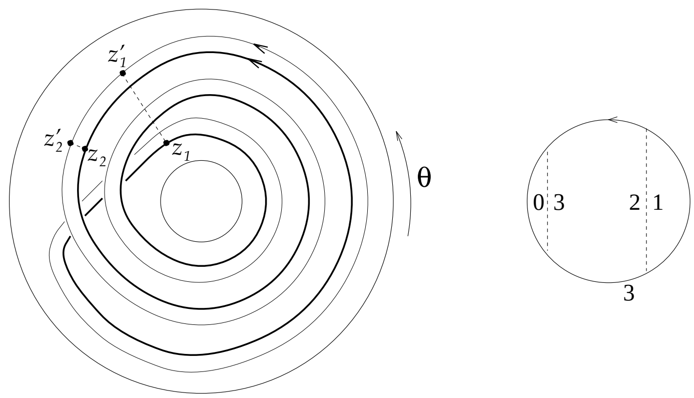


Figure 10: *The pairing on the knot with the blackboard framing and its marked chord diagram.*

D. We introduce:

Definition 6.2 $\widehat{Z}_n(K, K_\varepsilon) =$

$$= \frac{1}{(2\pi i)^n} \int_{0<\theta_1<\theta_2<\ldots<\theta_n<2\pi} \sum_{P=\{(z_j,z'_j)(\theta_j)\}} (-1)^{P_\downarrow} \bigwedge_{j=1}^{n} \frac{dz_j - dz'_j}{z_j - z'_j} D(P) \in \mathcal{M}_n \ ,$$

where P runs through all possible pairings on $\widehat{K} \cup \widehat{K}_\varepsilon$ and $P_\downarrow$ is the number of points in the n pairs at which the function θ is decreasing along $K \cup K_\varepsilon$.

Definition 6.3 $Z_n^f(K) = \lim_{\varepsilon\to 0} \widehat{Z}_n(K, K_\varepsilon)$.

Theorem 6.4 ([8], cf. [9]) *i) The limit that defines $Z_n^f(K)$ is finite.*
ii) $Z_n^f(K)$ is invariant under the homotopy in the class of framed Morse knots.
iii) $Z_n^f(K)$ is an invariant of order less than $n+1$.

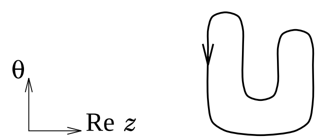
We set

$$Z^f(K) = \sum_{n\geq 0} Z_n^f(K) \in \overline{\mathcal{M}},$$

where $\overline{\mathcal{M}} = \prod_{n\geq 0} \mathcal{M}_n$.

E. The space $\mathcal{M} = \oplus_{n\geq 0}\mathcal{M}_n$ contains a subspace $\mathcal{A}$ spanned by the diagrams with all the markings vanishing. $\mathcal{A}$ is an algebra with respect to the connected sum operation. It is isomorphic to the algebra of non-marked chord diagrams modulo the non-marked 4-term relation [9, 4]. The connected summation provides an $\mathcal{A}$-module structure on the space $\mathcal{M}$ [8].

Let $\mathcal{U}$ be the curve



equipped with the framing $v = (v_z, v_\theta) = (i, 0)$. The curve lies in a sector of the annulus Im$z = 0$ of the solid torus $\mathbf{C} \times S^1$.

The series $Z^f(\mathcal{U}) \in \overline{\mathcal{A}} = \prod_{n\geq 0} \mathcal{A}_n$ is invertible since it starts with $1 \in \mathcal{A}_0$.

Let c be the number of critical points of the function θ on a framed knot K.

Definition 6.5 The element

$$\widetilde{Z}^f(K) = Z^f(K) \times Z^f(\mathcal{U})^{1-\frac{c}{2}} \in \overline{\mathcal{M}}$$

is called *the universal Vassiliev-Kontsevich invariant* of a framed Morse knot K in the solid torus.

Example 6.6 Let $\omega \in \mathcal{A}_1$ be the one-chord diagram with all three marks zero. Consider an unknot with the framing that makes one positive rotation around it. The value of $\widetilde{Z}^f$ on such unknot in ST is $\exp(\omega)$.

Theorem 6.7 ([8], cf. [9]) *For any framed Morse knot K, $\widetilde{Z}^f(K)$ depends only on the topological type of K and its framing.*

Theorem 6.7 implies Theorem 6.1.

The degree n component $\widetilde{Z}^f_n(K) \in \mathcal{M}_n$ of $\widetilde{Z}^f(K)$ is an invariant of order less than $n+1$.

7 Description of the space of finite order invariants in J^+-theory

7.1 The universal Vassiliev-Kontsevich invariant of plane curves without direct self-tangencies

We induce the universal invariant for J^+-theory from that for framed knots in the solid torus via the Legendrian lift.

Consider a parametrization $g : S^1 \to \mathbf{R}^2$ of an oriented regular curve $\mathcal{C}$ on an oriented plane. It defines the parametrization of the oriented regular curve $\check{\mathcal{C}}$ in the solid torus $ST^*\mathbf{R}^2 = \mathbf{R}^2 \times S^1$:

$$\check{g} : s \mapsto (g(s), \nu(s)).$$

Here $\nu(s)$ is the unit vector normal to the velocity $u(s)$ of the curve $\mathcal{C}$ at the point $g(s)$, such that the frame $\{\nu(s), u(s)\}$ is positive on the plane. $ST^*\mathbf{R}^2$ is identified here with $ST\mathbf{R}^2$ via a choice of a metric.

The curve $\check{\mathcal{C}} \in \mathbf{R}^2 \times S^1$ has the canonical framing

$$v(s) = (\nu(s), 0).$$

Assume a curve $\mathcal{C}$ has no direct self-tangencies and the derivative of its curvature does not vanish at its inflection points. Then $\check{\mathcal{C}}$ is a Morse knot in the solid torus $ST^*\mathbf{R}^2 = \mathbf{R}^2 \times S^1 = \mathbf{C} \times S^1$.

Definition 7.1 *The universal Vassiliev-Kontsevich invariant* $\widetilde{Z}^{J^+}(\mathcal{C})$ of the plane curve $\mathcal{C}$ is the universal invariant $\widetilde{Z}^f(\check{\mathcal{C}}) \in \overline{\mathcal{M}}$ of its canonically framed Legendrian lift $\check{\mathcal{C}}$ in $ST^*\mathbf{R}^2$.

Theorem 7.2 *The element $\widetilde{Z}^{J^+}(\mathcal{C}) \in \overline{\mathcal{M}}$ does not change during any generic regular homotopy of the curve $\mathcal{C}$ which does not involve direct self-tangency transformations.*

Proof. This is the direct corollary of Theorem 6.7, since such homotopy lifts to the isotopy of the framed Legendrian knot $\check{C}$ in the solid torus. □

7.2 The main theorems

We return to the spaces U_n of order less than $n+1$ invariants of regular plane curves without direct self-tangencies. We assume now the invariants to be **complex**-valued.

Theorem 7.3 $\quad U_n/U_{n-1} = \mathcal{M}_n^*$.

Proof. Proposition 5.7 embeds the quotient into $\mathcal{M}_n^*$.

Due to Theorem 7.2, to prove the equality it is sufficient, as in [4], to show that any marked n-chord diagram D appears as the lowest order term in the universal invariant of an appropriate plane curve.

As in [4], consider any oriented plane curve $\mathcal{D}$ whose marked chord diagram is D. Let $p_1, \ldots, p_n$ be the points of direct self-tangency of $\mathcal{D}$. Consider 2^n curves $\mathcal{D}_\sigma$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_j = 0, 1$, obtained from $\mathcal{D}$ by resolution of the direct self-tangencies (recall Section 2): the resolution of the point p_j is positive if $\sigma_j = 0$ and negative if $\sigma_j = 1$. Set $|\sigma| = \sigma_1 + \ldots + \sigma_n$. All the $\mathcal{D}_\sigma$ coincide with $\mathcal{D}$ outside small neighbourhoods of the points p_j.

According to the definitions

$$\widetilde{Z}^{J^+}(\mathcal{D}) = \sum_\sigma (-1)^{|\sigma|} \widetilde{Z}^{J^+}(\mathcal{D}_\sigma) = \sum_\sigma (-1)^{|\sigma|} \widetilde{Z}^f(\check{\mathcal{D}}_\sigma).$$

We orient the solid torus $ST^*\mathbf{R}^2 = \mathbf{R}^2 \times S^1$ by the frame {positive frame of the plane, positive direction of rotation on the plane}. The Legendrian lift of a positive (respectively negative) resolution of a direct self-tangency is a positive (respectively negative) resolution of the double point of the singular knot in $ST^*\mathbf{R}^2$ (Fig.11). The marked chord diagram of the singular framed knot $\check{\mathcal{D}} \in ST^*\mathbf{R}^2$ is the marked chord diagram of the plane curve $\mathcal{D}$. So, as in [4, 8], we get

$$\widetilde{Z}^{J^+}(\mathcal{D}) = \sum_\sigma (-1)^{|\sigma|} \widetilde{Z}^f(\check{\mathcal{D}}_\sigma) = \widetilde{Z}^f(\check{\mathcal{D}}) = 2^n D + \text{higher order terms.} \quad \square$$

Comparing Theorems 6.1 and 7.3 and using the above discussion, we obtain:

Theorem 7.4 *The graded spaces of finite order complex-valued invariants of oriented framed knots in a solid torus and of oriented regular plane curves without direct self-tangencies are isomorphic. The isomorphism is provided by the Legendrian lift of plane curves to the solid torus* $ST^*\mathbf{R}^2$.

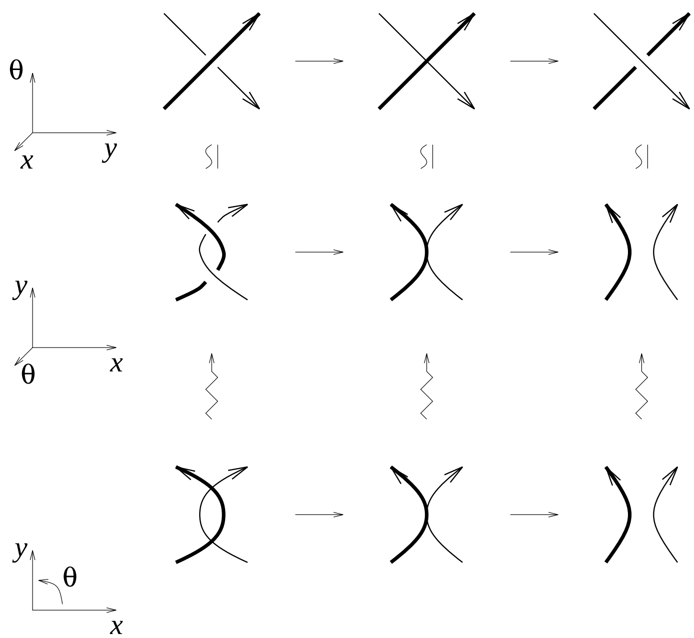


Figure 11: *The Legendrian lift of the positive direct self-tangency move of plane curves is the positive change-crossing move of curves in the solid torus.*

Remark 7.5 Similar to [5], coefficients of the polynomial invariants of knots in ST [12] are in a sense finite order invariants [8]. The Legendrian lift induces these polynomials to the J^+-theories of plane curves and wave fronts. Coefficients of the induced polynomials, properly understood, are finite order invariants as well [7].

Remark 7.6 The 4-term relation for Vassiliev type invariants of knots in 3-manifolds comes from the bifurcations of a triple point on a singular knot. The 4-term relation for invariants of curves on surfaces comes from the bifurcations of a triple self-tangency point. Both relations are locally isomorphic. In Fig.12 we show the bifurcation diagrams (that is germs of generic sections of the corresponding discriminants in the corresponding spaces of mappings) of the two degenerations. The diagrams are not diffeomorphic, but the stratifications of the parameter spaces do have a lot in common.

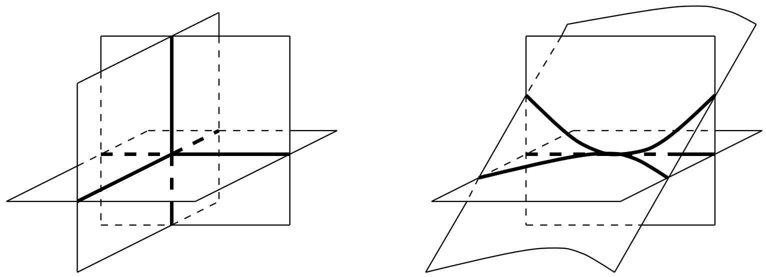


Figure 12: *Bifurcation diagrams of a triple point of a space curve and of a triple self-tangency of a plane curve.*

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