PREDICTING TRANSITION WITHOUT EMPIRICISM OR DNS

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ABSTRACT
A numerical procedure for predicting the receptivity of laminar boundary layers to freestream turbulence consisting of vortex arrays with arbitrary orientation has been developed. Results show that the boundary layer is most receptive to those vortices which have their axes approximately in the streamwise direction and vortex wavelengths of approximately 1.2δ. The computed near wall gains for isotropic turbulence are similar in magnitude to previously published experimental values used to predict transition. The new procedure is therefore capable of predicting the development of the fluctuations in the laminar boundary layer from values of the freestream turbulence intensity and length scale and hence determining the start of transition without resorting to any empirical correlation.

NOMENCLATURE

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<tr>
<td>δ</td>
<td>Boundary layer thickness</td>
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<td>v</td>
<td>Kinematic viscosity</td>
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<td>ρ</td>
<td>Fluid density</td>
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<td>ω_X, ω_Y, ω_Z</td>
<td>Dimensionless fluctuation frequencies in X, Y and Z directions</td>
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<td>x, y, z</td>
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<tr>
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<tr>
<td>a, b, c</td>
<td>Unit vector in freestream vortex axis direction</td>
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<td>Unit vector in freestream vortex binormal direction</td>
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<td>Fluctuating pressure</td>
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<td>Re</td>
<td>Boundary layer thickness Reynolds number</td>
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<td>Re_T</td>
<td>Streamwise distance Reynolds number</td>
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<td>Time</td>
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<td>T_u</td>
<td>Dimensionless time</td>
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<td>u</td>
<td>Streamwise mean velocity</td>
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<td>u', v', w'</td>
<td>streamwise, normal and spanwise fluctuating velocity components</td>
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<td>β</td>
<td>Fluctuation decay coefficient</td>
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INTRODUCTION
For many years, engineers have used empirical correlations (e.g. Abu-Ghannam and Shaw [1]) to predict transition on gas turbine blades. These correlations are generally fairly reliable for attached flows and where the flow is at least approximately two dimensional, but for separated flow or where strong blade curvature or sweep is present there are no reliable correlation procedures. Furthermore, the collection and correlation of experimental data to encompass all the possible flow scenarios encountered on a gas turbine blade would appear to be an exhaustive task. In the last few years, Direct Numerical Simulation techniques have been used successfully to predict transition for simple geometries. These methods do have the potential to predict transition accurately for situations where correlations cannot. However, currently DNS is computationally too expensive to be used for engineering design calculations.

Work by Johnson and co-workers [2,3] and Mayle and co-workers [4,5] has improved physical understanding of the bypass transition process. They recognised that the growth of low frequency streamwise velocity fluctuations in the pre-transitional boundary layer is directly responsible for the generation of turbulent spots. This resulted in the derivation of transition prediction procedures for both zero pressure gradient flows (Mayle et al. [5]) and favourable and adverse pressure gradients (Johnson and Erkan [3]). Although these procedures were successful, they still require empirical correlation procedures to predict the velocity fluctuations in the pre-transitional boundary layer. Therefore although the methods provide an improved understanding of bypass transition their
extension to more complex flow situations isn’t possible without further extensive empirical correlation data.

The purpose of the current work was to develop a numerical procedure to determine the velocity fluctuations in the pre-transitional boundary layer resulting from a prescribed freestream turbulence structure. The objective was to understand the mechanisms through which the fluctuations are generated and hence to identify a means of prediction for general three dimensional boundary layers.

**THEORY**

Experimental work (Johnson and Ercan [3]) has demonstrated that the amplitude of velocity fluctuations in the pre-transitional boundary layer scales with the amplitude of the freestream turbulence which induces them. This suggests that the fluctuations can be reasonably predicted through a linear perturbation method. Furthermore the majority of empirical correlations for start of transition use purely local boundary layer conditions. This suggests that although historical effects do exist (e.g. Leading edge geometry), these are of secondary importance. For simplicity therefore it has been assumed in the current work that the laminar fluctuations can be approximated as linear perturbations to a non-developing (inviscid) boundary layer whose profile is given by the polynomial

\[
U = 2\left(\frac{V}{c}\right)^3 - 5\left(\frac{V}{c}\right)^4 + 6\left(\frac{V}{c}\right)^5 - 2\left(\frac{V}{c}\right)^6
\]

The momentum equations governing the linear perturbation are then

\[
u_t + \frac{1}{Pr} p_s + \nu u_s' + \nu v' - \nu \nabla^2 u = 0
\]

\[
u_t + \frac{1}{Pr} p_s + \nu u_v' = -\nu \nabla^2 v = 0
\]

\[
u_t + \frac{1}{Pr} p_s + \nu w' = -\nu \nabla^2 w = 0
\]

and the continuity equation is

\[u_t + v_v' + w_z = 0
\]

The perturbation to be considered will be periodic in x, z and t, but must also decay in the streamwise direction through viscous dissipation. Therefore, if spanwise symmetry about \(z = 0\) is assumed, the perturbation will be given by

\[
u' = u_p e^{i\Omega_x(x-ct)} \cos(\Omega_z Z) e^{-\beta x}
\]

\[
u' = v_p e^{i\Omega_x(x-ct)} \cos(\Omega_z Z) e^{-\beta x}
\]

\[
u' = w_p e^{i\Omega_x(x-ct)} \sin(\Omega_z Z) e^{-\beta x}
\]

\[
u' = p_m e^{i\Omega_x(x-ct)} \cos(\Omega_z Z) e^{-\beta x}
\]

where the coefficients \(u_p, v_p, w_p\) and \(p_m\) are complex functions of \(Y, \Omega_x\) and \(\Omega_z\) are the angular frequencies in the streamwise and spanwise directions respectively. \(c\) is the perturbation convection velocity and \(\beta\) is the streamwise decay coefficient.

The dimensionless spatial and temporal co-ordinates \(X, Y, Z\) and \(T\) are defined in the nomenclature.

Substituting these expressions for the perturbation into the equations of motion (Equations 2 to 5) leads to

\[
-U_t + \frac{1}{Pr} p_s + \nu \nabla^2 u = 0
\]

\[
(v - \nu \nabla^2 v = 0
\]

\[
(w - \nu \nabla^2 w = 0
\]

where \(D = \frac{d}{dy}\)

**Freestream perturbations**

In the freestream \(Du = 0\) and so through the elimination of the velocity fluctuations between equations 2 to 5 it can be shown that the Laplacian of the pressure fluctuation is zero.

\[v^2p' = 0
\]

and hence substituting from equation 9 and assuming a periodic variation of frequency \(\Omega_x\) in the Y direction.

\[(\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2 - \beta^2 + i2\beta\Omega_x)p = 0
\]

The only viable solution for a freestream disturbance where \(\Omega_x \neq 0\) is that \(p_s = 0\). Equation 10 now becomes

\[U_t + \frac{1}{Pr} p_s + \nu \nabla^2 u = 0
\]

Identical forms of this equation also result from equations 11 and 12 for the \(v\) and \(w\) velocity components and so for a non-zero solution for the perturbation velocity field.

\[c = 1 + \frac{2\beta}{Pr}
\]

and

\[\beta^2 + \Re \beta - (\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2) = 0
\]

or for positive \(\beta\)

\[\beta = \frac{-\Re + (\Re^2 + 4(\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2))^{0.5}}{2
\]

For typical boundary layers and frequencies

\[\Re >> (\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2)
\]

so

\[\beta = \frac{(\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2)}{\Re}
\]

The convection velocity (Equation 17) can now be re-written as

\[c = 1 + 4(\Omega_{x1}^2 + \Omega_{x2}^2 + \Omega_{x3}^2)^{0.5}
\]
where \( F_x = \frac{\Omega_y}{Re} \) = \( f_{xu} \), \( F_y = \frac{f_{yu}}{U} \) and \( F_z = \frac{f_{zu}}{U} \).

The amplitude of a perturbation \( A \) with spatial frequencies \( F_x \), \( F_y \) and \( F_z \) is then given by

\[
A = A_0 \exp \left[ 0.5 \, \text{Re} \left( 1 - \left[ 1 + 4 \left( \frac{F_x^2 + F_y^2 + F_z^2}{F_x^2} \right)^{0.5} \right] \right) \right]
\]

(23)

where \( A_0 \) is its amplitude at some arbitrary \( X = 0 \) datum. For typical windtunnel conditions the dimensionless frequencies \( F_x \), \( F_y \) and \( F_z \) will be much less than unity and so \( A \) can be approximated as

\[
A \approx A_0 \exp \left[ - \text{Re} \left( F_x^2 + F_y^2 + F_z^2 \right) \right]
\]

(24)

This equation suggests that the frequency spectra for the perturbations is only a function of \( \text{Re} \left( F_x^2 + F_y^2 + F_z^2 \right) \) and hence that the turbulent length scale will vary as \( x^{0.5} \) which has been confirmed in wind tunnel experiments (Roach[6]). The overall turbulent decay can also be determined as

\[
Tu^2 = Tu_s \int_0^1 \int_0^2 \int_0^1 A^2 \, dF_x \, dF_y \, dF_z
\]

(25)

which results in \( Tu = Tu_s \, \text{Re}^{-0.75} \)

(26)

This result does not agree exactly with the experimental results of Roach who observed a power law decay but with a coefficient of -0.714.

**Boundary layer fluctuations**

The momentum equations (10-12) can now be simplified using the relationships 17 and 18 to obtain

\[
\left[ - \frac{1}{\text{Re}} (D^2 + \Omega \xi) + (i\Omega x - \beta)(\frac{U}{U} - 1) \right] U_p + \left( \frac{\partial U}{\partial y} \right) V_p + (i\Omega x - \beta)p_p = 0
\]

(27)

\[
\left[ - \frac{1}{\text{Re}} (D^2 + \Omega \xi) + (i\Omega x - \beta)(\frac{U}{U} - 1) \right] V_p + Dp_p = 0
\]

(28)

\[
\left[ - \frac{1}{\text{Re}} (D^2 + \Omega \xi) + (i\Omega x - \beta)(\frac{U}{U} - 1) \right] W_p - \Omega x p_p = 0
\]

(29)

The response of the laminar boundary layer to the freestream fluctuations defined by the ordinary differential equations 13, 27-29 can now be sought. Fourth order finite difference approximations using a uniform grid spacing of \( \Delta Y = 0.01 \) was used with a Gauss Seidel elimination procedure. This system of equations requires seven boundary conditions as the equations are second order in \( u_p \), \( v_p \) and \( w_p \) and first order in \( p_p \). Four of these boundary conditions are provided at the wall viz. \( u' = v' = w' = 0 \) and \( Dv' = 0 \). The remaining three boundary conditions result from the three velocity components in the freestream vortex array.

**Freestream Turbulence**

In the current work the response of the boundary layer to homogeneous isotropic freestream turbulence is sought. The momentum equations 27-29 are satisfied in the freestream by any periodic variation in \( Y \) of frequency \( \Omega_y \), but the amplitude of this variation must also satisfy the continuity equation 13. The solution for an array of vortices is

\[
\frac{v_i}{U} = e^{-\beta(\Omega x + \beta^2 n_x^2)} \frac{\cos(\Omega Y_1 - \alpha_Y) \cos(\Omega Z_1)}{(2\Omega x^2 + \beta^2(n_x^2 + b_x^2))^{0.5}}
\]

(31)

\[
\frac{w_i}{U} = e^{-\beta(\Omega x + \beta^2 n_x^2)} \frac{\sin(\Omega Y_1) \sin(\Omega Z_1 - \alpha_Z)}{(2\Omega x^2 + \beta^2(n_x^2 + b_x^2))^{0.5}}
\]

(32)

where \( U_1 \), \( v_1 \) and \( w_1 \) are the velocity components in the \( X_1 \), \( Y_1 \) and \( Z_1 \) co-ordinate directions, which are given by

**Figure 1. Definition of the vortex co-ordinate system.**

\[
\begin{align*}
X_i &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cos \theta \cos \phi \\ &\quad \cos \theta \sin \phi \\ &\quad \sin \theta \cos \phi \\
Y_i &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \cos \phi \sin \psi - \sin \theta \cos \phi \\ &\quad \cos \theta \sin \psi + sin \phi \\ &\quad -\cos \phi \sin \psi \\
Z_i &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \sin \phi \sin \psi - \cos \theta \cos \phi \\ &\quad \sin \phi \sin \psi + \sin \theta \cos \phi \\ &\quad \sin \theta \cos \phi \sin \psi + \cos \theta \cos \phi \\
\end{align*}
\]

(34)
such that the orientation of the vortex axes are given by the angles $\theta$ and $\phi$ and the relative orientation of each vortex axis to its neighbours in the array by the angle $\psi$ as shown in Fig. 1. The boundary layer response to this freestream vortex array is given by a combination of two solutions of the equations 13, 27-29 for frequencies of

$$
\begin{pmatrix}
\Omega_{\xi 1} \\
\Omega_{\eta 1} \\
\Omega_{\zeta 1}
\end{pmatrix} = \Omega
\begin{pmatrix}
x_x + b_x \\
x_y + b_y \\
x_z + b_z
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\Omega_{\xi 2} \\
\Omega_{\eta 2} \\
\Omega_{\zeta 2}
\end{pmatrix} = \Omega
\begin{pmatrix}
x_x - b_x \\
x_y - b_y \\
x_z - b_z
\end{pmatrix}
$$

Solutions were obtained for 100 vortex frequencies from 0 to $2U$ Hz, 40 $\phi$ angles from 0 to 90 degrees, 19 $\theta$ angles from 0 to 90 degrees and 45 $\phi$ angles from 0 to 45 degrees. The value of $\beta$ is such that freestream fluctuations above the upper limit frequency can be assumed to have decayed to a negligible amplitude.

**Near Wall Gain**

Johnson and Erkan [3] showed that in the Near Wall region (approximately $Y<0.2$) the near wall velocity fluctuations have an invariant frequency spectra and scale with the mean velocity. They therefore defined the Near Wall Gain as

$$G = \frac{u'U}{(U^2 + V^2)^{0.5}U} \quad (36)$$

as a measure of the receptivity of the boundary layer to freestream turbulence at a particular frequency. It should be noted that the hot wire used by Johnson and Erkan responded to both streamwise and normal velocity components but $V'=0$ near the wall.

**RESULTS**

Figure 2 shows the orientation averaged Near Wall Velocity Gain variation with the freestream vortex wavelength. The figure shows that the boundary layer is most responsive to a number of specific wavelengths The values of the gain are however strongly influenced by the vortex orientation as shown in Fig. 3. These results have only been averaged over the array orientation angle $\psi$ and clearly show that it is vortices with axes in or close to the streamwise direction that are most effective in generating near wall velocity fluctuations.

Figure 4 shows Y-Z plane velocity and pressure plots for a near streamwise vortex array at 25% and 50% points in the vortex cycle. The freestream vortex possesses a very low streamwise velocity component, but as the vortex interacts with the strong shear in the boundary layer, the normal velocity $V'$ component carries high $u$ velocity fluid into the boundary layer and although there is a rapid decay in the $V'$ component this induces a change in pressure which extends to the wall. This change in pressure results in a corresponding $u'$ component which is greatest near the wall where the steady velocity is least. It is therefore the pressure fluctuation which induces the strong streamwise velocity fluctuations near the wall. These streamwise ‘streaks’ actually possess only weak streamwise vorticity as indicated by the small $v'$ and $w'$ velocity components. The lower diagram in Fig. 4 shows that there is a phase lag between the pressure and streamwise velocity fluctuations. The amount of lag will depend on the relative magnitudes of $\Omega_x$ and $\beta$ which determine the relative magnitude of the real and imaginary terms in the governing equations 13 and 27-29.

The streamwise velocity fluctuations or streaks have also been observed in experimental boundary layer studies (e.g. Kittichacharn et al. [7] and Westin et al. [8]) and in DNS results Voke[9]). Westin et al. observed that the vortex spacing was approximately 1.2 $\delta$ which correlates closely with one of the peaks observed in Fig. 2. Vortices with wave numbers less than one will decay rapidly in the freestream as indicated by equation 24, so although the gain for these vortices is high their amplitude in the freestream and the boundary layer will be low.
Figure 4. Y-Z cross section of approximately streamwise vortices \( \theta = 0^\circ, \phi = 1^\circ \). Left hand diagrams show velocity perturbation and right hand diagrams show pressure perturbation. Contours: Blue - negative, Red - positive.

It is clear that the near wall streaks induced by freestream streamwise vortices have an important role in the transition process. It is also clear that if the receptivity of the boundary layer to these vortices can be reduced then transition can be delayed and it seems likely that this is how drag reduction techniques such as polymers, riblets and surface dendricles are successful.

**Streamwise velocity fluctuations.**

Equation 35 indicates that a vortex with a frequency \( \Omega \) will contribute velocity fluctuations in the three co-ordinate directions with frequencies between 0 and \( \sqrt{2} \Omega \) dependent on the vortex orientation. In particular, the vortices with streamwise or near streamwise axes (\( \theta \) small and \( \phi \) small) to which the laminar boundary layer is particularly receptive and have relatively high \( \Omega_y \) and \( \Omega_z \) frequencies will have very low streamwise frequencies. This results in the highest gains being observed at the lowest streamwise frequencies as indicated in Fig. 5, which shows the orientation averaged gain values plotted against the streamwise frequency \( \Omega_x \). The current results therefore identify the freestream streamwise vortices as being the source of these low frequencies. Mayle et al. [4,5] also determined that the low streamwise frequencies (wavelengths of 17-19 \( \delta \) ) were responsible for generating the majority of the near wall velocity fluctuations. The predicted gains in Fig. 5 are close to the measured values obtained by Johnson and Erkan shown in Fig. 6 for frequencies above 0.01. For lower frequencies the predicted values tend to decrease whereas the measured values are almost constant. The reason for this is probably that the frequency resolution of the current results is insufficient as the discrepancy at low frequencies was observed to worsen when lower numbers of vortex frequencies were used for the predictions. Fig. 2 indicated that resonant peaks exist in the frequency response and if these peaks are
turbulence levels which have their axes in approximate streamwise direction. These vortices induce strong pressure fluctuations within the boundary layer which induce streamwise velocity fluctuation streaks near the wall. These streaks have preferred spanwise spacings which concur with experimental observations.

3. The predicted near wall gain variation with streamwise fluctuation frequency agrees with previously published hot wire measurements.

4. The current results lead to a predictive procedure for start of transition from values of the freestream turbulence intensity and length scale which does not require any empirical information but is much simpler than DNS. The method can also be extended to predict spot generation rates and hence transition length.

**FUTURE WORK**

The present paper is restricted to zero pressure gradient boundary layers on flat plates, however the receptivity of any two dimensional boundary layer can be predicted simply by altering the boundary layer profile given in equation 1. Future work will therefore consider boundary layers subjected to streamwise pressure gradients including those that have undergone separation. The current procedure can be extended to three dimensional boundary layers and hence the effects of flow features such as secondary flow on receptivity and hence transition can be studied. It is also planned to use the procedure to investigate the effects of compliant surfaces and riblets on receptivity with a view to optimising drag reduction.

**REFERENCES**


