Projection-genericity of space curves

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Introduction

If \( \Gamma \) is a smooth space curve we can define a 3-parameter family of maps by considering the family of projections of \( \Gamma \) from a variable point of space to a fixed plane. It was shown by Soares David in [6] (see also [7]) that for a residual set of curves \( \Gamma \), this family is generic — i.e. transverse to the natural stratification of the family of singular smooth curves in the plane by equisingularity (and hence stable under perturbations). Thus only a short list (see (1) below) of singularities may occur. We recall these results in §1.

While attractive, this result leaves some business unfinished. A full statement ought in particular to produce a family of curves which is open as well as dense in the space of smooth maps \( \phi : S^1 \to \mathbb{R}^3 \) (with the \( C^\infty \)-topology). To obtain an openness result, one should compactify the parameter space. Thus, first, from now on we regard \( \mathbb{R}^3 \) as embedded in real projective space, which we denote \( P^3 \); and second, we have to consider projections from points of the curve itself. The projective setting offers further advantages, as the geometry is then simplified.

However while, provided that \( \Gamma \) is smoothly embedded, its projection from a point of itself is a well defined smooth curve, it is not now the case that varying the point gives a smooth, or even a continuous family of projections, and the study of this situation plays an essential rôle in the proof of openness.

We will see that the correct procedure gives a family of plane curves parametrised by the blow-up of \( P^3 \) along \( \Gamma \), in which a point lying over \( x \in \Gamma \) corresponds to the projection of \( \Gamma \) from \( x \), together with a straight line. Theorem 3.1, one of our key results, states that this gives a flat family of curves. We describe, in terms of the geometry, the restrictions to impose on this family, and show (in §4) that these hold for a dense open set of curves \( \Gamma \), defined by an explicit list of conditions, for which the family of projections is stable in a certain sense.

The details of proof require many calculations, which we defer to the final section.

1 Recall of earlier results

We recall the main results of [6], with details slightly altered to fit the context of this paper. Write \( f : S^1 \to P^3 \) for a \( C^\infty \)-map, with image a smooth curve \( \Gamma \). We will adhere to this notation throughout and, for example, have a family \( \{ f_u \} \) with image \( \Gamma_u \). Choose a neighbourhood \( U \) of \( \Gamma \), and fix a plane \( P^2 \subset P^3 \) and a compact subset \( K \) of \( P^3 \) disjoint from \( U \cup P^2 \). Then for any map \( f : S^1 \to U \) there is an induced map \( H_f : S^1 \times K \to P^2 \), where \( H_f(t,x) \) is the point where the straight line from \( f(t) \) to \( x \) meets \( P^2 \); the projection of \( f(t) \) from \( x \). It is easily seen that the singularities of this map do not depend on the choice of the plane \( P^2 \), which we
thus suppress from the notation. We regard $H_f$ as a $K$-parameter family of maps $S^1 \to P^2$, so that the jet extensions are taken as maps $j^f H_f : K \times S^1 \to j^f(S^1, P^2)$, and similarly for multijets.

**Theorem 1.1** [6, Theorem 1.1] For any submanifold $W$ of $j^f(S^1, P^2)$, the set $\{ f \in C^\infty(S^1, U) \mid j^f H_f \pitchfork W \}$ is residual in $C^\infty(S^1, U)$.

By applying this result to particular submanifolds $W$, Soares David obtains the complete list of singularities which appear in such families of projections. We shall use Arnold’s notations $A_n, D_n, X_9$, which usually refer to the equations defining these singularities, for the singularities of the image curve $H_f(S^1)$. Since a generic map from $S^1$ to $P^2$ has $A_1$ singularities, we call these non-degenerate and will usually ignore them. We list types of (degenerate) singularities as follows:

\[
\begin{array}{cccc}
\text{codimension} & 1 & 2 & 3 \\
A_2 & A_3 & D_4 \\
A_4 & A_5 & D_5 & D_6 & X_9. \\
A_6 & A_7 & D_8.
\end{array}
\]

**Theorem 1.2** [6, Theorem 5.1]; see also [5]. There is a residual set $U$ of maps $f : S^1 \to U$ such that for $f \in U$ the family $\{ C_x = H_f(S^1 \times x) \mid x \in K \}$ of image curves has the following properties. There is a smooth stratification of $K$ such that if $x$ belongs to a $(3 - c)$-dimensional stratum, $C_x$ has degenerate singularities, all in (1), only at a finite set, whose codimensions add up to $c$.

We next describe these conditions geometrically. We also wish to dispense with $K$ and $U$. Since $P^3$ is the set of lines through the origin in $\mathbb{R}^4$, we have a tautological bundle $\xi$ over $P^3$ which is a sub-bundle of the trivial bundle of rank 4. Write $\eta$ for the quotient $\mathbb{R}^3$ bundle, and $E = P(\eta)$ for the associated plane bundle. For any $x \in P^3$ there is a well-defined projection $\mathbb{R}^4 \to \eta_x$ and hence $\pi_x : P^3 \setminus \{ x \} \to E_x$. We can thus define our family of projections to be the family of restrictions $\{ \pi_x \Gamma : x \in P^3 \setminus \Gamma \}$. However, as our considerations are (multi-)local, we only deal at any time with points $x$ in a neighbourhood of one point, and a subset of $\Gamma$ in a neighbourhood of a fixed finite subset. Moreover, all conditions we impose are projectively invariant. We will thus continue to speak of a family of projections to $P^2$, and for calculations can take local co-ordinates with a fixed target plane $\mathbb{R}^2$.

We can now impose the conditions of Theorem 1.2, taking $K$ to be the complement of a (small) neighbourhood of $\Gamma$.

Suppose $\Gamma$ is an embedded curve with nowhere vanishing curvature (this is itself an open dense condition). Thus at each point $P \in \Gamma$ there are a well-defined tangent line $T_P \Gamma$ and osculating plane $O_P \Gamma$. We say $P$ is a stall of $\Gamma$ if the torsion vanishes at $P$, or equivalently, if the local intersection number $\Gamma.O_P \Gamma > 3$, and a transverse stall if $\Gamma.O_P \Gamma = 4$. An $r$-secant is a line, not a tangent, meeting $\Gamma$ in at least $r$ points: if $r = 2$ we omit the $r$. We will call it a $T$-$r$-secant if the tangents at two of its points $P, Q$ on $\Gamma$ lie in a plane (or equivalently, since we are in projective space, intersect); we call this plane the $T$-plane.

The projection of $\Gamma$ from a point $x \not\in \Gamma$ is an immersion at $P$ unless $x$ lies on $T_P \Gamma$. In this case the image has an $A_2$ singularity at the image of $P$ unless either $T_P$ meets $\Gamma$ again (when if $f \in U$ we have a $D_3$ singularity); or $P$ is a stall (if $f \in U$, all stalls are transverse). If $P$ is a transverse stall, there is a unique point $x_0$ — which we call the $P$-centre of the stall — on $T_P \Gamma$ such that the projection from a point $x \neq x_0$ on $T_P \Gamma$ has an $A_4$ type singularity, but projection from $x_0$ has a higher singularity (if $f \in U$, an $A_6$). It was pointed out by Soares David [6] that these points were omitted in [7].

If $f \in U$, $\Gamma$ has no bitangent, tangent which is a 3-secant, or tangent at a stall meeting $\Gamma$ again. It also follows that there is no 5-secant, T-4-secant, or T-3-secant with all 3 tangents coplanar.
Each T-secant contains a unique point $y_0$ — its T-centre — such that under the projection from a point $x \neq y_0$ (and $\neq P, Q$) on the T-secant, the intersection number of the images of the branches at $P$ and $Q$ is 2 (an $A_3$ singularity), but under projection from $y_0$, is $> 2$. In general, this is 3 (type $A_5$); otherwise it is 4 (type $A_7$) and we speak of a special T-secant (if $f \in U$, we do not have a higher singularity).

Projecting along a general 3-secant gives a $D_4$ singularity. A T-3-secant has a T-centre; the projection from another point has a $D_6$ singularity, but from the T-centre a $D_8$ (for $f \in U$ we do not have any special T-3-secant).

Finally, projecting along a 4-secant gives an $X_9$ singularity.

While there are $\infty^2$ secants, there are only $\infty^1$ tangents, T-secants or 3-secants, and only finitely many tangents meeting $\Gamma$ again, stalls, special T-secants, T-3-secants or 4-secants. Consideration of these lines explains why (1) does not include all singularities of $A_e$-codimension 3. For as only finitely many tangents meet $\Gamma$ again, in general none of them occur at a stall, meet $\Gamma$ a third time, or have the tangent at the second point contained in the osculating plane at the first; similarly of the finitely many 4-secants, in general none are T-4-secants or 5-secants. We may give this argument an alternative shape by observing that each of the remaining singularity types of codimension 3, if it occurred in our situation, would have to occur along a line, so occur in codimension 2.

## 2 Openness of transversality conditions

Given smooth manifolds $A$ and $B$ where $A$ (at least) is compact, we will use the $C^\infty$-topology on the space $C^\infty(A, B)$: since $A$ is compact, there is no distinction between the fine and the ordinary topology. Our crucial tool in proving openness will be the following.

**Proposition 2.1** [3, Corollary 3.4.12] Let $N$ and $P$ be $C^\infty$-manifolds, with $N$ compact. Then a subset $U$ is open in $C^\infty(N, P)$ if and only if for all 1-parameter families $\{f_u : N \to P \mid u \in U\}$ (for $U$ a neighbourhood of 0 in $\mathbb{R}$) defining a smooth map $F : N \times U \to P \times U$, the set $\{u \in U \mid f_u \in U\}$ is open in $\mathbb{R}$.

We begin with two easy applications.

**Lemma 2.2** The set $O_0$ of maps $f : S^1 \to P^3$ such that

- (Tim) $f$ is an immersion,
- (Tinj) $f$ is injective, and
- (Tcur) $\Gamma := f(S^1)$ has nowhere zero curvature

is dense and open in $C^\infty(S^1, P^3)$.

**Proof** Density follows from standard transversality theorems, since each condition involves avoidance of a subset of (multi-)jet space of codimension greater than the source dimension. Openness for the first two conditions follows since embeddings are open in the space of maps. As to (Tcur), let $\{f_u\}$ be a smooth family with $f_0 \in O_0$, and suppose there is a sequence $u_i \to 0$ such that for some $x_i \in \Gamma$, $f_{u_i}(S^1)$ has zero curvature at $x_i$. Passing to a subsequence, we may suppose (since $S^1$ is compact) that $x_i$ converges to a point $x_0$. But then it follows that $f_0(S^1)$ has zero curvature at $f_0(x_0)$, contrary to hypothesis.

**Lemma 2.3** The set of maps $f : S^1 \to P^2$ of immersions such that $f(S^1)$ has only non-degenerate singularities is dense and open in $C^\infty(S^1, P^2)$.

**Proof** Since the set $\Sigma_2$ of singular 1-jets has codimension 2, Thom’s transversality Theorem implies that the set of immersions are dense; since $\Sigma$ is closed, it
is open. The subset $\Sigma_4$ of $3j^0(S^1, P^2)$ corresponding to jets with a common target has codimension 4; the subset $\Sigma_3$ of $3j^1(S^1, P^2)$ of jets with common target and the two images tangent has codimension 3. Thus for a dense set of maps, we avoid both these, so have no degenerate singularities.

Now suppose $f_u$ a smooth family such that $f_0(S^1)$ has only non-degenerate singularities; given a sequence $u_i \to 0$ we will write $f_i$ for $f_{u_i}$ and $C_i$ for $f_i(S^1)$. Since immersions are open, we may suppose each $f_{u_i}$ an immersion. If, for some sequence $u_i \to 0$, $C_i$ has a triple point $f_i(P_i) = f_i(Q_i) = f_i(R_i)$ we may suppose, passing to a subsequence, that $P_i$ converges to a limit $P_0$; similarly for $Q_i$ and $R_i$. Were e.g. $P_0 = Q_0$, the curvatures of $C_i$ would be unbounded in the small intervals between $P_i$ and $Q_i$; whereas these must all converge to the curvature of $C_0$ at $P_0$, a contradiction. But if $P_0, Q_0$ and $R_0$ are all distinct, $C_0$ has a triple point, which is also a contradiction. Similarly if $C_i$ has a self-tangency at $f_i(P_i) = f_i(Q_i)$ we may suppose $P_i \to P_0$, $Q_i \to Q_0$ and $P_0 \neq Q_0$. But then $C_0$ has a self-tangency, a contradiction.

From now on, we will only consider maps $f \in O_0$. In this case, the curve $\Gamma$ determines $f$ up to composition with a diffeomorphism of $S^1$, and at each point $P \in \Gamma$, $\Gamma$ has a well defined tangent line $T_P \Gamma$ and osculating plane $O_P \Gamma$. For $\Gamma \in O_0$, the projection $\pi_P$ of $\Gamma$ from the point $P$ of itself is well defined and smooth: we write $C_P$ for its image. Also, the result depends smoothly on $P$, so defines a smooth, continuous map

$$pr_1 : O_0 \to C^\infty(S^1, C^\infty(S^1, P^2)).$$

Strictly speaking, we should be discussing sections of a bundle, but again this does not affect the local theory.

We thus need to consider 1-parameter families $F : U \times S^1 \to P^2$, where $U$ also is a copy of $S^1$. We define $F \in C_1$: if $f_u$ is an immersion with only non-degenerate singularities for all but finitely many $u \in U$, and for each $u$, $f_u$ has at most one degenerate singularity, which has codimension 1 (i.e. has type $A_2, A_3$ or $D_4$), and is versally unfolded by the family $\{f_u | u \in U\}$.

We define

$$O_1 := \{f \in O_0 | pr_1(f) \in C_1\}.$$

Lemma 2.4 (i) The set $C_1$ is dense and open in $C^\infty(U \times S^1, P^2)$.

(ii) The set $O_1$ is dense and open in $O_0$.

Proof We establish density by explicit listing of jet transversality conditions. It is convenient to take the two cases together. We begin with a list of strata we can avoid for dimensional reasons.

(TE1) $C_P$ has no quadruple point: $\Gamma$ has no 5-secant.

(TE2) $C_P$ has no triple point with 2 branches tangent: $\Gamma$ has no T-4-secant.

(TE3) $C_P$ has no double point with 1 branch singular: $\Gamma$ has no tangent 3-secant.

(TE4) $C_P$ has no double point where the branches have the same tangent and curvature: $\Gamma$ has no T-3-secant with T-centre on $\Gamma$.

(TE5) $C_P$ has no singular point of type higher than $A_2$: the tangent at a stall of $\Gamma$ does not meet $\Gamma$ again.

With these excluded, the only possible types of degenerate singularities are $A_2, A_3$ and $D_4$. For (i) we need each of these to be versally unfolded in the family $F$. This requires a slight variation of Mather’s multi-transversality theorem [2]. For a family $F : N \times U \to P$ we define partial jet extensions

$$r^{j_k}_U F : N^{(r)} \times U \to r^{j_k}(N, P)$$

by

$$r^{j_k}_U F(x_1, \ldots, x_r, u) := (j^k f_u(x_1), \ldots, j^k f_u(x_r)).$$
A singularity type corresponding to a submanifold \( \Sigma \) of \( rJ^k(N, P) \) is versally unfolded by the family \( F \) if and only if \( rJ^k F \) is transverse to \( \Sigma \). The set of \( F \) such that this holds is residual. It now suffices to apply this to the submanifolds \( \Sigma_2, \Sigma_3, \) and \( \Sigma_4 \) of multi-jet space defined above.

For (ii), \( C_P \) has a point of type \( A_2 \) if \( P \) lies on the tangent at some \( Q \in \Gamma \), of type \( A_3 \) if \( P \) is on a T-3-secant \( PQR \), and of type \( D_4 \) if \( P \) lies on a 4-secant \( PQRS \). We will show in Lemma 6.8 that the versality conditions in the definition of \( O_1 \) in cases \( A_2, A_3, \) and \( D_4 \) are equivalent respectively to

- (TD) If \( P \in T_Q \Gamma \) then \( T_P \Gamma \not\subset O_Q \Gamma \),
- (TC) No plane is tangent to \( \Gamma \) at 3 collinear points,
- (TX) The cross-ratio of the planes through a 4-secant \( L \) containing the 4 tangent lines is not equal to the cross-ratio of the 4 points on \( L \).

Each of these corresponds a subset of some jet space of such codimension that in general it is avoided.

We turn to openness. It suffices to consider \( C_1 \), since openness of \( O_1 \) then follows from the fact that \( pr_1 \) is continuous.

By Proposition 2.1 it suffices to consider a 1-parameter family \( \{F_t\} \) with \( F_0 \in C_1 \) and a sequence \( t_i \) converging to 0 such that \( F_{t_i} \not\in C_1 \), so there exist \( u_i \in S^1 \) and a ‘bad’ singular point \( P_i \) of \( F_{t_i},u_i(S^1) \). Passing to a subsequence if necessary, we may assume that \( u_i \) converges to \( u_0 \in S^1 \) and \( P_i \) to \( P_0 \in F_0,u_0(S^1) \). Arguing as for Lemma 2.3, we see that if \( P_i \) cannot be of one of the types excluded by (TE1)-(TE5), so has type \( A_2, A_3 \) or \( D_4 \). By hypothesis such a singularity is versally unfolded by the family \( F_0 \). But then, by openness of versality \([8, 3.7.1]\), the same is true for all small enough \( t_i \); a contradiction.

A simple consequence of the above conditions is

**Lemma 2.5** If \( \Gamma \in C_1 \), and the projection \( C_P \) from \( P \) has a singular point \( Z_P \), then

(i) \( Z_P \) is distinct from the image \( Y_P \) under projection of the tangent at \( P \);
(ii) the line \( Y_P Z_P \) is transverse to \( C_P \) at \( Z_P \).

**Proof** In the \( A_2 \) case, if \( Z_P = Y_P \), the line \( PQ \) is tangent to \( \Gamma \) at both \( P \) and \( Q \), contradicting (TD). If the line is tangent to \( C_P \) at \( Z_P \), we have \( T_P \Gamma \subset O_Q \Gamma \), again contradicting (TD).

In the \( A_3 \) case, if \( Z_P = Y_P \), \( T_P \Gamma \) would be a tangent 3-secant, contradicting condition (TE3). If tangency occurs, the \( T \)-plane would contain also the tangent at \( P \), contradicting (TC).

For \( D_4 \), if \( Z_P = Y_P \), \( T_P \Gamma \) would be a tangent 3-secant, again contradicting condition (TE3). If tangency occurs, we have a \( T \)-4-secant, contradicting (TE2).

We obtain further open conditions by using the arguments of [9]. Consider curves \( \Gamma \) in \( P^3 \) given as the image of maps \( f : S^1 \to P^3 \). If the local intersection number of \( \Gamma \) with a plane \( \Pi \) at a point \( P \) is \( k \), define the contact number of \( \Gamma \) with \( \Pi \) at \( P \) to be \( k - 1 \), and define the total contact number \( \kappa(\Gamma, \Pi) \) of \( \Gamma \) with \( \Pi \) to be the sum of the contact numbers at all their common points, or equivalently, the total intersection number \( \Gamma \cap \Pi \) diminished by the number \( \#(\Gamma \cap \Pi) \) of points of intersection.

Here we use ‘intersection number’ in the real sense: if \( \Pi \) is given locally by \( x = 0 \) and \( f \) by \( f(t) = (a(t), a'(t), a''(t)) \) then the intersection number at \( t = 0 \) is the multiplicity of 0 as root of \( a(t) \), i.e. the order of \( a(t) \) at \( t = 0 \). If \( a(t) \) has order \( k \) at \( 0 \), and we have a 1-parameter family \( \Gamma(u) \) of curves, leading to \( a_u(t) \), then the sum of orders of vanishing of \( a_u \) at values of \( t \) near \( 0 \) is \( \leq k \), as we can reduce to the case of a polynomial \( a_u(t) = t^k + \sum_{i=1} a_i(u)t^{k-i} \). We will use this semicontinuity property below.
Define a condition
(TK0) For each plane $\Pi$, $\kappa(\Gamma, \Pi) \leq 3$.

**Proposition 2.6** The family $\mathcal{O}_k$ of smooth curves $\Gamma$ satisfying (TK0) is a dense open set.

**Proof** Density follows by counting constants: the conditions on a multijet of a plane corresponding to having $\kappa(\Gamma, \Pi) \geq 4$ define a subset of multijet space such that multi-transversality to it implies avoidance.

Let $F = \{ f_t : S^1 \to P^3 \mid t \in U \}$ be a smooth 1-parameter family, with $f_0 \in \mathcal{O}_k$. Suppose there exist arbitrarily small values of $t$ for which there is a plane $\Pi_t$ whose total contact number with $\Gamma_t$ is at least 4. The set of all planes forms the dual projective space $P^{3V}$ and is, in particular, compact. Thus we may pick a sequence of values of $t$ converging to 0 for which $\Pi_t$ converges to a line $\Pi_0$.

For each point $P \in \Gamma_0 \cap \Pi_0$, with intersection number $a + 1$, say, choose a neighbourhood $U$ of $P$ not containing any other point of $\Gamma_0 \cap \Pi_0$. Then the sum of the intersection numbers of $\Pi_t$ with $\Gamma_t$ at points of $U$ is upper semicontinuous. For small $t$ either the number of points of $\Gamma_t \cap U \cap \Pi_t$ is 0, so the intersection number and contact number are also 0, or it is $\geq 1$, so as the total intersection number is $\leq a + 1$, the contact number is $\leq a$. So the total contact number also is upper semicontinuous. Hence the total contact number of $\Gamma_0$ with $\Pi_0$ is $\geq 4$, a contradiction.

We show in Proposition 6.12 that if $W_{111} \subset 3J^1(S^1, P^3)$, $W_{21} \subset 2J^2(S^1, P^3)$ and $W_3 \subset J^3(S^1, P^3)$ are the submanifolds corresponding to planes having $\kappa(\Pi, \Gamma) \geq 3$ with contact numbers 1+1+1, 2+1 and 3 respectively, the multijet of $f$ fails to be transverse if and only if either $\kappa(\Pi, \Gamma) \geq 3$ or (in the 1+1+1 case) the 3 contact points are collinear.

The condition (TK0) can be expressed as the union of conditions corresponding to partitions of 4:

1. (1111) no plane is tangent at more than 3 points,
2. (211) no plane is osculating at one point and tangent at 2 more,
3. (31) the osculating plane at a torsion zero point does not touch the curve again,
4. (22) no plane is osculating at more than 1 point,
5. (4) the zeros of the torsion are non-degenerate.

Define a curve $\Gamma$ to be section-generic, and write $\Gamma \in \mathcal{O}_s$ if (TK0), (TC) and (TD) hold.

**Proposition 2.7** $\mathcal{O}_s$ is dense and open in $C^\infty(S^1, P^3)$.

This result is proved in [1] by another method: the authors construct an auxiliary map $\phi_f$ and show that $\phi_f$ depends continuously on $f$, and that $\phi_f$ is $(C^\infty)$-stable if and only if $\Gamma$ is section-generic. The result then follows from the openness (due to Mather) of the set of stable maps.

**Lemma 2.8** Suppose $p$ not a double point of $C_P$. Then for each line $L$ through $p$, with pre-image the plane $\Pi$ through $T_P\Gamma$, $\Pi.\Gamma = L.C_P + 1$, so that if $\#(\Pi \cap \Gamma) = \#(L \cap C_P)$, we have $\kappa(\Pi, \Gamma) = \kappa(L, C_P) + 1$.

**Proof** At a point $Q \in \Pi \setminus T_P\Gamma$, the local equation of $\Pi$ is the composite of projection with the local equation of $L$, so the respective intersection numbers coincide. Under the hypothesis, the only point of $T_P\Gamma \cap \Gamma$ is $P$ itself.

At $P$ we need a local calculation. Take local coordinates with $\Gamma$ given by $(t, \phi(t), \psi(t))$, where $\phi, \psi$ each have order at least 2 at $t = 0$, so $T_P$ is the $x$-axis, and we may take $\Pi$ as the plane $y = 0$. The projection $C_P$ from $P$ is then parametrised by $(\phi(t)/t, \psi(t)/t)$. We thus have

$$L.C_P = \ord_t(\phi(t)/t) = \ord_t(\phi(t)) - 1 = \Pi.\Gamma - 1.$$
3 Flatness of the family of enhanced projections

The crucial stage in our analysis is the passage from the family of projections of \( \Gamma \) from points of itself to the projections from nearby points. First suppose in suitable local coordinates we have \( f(t) = (t, t^2(t), t^3(t)) \), where \( \phi \) and \( \psi \) vanish at \( t = 0 \) so that the tangent there is the \( x \)-axis. We consider projection from the point \((0, Y, Z)\) on the normal plane (any point close to \( \Gamma \) lies on a unique normal plane locally) to the fixed plane \( x = 1 \). This gives the map \( t \mapsto (\phi(t) + Y(1-t^{-1}), \psi(t) + Z(1-t^{-1})) \).

We are interested in what happens for \( Y \) and \( Z \) small. For \( t \) very small, the result is close to the line \( t \mapsto (Y(1-t^{-1}), Z(1-t^{-1})) \), or \( Zy = Yz \), and for \( t \) large, the result is close to \( t \mapsto (\phi(t) + Y, \psi(t) + Z) \), which is a translation of the projection from the origin. Thus in some sense, the image of projection from \((0, Y, Z)\) converges to the union of the line \( Zy = Yz \) with the image of the projection from \((0,0,0)\). In other terms, the projection of \( \Gamma \) from a point near \( P \in \Gamma \) is close to the union of the projection \( C_\pi \) of \( \Gamma \) from \( P \) and a line \( L \) through the point \( Y_P \) corresponding to \( P \) itself (the image of the tangent line \( T_P \Gamma \)).

To make this precise we need to deal with values of \( t \) which are neither very small nor very large. However, since \( Y \) and \( Z \) themselves are small, the image points in question are close to the origin, so that the crucial question is local.

Write \( B : X \to P^3 \) for the blow up of \( P^3 \) along \( \Gamma \), and \( E \) for the \( P^2 \) bundle over \( X \) given by pulling back the universal bundle over \( P^3 \). Define a family of curves in the fibres of \( E \) as follows. If \( z = B(x) \not\in \Gamma \), then \( \Gamma_x = \pi_x(\Gamma) \). If \( z \in \Gamma \), then \( x \) defines a plane \( P_x \) through the tangent \( T_x \Gamma \), and we define \( \Gamma_x := \pi_x(\Gamma \setminus \{z\}) \cup \pi_x(P_x) \).

**Theorem 3.1.** Suppose that no plane has infinite order of contact with \( \Gamma \). Then near the exceptional hypersurface \( \pi^{-1}\Gamma \), the above is a flat family. This means that near any point there is a smooth function whose zero set intersects each fibre in the given curve.

**Proof** The assertion is local, so we will work in local coordinates. We may suppose \( \Gamma \) given by a smooth parametrisation

\[(x, y, z) = (t, \phi(t), \psi(t)),\]  

where each of \( \phi, \psi \) has order at least 2 at \( t = 0 \). We define local coordinates on the blow-up by setting

\[Z = \psi(X) = U(Y - \phi(X)).\]  

The behaviour of the projection is unaltered if we regard it as projection onto the plane \( x = 1 \): denote coordinates in this image plane by \((v, w)\) (rather than \((y, z)\)) to avoid confusion. Thus the image of the projection of the point \((x, y, z)\) from the point \((X, Y, Z)\) is given by

\[v = \frac{(1 - x)Y + (X - 1)y}{X - x}, \quad w = \frac{(1 - x)Z + (X - 1)z}{X - x}.\]  

Substituting for \( Z \) from (3) and for \( x, y, z \) from (2) we find

\[v = \frac{(1 - t)Y + (X - 1)\phi(t)}{X - t}, \quad w - Uv = \frac{(1 - t)(\psi(X) - U\phi(X)) + (X - 1)(\psi(t) - U\phi(t))}{X - t}.\]  

Clearing the denominator in (4) and rearranging gives

\[(vX - Y) + t(Y - v) = (X - 1)\phi(t).\]  

7
On the other hand, since $\phi(t)$ and $\psi(t)$ are $C^\infty$ near $t = 0$, the right hand side of (5) extends to a $C^\infty$ function $\Psi(X,U,t)$ near the origin. Letting $t$ tend to $X$, we find that $\Psi(X,U,X) = (1 - X)(\psi'(X) - U\phi'(X)) + (\psi(X) - U\phi(X))$. Moreover, $\Psi(0,0,t)$ reduces to $\psi(t)/t$. We now distinguish cases according to the order of $\psi(t)$ at $t = 0$.

First let the order be 2. Then $\Psi(X,U,t)$ has non-zero coefficient of $t$. By the implicit function theorem, there is a $C^\infty$ function $\tau(W,X,U)$ (defined near the origin) such that $t = \tau(W,X,U)$ if and only if $W = \Psi(X,U,t)$. We can thus substitute $t = \tau(w - Uv,X,U)$ in (6), and obtain

$$v(X - \tau(w - Uv,X,U)) = Y(1 - \tau(w - Uv,X,U)) + (X - 1)\phi(w - Uv,X,U)).$$

(7)

Conversely, suppose (7) holds. Define $t := \tau(w - Uv,X,U)$. Then by the definition of $\tau$, (5) holds, and substituting in (7) gives (6).

If $t \neq X$, we can divide by $X - t$ to obtain (4); hence $(v,w)$ is indeed the projection of $(t,\phi(t),\psi(t))$ from $(X,Y,Z)$ with $Z$ given by (3).

If, however, the solution is $t = X$, then (6) reduces (since $X$ is small, so $X \neq 1$) to $Y = \phi(X)$. Thus $Z - \psi(X) = U(Y - \phi(X)) = 0$, and the point of projection lies on the curve $\Gamma$. Our other equation is $w - Uv = \Psi(X,U,X)$, which we can rewrite as

$$w - \psi(X) - (1 - X)\psi'(X) = U(v - \phi(X) - (1 - X)\phi'(X)).$$

(8)

But the plane through the tangent line at $t = X$ to $\Gamma$ corresponding to the auxiliary parameter $U$ is given by

$$z - \psi(X) - (x - X)\psi'(X) = U(y - \phi(X) - (x - X)\psi'(X)),$$

and its projection is given by setting $x = 1$, $y = v$, $z = w$, and thus coincides with (8).

We follow a similar plan in general. Let the order of $\psi(t)$ at $t = 0$ be $n + 1$. Since this is the order of contact of the curve with the plane $Z = UY$, it is finite by hypothesis. Then by Malgrange’s preparation theorem (a generalisation of the implicit function theorem), for any $C^\infty$ function $h(t,X,U)$ of $t,X$ and $U$ we can find $C^\infty$ functions $\alpha_i(W,X,U)$ (for $1 \leq i \leq n$) such that $h(t,X,U) = \sum_{i=1}^{n} \alpha_i(W,X,U)t^{n-i}$ when $W = \Psi(X,U,t)$. Since the difference vanishes when $W = \Psi(X,U,t)$ it is divisible by it, so there exists a further $C^\infty$ function $C(W,X,U,t)$ such that

$$h(t,X,U) - \sum_{i=1}^{n} \alpha_i(W,X,U)t^{n-i} = C(W,X,U,t)(W - \Psi(X,U,t)).$$

(9)

We apply this twice, first taking $h$ to be $(X - 1)\phi(t)$ and second, taking $h$ to be $t^n$: denote the functions $\alpha_i$ and $C$ in the second case by adding a prime. Replacing $(X - 1)\phi(t)$ using (6), we have two polynomial equations for $t$ which hold when $W = \Psi(X,U,t)$:

$$(vX - Y) + t(Y - v) = \sum_{i=1}^{n} \alpha_i(W,X,U)t^{n-i},$$

(10)

$$t^n = \sum_{i=1}^{n} \alpha'_i(W,X,U)t^{n-i}.$$ 

(11)

Form the resultant of these two equations (e.g. write down the Sylvester determinant), and denote it by $R(W,X,Y,U,v)$. Then we claim that the equation $R(w - Uv,X,Y,U,v) = 0$ defines the required locus.
Suppose given a point \((X,Y,U,v,w)\) on which \(R\) vanishes. Since \(R\) was defined as a resultant, there exists at least one value of \(t\) for which equations (10) and (11) both hold, with \(W = w - Uv\). Hence \(C'(w - Uv, X, U,t)(w - Uv - \Psi(X,U,t)) = 0\).

We will see below that \(C'\) does not vanish at, hence near, the origin. It then follows that (5) holds. We also have the identity

\[
(X - 1)\phi(t) - \sum_{i=1}^{n} \alpha_i(W,X,U)t^{n-i} = C(W,X,U,t)(W - \Psi(X,U,t)).
\]

Substituting \(W = w - Uv\) we see that the right hand side vanishes, hence so does the left: combining with (10) we deduce that (6) also holds.

It follows, as before, from (5) and (6) that if \(t \neq X\) the point \((v, w)\) lies on the projection of \(\Gamma\) from \((X, Y, Z)\); while if \(t = X\) then \((X, Y, Z) \in \Gamma\) and the point lies on the line (8) which is the projection of the plane (parametrised by \(U\)) through the tangent to \(\Gamma\).

It remains to show that \(C'\) does not vanish. Setting \(W = X = U = 0\) in the defining identity of \(C'\) gives \((t^n - \sum \alpha_i(0,0,0)t^{n-i}) = C'(0,0,0,t)(-\phi(t)/t)\). Here the left hand side has order at most \(n\); the right hand side has order exceeding \(n\) unless \(C'(0,0,0,0) \neq 0\).

The parameter \(n\) in this proof is the local intersection number of \(\Gamma\) with the plane \(\Pi : z = 0\) which projects to the line \(L\). Since \(\Pi\) passes through the tangent line, \(n \geq 2\). We have \(n = 2\) if \(\Pi\) is not the osculating plane; if it is, \(n = 3\) if the point is not a stall, and \(n = 4\) if it is a transverse stall.

The elimination can be written down more easily in the case \(n = 2\), but this does not essentially simplify the argument. The equations obtained are too complicated to be of direct use (even for the case when \(\Gamma\) is given by \((t, t^2, t^3)\)), the resultant has 85 terms). When \(n = 2\) we begin with two transverse smooth branches, forming an \(A_1\), so no degenerate singularity occurs. When \(n = 3\) we begin with an \(A_3\) singularity: we can illustrate what happens by taking \(\Gamma\) to be a twisted cubic. Then the projection from \(X \not\in \Gamma\) is a rational cubic, nodal if \(X\) does not lie on the surface of tangents, cuspial if it does. The projection from \(X \in \Gamma\) is a smooth conic; \(L\) touches this if and only if \(\Pi\) osculates \(\Gamma\). When \(n = 5\) we have a deformation of an \(A_3\) singularity, which cannot be versal for dimensional reasons (indeed the above example illustrates that versality fails also for the \(A_3\) case).

Note that if the tangent at \(P\) passes through another point \(Q\) of \(\Gamma\), we have constructed a flat family by considering a neighbourhood of \(P\). The projection of the part of \(\Gamma\) near \(Q\) gives a branch \(B_Q\) (which is smooth as we exclude bitangents), so the total projection from points near \(P\) is the union of the relevant member of the flat family and a smooth deformation of \(B_Q\).

### 4 Enhanced projections from points of \(\Gamma\)

We saw in Theorem 3.1 that as \(X\) converges to a point \(P \in \Gamma\), the image of the projection \(C_X\) of \(\Gamma\) from \(X\) converges, not to the projection \(C_P\) of \(\Gamma\) from \(P\), but to the union of \(C_P\) with a line \(L\) through the image point \(Y_P\) of the tangent \(T_P\) of \(\Gamma\). We thus have a 1-parameter family of based curves \(\{C_P, Y_P\}\) and the 2-parameter family obtained by adjoining a variable line through \(Y_P\).

It turns out that the families obtained as above by projection have properties not shared by general 1-parameter families of based curves, so we confine ourselves to families \(\{C_P, Y_P\}\) obtained by projection from a space curve \(\Gamma \in \mathcal{O}_1\). Thus \(C_P\) is non-degenerate for all but finitely many \(P\), in which cases it has a single singularity of one of the types \(A_2\), \(A_3\) or \(D_4\). Denote these cases \(\alpha, \beta\) and \(\gamma\) respectively. By
Lemma 2.5, in each of these cases the degenerate singular point $Z_P$ is distinct from $Y_P$, and the line $Y_P Z_P$ is transverse to $C_P$ at $Z_P$. Thus if $Y_P$ is a singular point of $C_P$, it is an ordinary double point. We will denote by $\delta$ the case when it is.

For a single based curve $(C,Y)$, we consider the family of lines $L$ through $Y$ and their contact with $C$. The condition $\kappa(L, C) = 0$ is open: not only for nearby lines $L$ with fixed $C$, but also for nearby pairs $(L, C)$.

Next suppose $C$ non-degenerate and $\kappa(L, C) = 1$. The contact occurs at a unique point $u \in C$, and either

a: is a simple tangency at a smooth point of $C$, or
b: is a transverse intersection at a double point (which must be ordinary).

We distinguish the cases when $u = Y$ and denote them by $a^*$ and $b^*$; thus the latter only occurs in case $\delta$.

The condition $\kappa(L, C) = 1$ also has a certain stability property. In case $a$, $u \neq Y$ is a smooth point of $C$, and for any $(C', Y')$ near $(C, Y)$, there is a unique line $L'$ through $Y'$ and near $L$ which touches $C'$, and other nearby lines are transverse to $C$. In case $a^*$, $L$ is the tangent to $C$ at $Y$ and so deforms smoothly as we deform the pair $(C, Y)$. In case $b$, as we deform $C$, the point of transverse self-intersection will persist for nearby curves, and again there is a unique way to deform $L$, while for other nearby lines, the total contact number is 0. We will give more detail in the proof of Proposition 6.15.

We now impose two further conditions

(TK1) A line $L$ through $Y_P$ and a degenerate singular point of $C_P$ passes through no other singular point, and is transverse to $C_P$.

Transversality at the singular point itself was analysed in Lemma 2.5.

(TK2) For any $P$, and any $L$ through $Y_P$, $\kappa(L, C_P) \leq 2$.

If $C_P$ has no double point on $L$, this condition is equivalent to (TK0), as follows from Lemma 2.8, but the new condition excludes many further possibilities.

Now suppose that $\kappa(L, C_P) = 2$, and that $L$ does not pass through a degenerate singular point of $C_P$. If the contact occurs at two points, the cases may be enumerated as $aa, aa^*, ab, ab^*, ba^*, bb, bb^*$. If $L$ has contact number 2 at one point $u$, we may have

- $a_2$: $L$ is an inflexional tangent at $u \neq Y_P$,
- $a_2^*$: $L$ is an inflexional tangent at $Y_P$,
- $b_2$: $L$ is tangent to one branch at a double point $u \neq Y_P$, or
- $b_2^*$: $L$ is tangent to one branch at the double point $Y_P$; when necessary we distinguish the case $b_2^*+$, when $L$ corresponds to the osculating plane at $P$ from $b_2^*$, when $L$ is tangent to the other branch.

Thus in case $\delta$ there are several lines with $\kappa(L, C_P) = 2$: the two tangents at $Y_P$ (each in case $b_2^*$), tangents from $Y_P$ to $C_P$ (each of type $ab^*$) and lines from $Y_P$ to other double points (each of type $bb^*$).

The singularities of $C_P \cup L_P$ are thus as follows, where for the cases $\alpha, \beta$ and $\gamma$ we take $L_P$ as the line joining $Y_P$ to the degenerate singular point, in the rest, $L_P$ is the line through $Y_P$ with $\kappa(L_P, C_P) = 2$:

<table>
<thead>
<tr>
<th>Type</th>
<th>Sings</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$D_5$</td>
<td>$D_6$</td>
</tr>
</tbody>
</table>

We next interpret the cases in terms of the geometry of the space curve. In each of the following list of 12 cases, we first describe the configuration of the projected curve $C_P$, and give the singularities of $C_P \cup L_P$. Here we write $p, q$ etc. for the images of $P, Q$ etc.

$\alpha$: The curve $C_P$ has a cusp at $q$. The point $P$ lies on the tangent to $\Gamma$ at another point $Q$. 

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\(\beta\): The curve \(C_P\) has a self-tangency at \(q = r\).

The point \(P\) lies on a T-trisecant \(PQR\) with the tangents at \(Q, R\) coplanar.

\(\gamma\): \(C_P\) has a triple point \(q = r = s\).

The point \(P\) lies on a 4-secant \(PQRS\).

\(\delta\): \(p = q\) is a double point of \(C_P\).

The tangent to \(\Gamma\) at \(P\) meets \(\Gamma\) again in \(Q\).

\(aa\): \(L_P\) touches \(C_P\) at \(q\) and \(r\).

The tangents to \(\Gamma\) at \(P, Q\) and \(R\) are coplanar.

\(aa^*\): \(L_P\) touches \(C_P\) at \(p\) and \(q\).

The tangent to \(\Gamma\) at \(Q\) lies in the osculating plane \(O_P\).

\(ab\): \(L_P\) touches \(C_P\) at \(q\) and passes through a double point \(r = s\).

The plane \(\pi_P\) contains the tangents to \(\Gamma\) at \(P, Q\) and the trisecant \(PRS\).

\(ba\): \(L_P\) touches \(C_P\) at \(p\) and passes through a double point \(r = s\).

The osculating plane \(O_P\) contains the trisecant \(PRS\).

\(bb\): \(L_P\) passes through double points \(q = r\) and \(s = t\).

The plane \(\pi_P\) contains the tangent to \(\Gamma\) at \(P\) and the trisecants \(PQR\) and \(PST\).

\(a_2\): \(L_P\) is an inflexional tangent at \(q\).

The tangent to \(\Gamma\) at \(P\) lies in the osculating plane \(O_Q\).

\(a_2^*\): \(L_P\) is an inflexional tangent at \(p\).

\(P\) is a stall on \(\Gamma\).

\(b_2\): \(L_P\) is tangent at \(q\) to a double point \(q = r\) of \(C_P\).

We have a T-trisecant \(PQR\): the tangents at \(P, Q\) lie in a plane \(\pi_P\).

Observe that the same geometrical situation may give rise to 2 cases by permuting the roles of the points \(P, Q\) etc: these pairs are \(\alpha\) and \(\delta\) (a tangent to \(\Gamma\) meets the curve again), \(\beta\) and \(b_2\) (there is a T-trisecant), and \(a_2\) and \(aa^*\) (the osculating plane at one point contains the tangent at another).

We will find that the cases with no \(b\) in the notation (no collinearity condition) are easier to treat. In these cases the plane \(\pi_P\) satisfies \(\kappa(\pi_P, \Gamma) = 3\), where 3 is partitioned as 3 (case \(a_2^*\)), 21 (cases \(a_2, aa^*\)), and 111 (case \(aa\)) These cases occurred in the discussion of section genericity, as did \(\alpha\) and \(\delta\). The cases \(ab, bb, a^*b\) and \(b_2\) will require deeper calculations.

We require each case to occur transversely. A convenient precise formulation is (TMS) the multi-jet extensions of \(f\) are transverse to the submanifolds of jet space defining each of the 12 cases.

However in all cases except \(ab, a^*b, bb\) this already follows from conditions previously considered.

**Proposition 4.1** In the cases below, (TMS) is equivalent to the named condition:

\[
\begin{array}{cccccccccc}
aa & aa^* & a_2 & a_2^* & \alpha & \beta & \gamma & \delta & b_2 \\
(TK0) & (TK0) & (TK0) & (TK0) & (TD) & (TE4) & (TX) & (TD) & (TE4)
\end{array}
\]

These follow from the more detailed statements of Propositions 6.12 and 6.13. Thus for these 9 cases, the conditions hold for all \(\Gamma \in \mathcal{O}_1 \cap \mathcal{O}_2\). We will also show in Proposition 6.14 that in case \(a^*b\), (TMS) is a consequence of (TK2).

If a 2-point singularity \(aa, ab\) or \(bb\) occurs at \(t = t_0\), as we deform \(t\) the two singularities on \(L_0\) deform continuously, to \(q_t, r_t\), say. Write \(\theta_t\) for the angle between \(p_0 q_0\) and \(p_0 r_t\) and \(\Delta_t\) for the area of the triangle \(p_0 q_0 r_t\).

(Trot): \(\partial \theta_t/\partial t \neq 0\) at \(t = t_0\), or equivalently, \(d \Delta_t/dt \neq 0\) at \(t = t_0\).

We show in Proposition 6.15 that in cases \(ab, bb\), (TMS) is equivalent to (Trot).

We define \(\Gamma \in \mathcal{O}_2\) if \(\Gamma \in \mathcal{O}_1\), the family \((\Gamma, Y_P)\) of its self-projections satisfies (TK1) and (TK2), and (TMS) (or, equivalently, (Trot)) holds.

**Proposition 4.2** \(\mathcal{O}_2\) is dense and open in the set of families of space curves.
Proof As to density, we can suppose \( \Gamma \in \mathcal{O}_1 \). Then the values of \( t \) corresponding to types \( \alpha, \beta \) and \( \gamma \) are isolated, and they are stable under deformation of the family. For a single based curve of one of these types, the condition that \( Y_PZ_P \) is transverse to \( CP \) hold for a dense set of such based curves. It follows that (TK1) holds on a dense set.

The condition that for some \( P \) there exists a line \( L \) through \( Y_P \) with \( \kappa(L,C_P) \geq 3 \) defines subsets of the relevant multijet spaces of codimension exceeding that of the source. Hence (TK2) holds on a dense set.

Since (TMS) is a finite collection of multi-transversality conditions, it too holds densely.

We turn to openness. By Lemma 2.4, \( \mathcal{O}_1 \) is open. Now again cases \( \alpha, \beta, \gamma \) occur discretely in the family, and for a single curve of one of these types, the condition that \( Y_PZ_P \) is transverse to \( CP \) defines open sets of such based curves. It follows that (TK1) gives an open set.

For (TK2), we use the criterion of Proposition 2.1. Thus suppose given a 1-parameter family \( \Gamma_n \) of space curves, projecting to a 1-parameter family \( \{ (C_{t,n}, Y_{t,n}) \} \) of based families, with \( \Gamma_0 \) satisfying (TK2), and suppose if possible that there is a sequence of values \( u = u_n \) tending to 0 for which this condition fails. Thus there is a line \( L_n \) through \( Y_n := Y_{(t_n,u_n)} \) having total contact number at least 3 with \( C_n := C_{(t_n,u_n)} \). Passing to a subsequence, we may assume (since the \( t_n \) belong to the compact set \( S^1 \)) that \( t_n \) tends to a limit \( t_0 \), and similarly that \( L_n \) converges to a line \( L_0 \) through \( Y_0 \). By semi-continuity, \( L_0 \) has total contact number \( \geq 3 \) with \( C_0 \). But this contradicts our hypothesis.

Now suppose \( \Gamma_n \) a 1-parameter family such that \( \Gamma_0 \) satisfies (TMS), and that there is a sequence of values \( u = u_n \) tending to 0 for which this condition fails. Passing to a subsequence, we may suppose that the clause of (TMS) which fails is the same on each occasion, that the corresponding values \( t_n \) converge to a limit, and also that the corresponding lines \( L_n \) converge to a line \( L_0 \), so that the contact points of \( L_n \) with \( C_n \) converge to those of \( L_0 \) with \( C_0 \). By hypothesis, the transversality condition holds for \( (C_n, L_n) \). Hence it also holds for \( (C_n, L_n) \) for \( n \) large enough.

We thus have a contradiction. \( \Box \)

For a family in \( \mathcal{O}_2 \), the exceptional cases \( \alpha, \beta, \gamma, \delta, aa, aa^*, ab, a^*b, bb, a_2, a_2^*, b_2 \) each occur for isolated points \( P \) on \( \Gamma \) (and the subcases involving \( b^* \) only occur for \( P \) in case \( \delta \)). We can thus define a further condition

(Tdis) For each \( P \) not of type \( \delta \), there is at most one line \( L_P \) through \( Y_P \) which either passes through a degenerate singular point of \( C_P \) or satisfies \( \kappa(L_P, C_P) = 2 \).

Addendum 4.3 The families in \( \mathcal{O}_2 \) satisfying (Tdis) form an open dense set.

Density follows since for a based curve \( (C_P, Y_P) \) to admit 2 such lines imposes a codimension 2 condition. Openness is immediate since under the hypothesis \( \mathcal{O}_2 \) the values of \( P \) giving exceptional behaviour form a finite set and each deforms continuously.

It is optional whether or not we require (Tdis). It is not essential for our main arguments, but clarifies the geometry: our list of cases for pairs \( (P, L) \) can now be regarded as a list of cases for \( P \).

5 Conclusion

We define \( \mathcal{O}_3 \) to be the class of curves \( \Gamma \in \mathcal{O}_2 \) such that the family of projections from points not on \( \Gamma \) satisfies the conditions of Theorem 1.2. We aim to show that \( \mathcal{O}_3 \) is dense and open in \( C^\infty(S^1, P^3) \). First we need to strengthen the conclusions of § 3.
Proposition 5.1 For any curve in $\Gamma \in \mathcal{O}_2$, and any smooth family $\{f_u\}$ with $f_0(S^3) = \Gamma_0 = \Gamma$, there is a neighbourhood $U$ of $\Gamma \times 0$ in $\mathbb{P}^3 \times \mathbb{R}$ such that the family of projections of $\Gamma_u$ from points $(P, u) \in U$ (with $P \not\in \Gamma_u$) satisfies the conditions of Theorem 1.2.

Proof It follows from the $\mathcal{O}_2$ condition that the singularities occurring in the family $C_P \cup L$ belong to the list enumerated above; in particular, they occur in (1) and have codimension at most 2. By Theorem 3.1, the family of deformations of $\Gamma_u$ from nearby points of $\mathbb{P}^3$ is a flat deformation of this family. But any such flat deformation can only produce singularities which again occur in (1) and have codimension at most 2.

We have seen in Lemmas 6.1, 6.2, 6.4 that those of codimension 1 ($A_2, A_3, D_4$) are automatically versally unfolded by the family of projections from all points of 3-space.

For one-point singularities of codimension 2, the conditions for versality are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_4$</td>
<td>(TK0) (4 case)</td>
<td>Lemma 6.1</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(TK0) (22 case)</td>
<td>Lemma 6.2</td>
</tr>
<tr>
<td>$D_5$</td>
<td>No condition</td>
<td>Lemma 6.3</td>
</tr>
<tr>
<td>$D_6$</td>
<td>(TE4)</td>
<td>Lemma 6.4</td>
</tr>
<tr>
<td>$X_5$</td>
<td>(TX)</td>
<td>Lemma 6.5</td>
</tr>
</tbody>
</table>

It follows from the definition that these conditions always hold for curves $\Gamma \in \mathcal{O}_2$.

For two-point singularities, it follows from Lemma 6.7, that the condition for versality is that the tangent planes to the two codimension 1 strata meet transversely.

Under (TK0), versality always holds in the cases $2A_2, A_2 + A_3$ and $2A_3$. For the tangent plane to the $A_2$ stratum is the osculating plane; to the $A_3$ stratum is the T-plane. Thus a plane $\Pi$ satisfying two such conditions has $\kappa(\Pi, \Gamma) \geq 4$, contradicting (TK0).

In the case $A_2 + D_4$, $X$ lies on a tangent $TP$ and a 3-secant $QRS$; versality fails only if the osculating plane at $P$ coincides with the tangent plane to the $D_4$ stratum, so in particular passes through $Q$, $R$ and $S$. As $X$ approaches $\Gamma$, $P$ and $S$ (say) both converge to the same point $P_0$ on the space curve. By Lemma 6.11(iii), a point which is not a stall has a neighbourhood $U$ such that the osculating plane at a point of $U$ does not meet $\Gamma$ again in $U$. Thus $P_0$ is a stall, and the osculating plane at $P_0$ contains a trisecant $P_0Q_0R_0$. But then the projection from $P_0$ has a line $L$ with $\kappa(L, C_{P_0}) \geq 3$: 2 at $Y_P$ and 1 at the image of $Q_0$, thus contradicting (TK2).

Next consider the case $A_3 + D_4$. We see from the list (12) that for this to arise for $X$ arbitrarily close to $P_0 \in \Gamma$, the limiting plane $\Pi$ must have contact of type $ab$, $ab^*$ or $a^*b$. However, case $a^*b$ does not arise since in a neighbourhood of a projection of type $a^*$, projections from points off $\Gamma$ cannot produce an $A_3$. Suppose the $A_3$ comes from a T-secant $XPQ$ and the $D_4$ from a 3-secant $XRST$, and that as $X$ approaches $P_0$, $P$ and $R$ approach $P_0$. Then $PQ$ converges to a T-secant $P_0Q_0$ and $RST$ to a 3-secant $P_0S_0T_0$; $H$ is the T-plane of $P_0Q_0$, and the limit of the tangent plane to the $D_4$ stratum at $X$, which contains the tangent $T_{P_0}\Gamma$. Since by (TE3) $\Gamma$ has no tangent 3-secant, case $ab^*$ does not arise, so we have $ab$.

However, it is shown in Proposition 6.15 that in case $ab$, (TMS) is equivalent to condition (Trot) for the family of curves $\Delta_4 \cup L$, and in Lemma 6.10 that (Trot) is equivalent to such compound singularities being versally unfolded in that family. It follows by openness of versality that they are then also versally unfolded in the family of nearby projections which deforms this family: note that since the projection is not singular at $Y_{P_0}$ this is indeed a parametrised family of curves.
The argument for $2D_1$ is very similar. The limiting plane $\Pi$ must correspond to case $bb$ or $bb^*$; again $bb^*$ is excluded by (TE3); and the argument for $bb$ is the same as for $ab$. 

We are now ready to prove our main result.

**Theorem 5.2** The class $\mathcal{O}_3$ is dense and open in $C^\infty(S^1, P^3)$.

**Proof** We have already seen that $\mathcal{O}_2$ is dense, and that for any curve in $\Gamma \in \mathcal{O}_2$ and any smooth family $\{f_u\}$ with $f_0(S^1) = \Gamma_0 = \Gamma$, there is a neighbourhood $U$ of $\Gamma \times 0$ in $P^3 \times \mathbb{R}$ such that the family of projections of $\Gamma_u$ from points $(P,u) \in U$ (with $P \notin \Gamma_u$) satisfies the conditions of Theorem 1.2. By that theorem, we may approximate $\Gamma$ by a curve which in addition satisfies the conditions for projections from other points; since $\mathcal{O}_2$ is open, this is still in $\mathcal{O}_2$, hence in $\mathcal{O}_1$.

To prove openness, consider a smooth family $\{f_u : S^1 \to P^3\}$ with $f_0 \in \mathcal{O}_3$. By Proposition 4.2, for all small enough $u$ we have $f_u \in \mathcal{O}_2$. Were there arbitrarily small $u$ for which the projection from some point $X_u \notin \Gamma_u$ had singularities otherwise than allowed in Theorem 1.2, we could pick a subsequence with $X_u$ converging to a point $X_0$ (here, of course, we use compactness of $P^3$). If $X_0 \notin \Gamma_0$ this contradicts the hypothesis $f_0 \in \mathcal{O}_3$. Otherwise, for small enough $u$, $X_u$ will be in the neighbourhood of $\Gamma_u$ provided by Proposition 5.1, again giving a contradiction.

Now suppose that for arbitrarily small $u$ there is some point $X_u \notin \Gamma_u$ for which the projection from $X_u$ has singularities not versally unfolded. Again we may suppose $X_u$ convergent to $X_0$; we may also suppose that each $X_u$ corresponds to the same list of singularities. But the projection of $\Gamma_0$ from $X_0$ is versally unfolded by hypothesis. Openness of versality thus gives a contradiction. 

The condition $\mathcal{O}_3$ is equivalent to $\mathcal{O}_2$ together with the requirement that, away from $\Gamma$, the various strata meet transversely. For we have verified that under $\mathcal{O}_2$ each one-point singularity that occurs belongs to (1) and is versally unfolded in the family of projections. We have also shown this for the 2-point singularities $2A_2, A_2A_3$ and $2A_3$. There are, however, numerous other cases.

Our arguments allow us to infer properties of the stratification induced on $P^3$ by equisingularity type of the corresponding projection of a given curve $\Gamma \in \mathcal{O}_3$. Away from $\Gamma$ itself, the stratification near any point is locally equivalent to that of the versal unfolding of the family of degenerate singularities on the projected curve. Thus, for example, the $A_3$ surface has a cuspidal edge along the $A_5$ curve. This extends to a stratification of the blow-up $X$ of $P^3$ along $\Gamma$. We expect that there is a unique local model for each of our named cases (in some cases, depending on parameters). To give full details would, however, require extensive additional local calculations.

### 6 Calculations

For our calculations, which are local, we work throughout in $\mathbb{R}^3$. We denote a typical point by $X = (x, x', x'')$; points of $\Gamma$ are denoted $P = (p, p', p'')$, $Q$, $R$ etc. We regard the co-ordinates $p, p', p''$ as functions of a local parameter $t_p$ on $\Gamma$ which vanishes at $P$ (we omit the subscript $p$ if there is no ambiguity). Their Taylor expansions are denoted $p = \sum_0^\infty p_t t_p^r$, $p' = \sum_0^\infty p'_t t_p^r$, etc. Successive derivatives of the vector $P$ with respect to $t_p$ are denoted by suffixes: $P_1, P_2, \ldots$. Thus at $t_p = 0$ we have $P_r = r!(p_r, p'_r, p''_r)$. We regard $P, P_1, \text{etc.}$ as vectors in $\mathbb{R}^3$, and use the ordinary notations of vector calculus.

Projections are made onto the plane $x'' = 0$, and where possible along a line close to the $x''$-axis. Consider a neighbourhood of $a$, where $a_0 = a_0' = 0$. We
project from a point $X$, considered as a deformation of $(0,0,x''_0)$ (where $x''_0 \neq 0$), to the plane $x'' = 0$. Then the image point has coordinates
\[
a''(t)(x,x',x'') - x''(a(t),a'(t),a''(t)) = \left( \frac{xa''(t) - x''a(t)}{a''(t)} - \frac{x'a'(t) - x''a'(t)}{a''(t)}, 0 \right).
\]

(13)

### 6.1 Versality criteria

In this section we give direct calculations of the conditions for transversality of the family of projections of $\Gamma$ to the given one-point singularities (at the end we deal with compound singularities). We begin by recalling the situation when the singularities are in normal form. We use the notations $E_x$ for the ring of germs of $C^\infty$-functions of $x$ and $x'$, $\theta$ to denote tangent spaces, $tf$ and $\omega f$ for the maps induced by $f$; see e.g. [8] for a full introduction. In particular, we write $T_A(f) = tf(\theta) + \omega f(\theta,x')$ for the extended $A$-tangent space (in the case when we have a bi-germ, with source variables $t$ and $u$), and use the symbol $\equiv$ to denote congruence of elements of $\theta(f)$ modulo it.

First consider the case $A_{2k}$, with normal form $t \mapsto (t^2,t^{2k+1})$. The ring $f^*E_x$ of function germs induced on $R$ by $f$ has codimension $k$ in $E$: it contains all monomials in $t$ except the $t^{2r-1}$ ($1 \leq r \leq k$). This describes $\omega f : \theta_x \rightarrow \theta(f)$; we obtain $tf : \theta_x \rightarrow \theta(f)$ by differentiating $f$ with respect to $t$, and the quotient $\theta(f)/\{tf(\theta_x) + \omega f(\theta_x)\}$ also has dimension $k$. We regard $\theta(f)$ as the free $E_x$-module with basis $\partial/\partial x, \partial/\partial x'$, and write a typical element as $V := A\partial/\partial x + A'\partial/\partial x'$ where, as usual, we have Taylor expansions $A = \sum A_t^i, A' = \sum A'_t^i$. All monomials $t^r\partial/\partial x, t^r\partial/\partial x'$ lie in $T_A(f)$ with the exception of the $t^{2r-1}\partial/\partial x'$ ($1 \leq r \leq k$), which thus give a basis for the quotient. The projection onto the quotient is thus given by the maps $\pi_r(V) = A_{r-1} - B_{r-1}$ for $1 \leq r \leq k$.

Secondly consider $A_{2k+1}$, with normal form given by the bi-germ $t \mapsto (t,t^k)$, $u \mapsto (u,0)$. Here $f^*E_x$ has codimension $k$ in $E_x \oplus E_u$; the image, generated by $(t,u)$ and $(t^k,0)$, contains all monomials except $(t^r,0)$ and $(0,u^r)$ with $0 \leq r \leq k - 1$, and contains also $(t^r,u^r)$ for these values of $r$. Here we denote a typical element of $\theta(f)$ by $V = (A,B)\partial/\partial x + (A',B')\partial/\partial x'$. All monomials in $\theta(f)$ belong to $T_A(f)$ with the exception of $t^r\partial/\partial x' \equiv u^r\partial/\partial x'$ for $0 \leq r \leq k - 2$, and the quotient has dimension $k - 1$. Projections to the quotient are given by $\pi_r(V) = A_{r-1}' - B_{r-1}'$ for $1 \leq r \leq k - 1$.

Thirdly we have $D_{2k+1}$, with normal form $t \mapsto (t^2,t^{2k+1})$, $v \mapsto (0,v)$. The monomials not in $f^*E_x$ are $(1,0) \equiv (-0,1), (t^{2r-1},0) (1 \leq r \leq k)$ and $(t^{2k+1},0) \equiv (0,v)$. All monomials in $\theta(f)$ belong to $T_A(f)$ with the exception of $(1,-1)\partial/\partial x$ and $t^{2r-1}\partial/\partial x'$ ($1 \leq r \leq k$), so we have codimension $k + 1$. Projection to the quotient is given, for $V = (A,C)\partial/\partial x + (A',C')\partial/\partial x'$, by $\pi_1(V) = A_0 - C_0$ and $\pi_{r+1}(V) = A_{r-1}' - B_{r-1}'$ for $1 \leq r \leq k$.

Fourthly we have $D_{2k+2}$, with normal form given by the tri-germ $t \mapsto (t,t^k)$, $u \mapsto (u,0), v \mapsto (0,v)$. Here $f^*E_x$ omits the monomials $(t^r,0,0), (0,v^r,0) (0 \leq r \leq k)$ and $(0,0,1), (0,0,v)$ but contains $(1,1,1), (t^r,u^r,0) (1 \leq r \leq k)$ and $(t^k,0,v)$, so has codimension $k + 2$. The monomials in $\theta(f)$ not in $T_A(f)$ are $(t^r,0,0)\partial/\partial x \equiv (0,u^r,0)\partial/\partial x'$ ($0 \leq r \leq k - 1$) and $(0,0,v)\partial/\partial x \equiv (1,0,0)\partial/\partial x' k^{(k-1),0}\partial/\partial x'$, so the codimension here is $k$. We take $\pi_{r+1}(V) = A_{r-1}' - B_{r-1}'$ for $0 \leq r < k - 1$ and $\pi_k(V) = A_{k-1}' - B_{k-1}' - k(A_0 - C_0)$.

For our calculations we need to consider maps not in normal form; fortunately only the cases with codimension at most 3 are required; moreover, the above give a useful guide, e.g. giving the degrees of determinacy of these germs. In each case we give explicit maps $\pi_r : \theta(f) \rightarrow \mathbb{R}$ for $1 \leq r \leq k$ such that $(\pi_1, \ldots, \pi_k)$ induces an isomorphism $\frac{\theta(f)}{T_A(f)} \rightarrow \mathbb{R}^k$. Since our maps are clearly independent, to check the
calculation it suffices to verify that each map vanishes on the images $\omega f(\theta_x)$ and, for each monomerm, $tf(\theta_t)$.

For $A_{2k}$ we write the germ as $f(t) = (\alpha(t), \alpha'(t))$, where, as usual, $\alpha(t) = \sum_\pi \alpha^t \pi$. In this case we may suppose $\alpha_1 = \alpha'_1 = \alpha'_2 = 0$ and $\alpha_2 \neq 0$. We have $k = 1 \Leftrightarrow \alpha'_3 \neq 0$. If $\alpha'_3 = 0$, we subtract a suitable multiple of $\alpha^2$ from $\alpha'$ to eliminate the coefficient of $t^4$; then the condition that we have an $A_4$ is that the coefficient of $t^5$ does not vanish, i.e. that $2\alpha_3\alpha'_4 \neq 2\alpha_2\alpha'_3$. Similarly we may obtain the condition for $k = 2$ to equal 3, which involves $\alpha'_3$: we do not need it explicitly.

To find the corresponding projections $\pi_t$ we proceed as follows. If we have $A_2$, then any element of $E_t$ of order $\geq 2$ is in $f^*E_t$. The term $tf(\theta_t)$ gives multiples of $(\partial \alpha/\partial t, \partial \alpha'/\partial t)$, which has a non-zero coefficient of $(t, 0)$, so as before we may take $\pi_1 = A'_1$.

Now suppose $\alpha'_1 = 0$, so any element of $E_t$ of order $\geq 2$ is in $f^*E_t$, and we must consider coefficients of $(t, 0), (t^2, 0), (0, t)$ and $(0, t^2)$. For $V \in \theta_t$, we may suppose $A'_1 = 0$. Subtract $\frac{A_1}{\alpha_2^2} \partial f/\partial t$ to remove the coefficient of $(t, 0); \alpha_2^{-1}(A_2 - \frac{2\alpha_1}{\alpha_2} A_1)(\alpha_0, 0)$ to remove the coefficient of $(t^2, 0)$, and $(A_3 - \frac{2\alpha_1}{\alpha_2} A_2 - \frac{3\alpha_1}{\alpha_2^2} A_1) \frac{2\alpha_2}{\alpha_2^3} \partial f/\partial t$ to remove the coefficient of $(t^3, 0)$, and $A_2^2(0, \alpha)$ to remove the coefficient of $(0, t^2)$. This leaves the coefficient of $(0, t^3)$ as $A^3_3 - \frac{2\alpha_1}{\alpha_2} A^2_2 - 2\frac{\alpha_1}{\alpha_2^2} A_1$, which we can take as $\pi_2$.

We can proceed similarly in the $A_6$ case, or suppose $A_0 = 0$ and subtract $\{A(t)\partial f/\partial t\}/\partial \alpha'/\partial t$ from $V$ to eliminate the first component. Then subtract appropriate multiples of $\alpha, \alpha^2$ from the second to eliminate the coefficients of $t^2$ and $t^4$. The coefficients of $t, t^3$ and $t^5$ in the result give possible choices for $\pi_1, \pi_2$ and $\pi_3$ (the latter may be changed by adding a multiple of $\pi_2$). The details can be done by hand or with a computer algebra programme. In summary,

$$
\begin{align*}
\pi_1 &= A'_1, \\
\pi_2 &= \alpha_2 A'_3 - \alpha_3 A'_2 - 2\alpha_4' A_1 \\
\pi_3 &= 2\alpha_3 A'_4 + 2\alpha_4 A'_3 - \alpha_5 A'_2 - 2\alpha_4' A_3 + \alpha_5' A_2 + (3\alpha'_6 + 5\alpha_3\alpha'_5) A_1.
\end{align*}
$$

(14)

For $A_{2k+1}$ we write the germs as $f(t) = (\alpha(t), \alpha'(t)), f(u) = (\beta(u), \beta'(u))$. We may suppose $k \geq 1$ and then that $\alpha'_1 = \beta'_1 = 0$, while $\alpha_1 \neq 0, \beta_1 \neq 0$. Then $k = 1 \Leftrightarrow \alpha_1^{-1} 2\alpha_2' \neq \beta_1^{-1} 2\beta_2'$. If equality does hold, then $k = 2 \Leftrightarrow \alpha_1^{-1} (\alpha_1 \alpha'_3 - 2\alpha_1 \alpha'_2) \neq \beta_1^{-1} (\beta_1 \beta'_3 - 2\beta_1 \beta'_2)$. We do not require the precise condition for $k$ to equal 3.

The corresponding projections $\pi_t$ may be taken as given by

$$
\begin{align*}
\pi_1 &= A'_0 - B'_0, \\
\pi_2 &= -\alpha_1^{-1} A'_1 + \beta_1^{-1} B'_1 + 2\alpha_1^{-1} \alpha_2' A_0 - 2\beta_1^{-1} \beta_2' B_0, \\
\pi_3 &= (\alpha_1^{-2} A'_2 - \alpha_1^{-3} (\alpha_2 A'_1 + 2\alpha_2 A'_1)) - 3\alpha_1^{-4}(\alpha_1\alpha'_3 - 2\alpha_2\alpha'_2) A_0 \\
&\quad - (\beta_1^{-2} B'_2 - \beta_1^{-3} (\beta_2 B'_1 + 2\beta_2 B'_1)) - 3\beta_1^{-4}(\beta_1\beta'_3 - 2\beta_1\beta'_2) B_0.
\end{align*}
$$

(15)

For $D_{2k+1}$ we take the first germ as for $A_{2k-2}$ and the second as $(\gamma(v), \gamma'(v))$, with $\gamma_1 = 0 \neq \gamma'_1$, so the condition for $k = 2$ is $\alpha'_3 \neq 0$. In fact we only require the case $k = 2$, and here we have (as before)

$$
\begin{align*}
\pi_1 &= A_0 - C_0, \\
\pi_2 &= A'_1,
\end{align*}
$$

(16)

though it is easy to calculate further that when $\alpha'_3 = 0$, we can take $\pi_3 = \alpha_2 A'_3 - 2\alpha'_4 A_1$.

Finally for $D_{2k+2}$ we have a third germ $(\gamma(v), \gamma'(v))$ in addition to two as for $A_{2k-1}$. The condition for $k = 1$ is that no two of the vectors $(\alpha_1, \alpha'_1), (\beta_1, \beta'_1), (\gamma_1, \gamma'_1)$ are proportional. If the perturbation parameter is denoted $s$, then to first
order, the perturbed lines are given by \((\alpha_1 t + A_0 s, \alpha_1' t + A_0' s)\) etc., i.e. by \(\alpha_1 x - \alpha_1 y = (\alpha_1' A_0 - \alpha_1 A_0') s\) etc., and we can define \(\pi_1\) as the invariant
\[
\begin{bmatrix}
\alpha_1 & \alpha_1' & \alpha_1 A_0 - \alpha_1 A_0' \\
\beta_1 & \beta_1' & \beta_1' B_0 - \beta_1 B_0' \\
\gamma_1 & \gamma_1' & \gamma_1' C_0 - \gamma_1 C_0'
\end{bmatrix},
\]
which is proportional to the area of the triangle cut by the three lines.

If we require \(\alpha_1' = \gamma_1 = 0\), then \(\alpha_1\) and \(\gamma_1'\) are non-zero; the condition for \(D_1\) is that neither \(\beta_1\) nor \(\beta_1'\) vanishes; and we normalise \(\alpha_1 = \gamma_1' = 1\). Here \(\pi_1\) reduces to
\[
\beta_1 A_0' + \beta_1' B_0 - \beta_1 B_0' - \beta_1' C_0.
\]

If \(k \geq 2\) we further normalise \(\beta_1' = 0\); then \(\beta_1 \neq 0\) and the formula for \(\pi_1\) (divided by \(\beta_1\)) simplifies to the same formula \(A_0' - B_0'\) as for \(A_3\). The condition defining \(k = 2\) \((D_0)\) is, as in the case of \(A_3\), that \(\alpha_2'/(\alpha_1)^2 \neq \beta_2'/(\beta_1)^2\). As in the normal form case, we see that the contribution of the third component only involves the constant term \(C_0\), and we find that the formulae for \(\pi_2\) and \(\pi_3\) are the same as in the cases of \(A_3\) and \(A_7\) with \(A_0\) and \(B_0\) replaced by \((A_0 - C_0)\) and \((B_0 - C_0)\) respectively in \(\pi_2\) for \(A_5\) and in \(\pi_3\) for \(A_7\):

\[
\pi_2 = -\alpha_1^{-1} A_1' + \beta_1^{-1} B_1' + 2\alpha_1^{-2} \alpha_2'(A_0 - C_0) - 2\beta_1^{-2} \beta_2'(B_0 - C_0),
\]

\[
\pi_3 = (\alpha_1^{-2} A_1' - \alpha_1^{-3} \alpha_2 A_1' + 2\alpha_1^{-2} \alpha_2 A_1) - 3\alpha_1^{-4} (\alpha_1 A_3 - \frac{2}{3} \alpha_2 A_2)(A_0 - C_0)
\]

\[-(\beta_1^{-2} B_1' - \beta_1^{-3} (\beta_2 B_1' + 2\beta_2' B_1) - 3\beta_1^{-4} (\beta_1 B_3 - 2\beta_2 B_2)(B_0 - C_0)).
\]

We observe that the condition defining the case \(D_6\) is just that the coefficient of \(C_0\) in the expression for \(\pi_2\) is non-zero; similarly, the condition defining \(D_8\) is that the coefficient of \(C_0\) in the expression for \(\pi_3\) does not vanish.

## 6.2 Projections from points off \(\Gamma\)

This completes our discussion of local conditions: we turn to the formula defining central projection. We project \(\Gamma\) from a point \(X\) (which in this section does not lie on \(\Gamma\)) to the plane \(x'' = 0\). The image \(\pi_X(P)\) of \(P\) is given by \((13)\).

We will take this as the curve \((\alpha(t), \alpha'(t))\) of the preceding section; to check the transversality conditions of \((14)\) etc. we will require the values obtained on substituting \((0, 0, x_0')\) for \((x, x', x'')\) not only in this expression, but also in its partial derivatives with respect to \(x, x'\) and \(x''\). Dropping the superficial third coordinate, these are

\[
\frac{\partial \pi_X(P)}{\partial x} = \left(\frac{\alpha''(t)}{a''(t) - a'(t)}), \frac{\partial \pi_X(P)}{\partial x'} = \left(0, \frac{\alpha''(t)}{a''(t) - a'(t)}\right)
\]

\[
\frac{\partial \pi_X(P)}{\partial x''} = \frac{-a''(t)}{(a''(t) - a'(t))}(a(t), \alpha'(t)).
\]

To perform calculations, we may substitute \(\alpha(t) = \frac{x_0'' a(t)}{x_0'' - a'(t)}\), \(\alpha'(t) = \frac{x_0'' a'(t)}{x_0'' - a'(t)}\), take in turn \(\partial \pi_X(P)/\partial x, \partial \pi_X(P)/\partial x', \partial \pi_X(P)/\partial x''\) (which we abbreviate to \(\partial \alpha, \partial \alpha', \partial \alpha''\)) as \((A, A')\), and evaluate the \(\pi_r\) on such of them as we need; and correspondingly if we have more than one branch.

Thus, for example, if \(a_0 = a_1 = 0\) we have

\[
\alpha = x_0'' \rho_a \{a_2 t^2 + (a_3 + a_2 a_1'' \rho_a) t^3 + (a_4 + a_3 a_1'' \rho_a + a_2 (a_2'' \rho_a + a_1''^2 \rho_a^2)) t^4 + \ldots\},
\]

where \(\rho_a\) denotes \(1/(x_0'' - a''_0)\) (we will define \(\rho_b\) etc. similarly). And for \(V_x\) we have \(A' = 0\) and \(A_0''/a''_0 - a''_0 = -a''_0 \rho_a\).

First consider projection along a tangent: this will give rise to singularities of type \(A_{2k}\). We thus have \(a_1 = a_1' = 0\), so by \((\text{Tim}), a_1'' \neq 0\). Setting \(a_2'' = 0\)
(so that the osculating plane is \(x' = 0\)), it follows from (Tcur) that \(a_2 \neq 0\). We have an \(A_2\) iff \(\alpha_3' \neq 0\), i.e. \(\alpha_3' \neq 0\), which is equivalent to the point not being a stall. If, however, \(\alpha_3' = 0\) we have an \(A_4\) unless \(a_2a_1' = 2\alpha_3\alpha_4'\), which reduces to \(a_2a_1' = a_4(a_2a_1'' + 2a_3)\), which is equivalent to \(x_0'' = a_0'' + a_2a_1''/a_2a_1' - 2a_3a_4'\), giving the co-ordinate of the T-centre of the stall (if the denominator vanishes, the T-centre is at infinity in the chosen co-ordinates). In this case we have an \(A_6\) unless a rather complicated expression vanishes, which can be regarded as giving an expression for \(a_3'\) in terms of lower coefficients.

We consider only the cases \(k \leq 3\); we have defined \(\pi_i\) for \(i \leq k\). In the following (and later) lemmas, the values noted ? are not needed to determine whether or not the matrices are nonsingular, so we ignore them.

**Lemma 6.1** The values of \(\pi_i\) on the basis elements for \(A_{2k}\) are given by

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\pi_i(\partial_x'))</th>
<th>(\pi_i(\partial_y))</th>
<th>(\pi_i(\partial_y'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-a''_0x_0'\rho_0)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(2a_4a_1''x_0''\rho_0^2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(a_2a_4a_1''x_0''\rho_0^2)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

An \(A_2\) is always versally unfolded; an \(A_4\) is if and only if \(a_3' \neq 0\), i.e. the stall is transverse; and when this holds, an \(A_6\) also is versally unfolded.

**Proof** The values of \(\pi_i\) given are obtained by substituting as above and simplifying. We have already seen that, in the relevant cases, \(a''_0\) and \(a_2\) do not vanish; neither do \(x_0''\) and \(\rho_0\). Thus transversality always holds in the \(A_2\) case, and holds in both the others if and only if \(a_3' \neq 0\), which is equivalent to the stall being transverse.

Next consider projection along a T-secant: this will give singularities of type \(A_{2k+1}\) with \(k \geq 1\). We may assume the T-secant not a tangent, and that the T-plane is \(x' = 0\), so that the germs \((a(t_0), a'(t_0), a''(t_0)), (b(t_0), b'(t_0), b''(t_0))\) satisfy \(a_0 = a_0' = b_0 = b_0', a_1 \neq 0, b_1 \neq 0\) and \(a_1' = b_1' = 0\), as well as having \(x_0'', a_0''\) and \(b_0''\) all distinct. Thus we have

\[
\alpha = x_0''\rho_0\{a_1t_0 + (a_2 - a_1a_1'\rho_0)t_0^2 + \ldots\},
\]

\[
\alpha' = x_0''\rho_0\{a_2a_1''t_0^2 + (a_3' - a_2a_1''\rho_0)t_0^3 + \ldots\},
\]

and the leading terms of \(V_x\) are \((a''_0\rho_0, 0)\), of \(V_y\) \((0, a_0'\rho_0)\) and those of \(V_z\) are \((-a_1a_0''\rho_0t_0, -a_2a_0''\rho_0^2t_0^2, 0)\). The condition for an \(A_5\) is \(\alpha_2'/\alpha_2^2 = \beta_2'/\beta_2^2\), which reduces to \(a_2'(x_0'' - a_0'')/a_2^2 = b_2'(x_0'' - b_0'')/b_2^2\), giving the value

\[
x_0'' = (b_2'x_0'' - a_2'b_2'')/(b_1'a_2' - a_2^2b_2')
\]

for the T-centre. The singularity is higher than an \(A_5\) iff

\[
(a_1a_3' - 2a_2a_2'')/(a_1^2) = (\beta_1\beta_3' - 2\beta_1\beta_2'')/(\beta_1)^2,
\]

giving the condition for a special T-secant.

**Lemma 6.2** For \(A_{2k+1}\), the values of \(\pi_i\) (when \(1 \leq i \leq k \leq 3\)) on the basis elements are

<table>
<thead>
<tr>
<th>(i)</th>
<th>(\pi_i(\partial_x'))</th>
<th>(\pi_i(\partial_y))</th>
<th>(\pi_i(\partial_y'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-x_0''(a_0'' - b_0'')\rho_0)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(2a_2'(a_0'' - b_0'')\rho_0/a_1^2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(a_2'(a_0'' - b_0'')\rho_0/a_1^2)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

An \(A_3\) is always versally unfolded; an \(A_5\) is if and only if \(a_3' \neq 0\), i.e. the T-plane does not osculate at one, hence both points of contact; and then an \(A_7\) also is versally unfolded.
Proof  Clearly $\pi_1$ takes the values $-a''_0 \rho_a + b''_0 \rho_b$, 0 and 0, so the first assertion follows. Next, $\pi_2(\partial''_x)$ is 0, while $\pi_2(\partial_x) = 2\alpha_1^{-2} \alpha_2 a''_0 \rho_a - 2\beta_1^{-2} \beta'_2 b''_0 \rho_b$. Since $\alpha_2'/\alpha_1^2 = \beta'_2/\beta_1^2$, this is equal to $2\alpha_2(\alpha_1)^{-2} = 2\alpha_2/(a''_0 \rho_a)$ multiplied by $a''_0 \rho_a - b''_0 \rho_b$, giving the stated result.

For $\pi_3(\partial''_y)$, since $A_0 = 0, A_1 = -a_1 a''_0 \rho_a, A'_1 = 0$ and $A_2 = -a_2 a''_0 \rho_a^2$ (and similarly for $b$), we obtain $\alpha_1^{-2} A_2 - 2\alpha_2 \alpha_1^{-3} A_1 - \beta_1^{-2} B'_2 + 2\beta'_2 \beta_1^{-3} B_1$, which reduces to $(2-1)(\alpha_2'/\alpha_1^2)(a''_0 \rho_a/x''_0) - (2-1)(\beta'_2/\beta_1^2)(b''_0 \rho_b/x''_0)$, and hence to $(\alpha_2'/\alpha_1^2)(a''_0 \rho_a - b''_0 \rho_b/x''_0)$.

For projection along a tangent meeting the curve again we normalise the first (tangent) component as before and take the second as $(c, c', c'')$. Then we have

**Lemma 6.3** The values of $\pi_1$ and $\pi_2$ for $D_5$ are

$$
\begin{array}{ccc}
 i & \pi_i(\partial'_x) & \pi_i(\partial''_x) - \pi_i(\partial_x) \\
 1 & x''_0(b'_0 - a'_0)\rho_a \rho_c & 0 \\
 2 & ? & -a'_1 a''_0 \rho_a^2 \\
 3 & ? & 0 \\
\end{array}
$$

Thus a $D_5$ is always versally unfolded.

Next we project along a trisecant.

**Lemma 6.4** If $a'_1 = c_1 = 0$, we have $\pi_1(\partial''_x) = 0$ and

$$
\pi_1(\partial_x) = a_1 b_1 c'_1 (b''_0 - c'_0) x''_0 \rho_a \rho_b \rho_c^2, \quad \pi_1(\partial'_x) = -a_1 b_1 c'_1 (b''_0 - c'_0) x''_0 \rho_a \rho_b \rho_c
$$

If $b'_1 = 0$, we have

$$
\begin{array}{ccc}
 i & \pi_i(\partial_x) & \pi_i(\partial''_x) - \pi_i(\partial_x) \\
 2 & ? & 0 \\
 3 & 2(\beta'_2 b'_2 - a'_2 b''_0 + c''_0) a_1^{-2} b'_1^{-2} \rho_c & a'_2 (a''_0 - b''_0) \rho_b/a'_1. \\
\end{array}
$$

A singularity of type $D_4$ is always versally unfolded; a $D_6$ is provided that the T-centre is not the third point on $\Gamma$; and then a $D_8$ is versally unfolded unless the T-plane osculates the two branches.

Proof  The calculation of $\pi_1$ is obtained by substituting in the determinant, which has several zero entries. Now $a_1$ and $c'_1$ must both be non-zero, so $\pi_1(\partial_x) = \pi_1(\partial'_x) = 0$ if and only if $b_1 = b'_1 = 0$, i.e. the trisecant is tangent at the point $P_v$.

Now suppose $b'_1 = 0, b_1 \neq 0$; then $\pi_1(\partial_x) = 0$ and $\pi_1(\partial'_x) \neq 0$. Then $\pi_2(\partial''_x) = 0$, since all the relevant coefficients of $\partial''_x$ vanish. For $\pi_2(\partial'_x)$, since $A'_1 = B'_1 = 0$ we have

$$
2 a'_2/(a_1^3) (-a''_0 \rho_a + c''_0 \rho_c) - 2 b'_2/b_1^2 (-b''_0 \rho_b + c''_0 \rho_c),
$$

and as $\alpha'_2/\alpha_1^2 = \beta'_2/\beta_1^2$, this reduces to the expression shown. We recognise the vanishing of this as the condition that $(0, 0, c''_0)$ is the T-centre of the bi-germ given by the two other components.

Finally suppose $\alpha'_2/\alpha_1^2 \beta'_2/\beta_1^2$; then we need to evaluate $\pi_3(\partial''_x)$. However, since here the term $C_0 = 0$, this is exactly the same as the calculation of $\pi_3(\partial''_x)$ for the $A_7$ case, and we have seen above that this vanishes only if the T-plane osculates both the first two branches.

We will also require the calculation for a 4-secant. The conclusion is:

**Lemma 6.5** The family of projections fails to versally unfold the $X_6$ singularity stratum corresponding to a given 4-secant if and only if the cross-ratio of the planes through the 4-secant containing the 4 tangent lines is equal to the cross-ratio of the 4 points on the line.

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Proof  For the normal form, we choose coordinates so that the 1-jets satisfy
\[ \gamma_1 = \delta_1 = 0 \] (so that \( \gamma_1 \neq 0, \delta_1' \neq 0 \)). Then \( f^*\mathcal{E}_{x,x'} \) is generated in degrees \( \leq 1 \) by \( (1,1,1,1) \), \( (\alpha_1 t, \beta_1 u, \gamma_1, v, 0) \) and \( (\alpha_1', t, \beta_1' u, 0, \delta_1' w) \). To calculate \( T_xA \) it suffices to study the constant terms.

The tangent lines to the strata \( A \) through the tangent at \((0,0,0,0)\) reduce to (a multiple of) \( \pi \) if and only if the tangent planes to the \((2 \text{ or } 3)\) strata at \((0,0,0,0)\) are projectively related to the point \((\alpha_1, \beta_1, \gamma_1, 0)\). This is linear in \( x \), \( \beta_1 \) and \( \gamma_1 \).

\[ \gamma_1 \delta_1 = 0 \]

Proof  The tangent plane is essentially the kernel of the map \( \pi_1 \) above, so the conclusion is immediate in cases (i), (ii) and (iv). For case (iii) we calculate as follows. We see from Lemma 6.4 that the kernel of \( \pi_1 \) is generated by
\[ \beta_1 \partial_x (c_0'' - x_0')(a_0'' - b_0') - \beta_1' \partial_x' (a_0'' - x_0')(c_0'' - b_0'). \]

This is linear in \( x_0' \), so the plane — which, of course, passes through the 3-secant — is projectively related to the point \( x_0' \) on it. Moreover, at \( x_0' = c_0' \) this expression reduces to \( 0 \). The tangent plane is isomorphic to the plane \( \langle 0,0,0,0 \rangle \); at \( x_0' = c_0' \) to \( \partial_x \), giving the plane \( x' = 0 \) through the tangent at \( (0,0,0,0) \); and at \( x_0' = b_0' \) to \( \beta_1 \partial_x - \beta_1' \partial_x' \), corresponding to the plane \( \beta_1 x = \beta_1' x' \) through the tangent at \( (0,0,b_0') \).
Proof This is an immediate consequence of the fact that each of the singularities by itself is versally unfolded.

6.3 Projections from points on $\Gamma$

Next we apply the criteria of §6.1 to versality for the 1-parameter family of based curves $C_t$ obtained by projecting $\Gamma$ from a variable point of itself. Thus the point $X = (x',x'',x')$ above must be taken as a variable point of $\Gamma$, thus depends only on one parameter. We need only consider possible singularities of types $A_2, A_3$ and $D_4$.

Lemma 6.8 An $A_2$ in this family is versally unfolded if and only if $T_X \not\subset O_P \Gamma$.

An $A_3$ in this family is versally unfolded if and only if $T_X \not\subset O_P \Gamma$ is not contained in the $T$-plane.

$A_4$ in this family is versally unfolded if and only if $(TX)$ holds for this 4-secant.

Proof By Lemma 6.1, the condition for versal unfolding of $A_2$ is (in the notation of that lemma) that we have a non-zero coefficient of $\partial'_t$, i.e. that $T_X \not\subset O_P \Gamma$ has a non-zero component in the $x'$ direction. Since we have normalised co-ordinates so that $O_P \Gamma$ is the plane $x' = 0$, this gives the condition stated.

An essentially identical argument holds in the $A_3$ case; here, however, it is the $T$-plane which is given by $x' = 0$.

Applying Lemma 6.4, and substituting $\beta'_1 = b'_1 = -x''_0\frac{a''_0}{b''_0 - x''_0}, \beta_1 = b_1 = -x''_0\frac{a''_0}{b''_0 - x''_0}$ shows that versality fails in the $D_4$ case if and only if:

$$b'_1 \left( \frac{b''_0}{b''_0 - x''_0} - \frac{c''_0}{c''_0 - x''_0} \right) x_1 + b_1 \left( \frac{b''_0}{b''_0 - x''_0} - \frac{a''_0}{a''_0 - x''_0} \right) x'_1 = 0,$$

which reduces to

$$\frac{(a''_0 - x''_0)(c''_0 - b''_0)}{(c''_0 - x''_0)(a''_0 - b''_0)} = -b_1 x'_1/b'_1 x_1.$$

Since the planes through the respective points $A, B, C, X$ containing the tangent lines at those points are $x = 0$, $b_1 x' = b'_1 x$, $x''_0 = 0$ and $x_1 x' = x'_1 x$, the corresponding cross-ratio is $-b_1 x'_1/b'_1 x_1$, and the result follows.

Next we consider versality for the 2-parameter family consisting of a 1-parameter family of based curves $(C_t, Y_t)$ with a line $L$ through $Y_t$. We may choose co-ordinates so that for each $t, Y_t$ is at the origin, and the line $L$ is the $x-$axis. Write $u$ for the co-coordinate on $L$. Thus for deformations at the origin we have $A_0 = A'_0 = A_1 = A'_1 = 0$; at a point other than the origin on the $x-$axis, there is no restriction on $A$ and $A'$. Rotating $L$ gives $(u \cos \theta, u \sin \theta)$. Differentiating and setting $\theta = 0$ gives $B = 0$ and $B' = u$.

First we consider the cases $\kappa(L, C_0) = 1$.

Lemma 6.9 For cases $a$ and $b$, the singularity $(A_3$ or $D_4$ respectively) is versally unfolded by rotating the line $L$. For cases $a^*$ and $b^*$, it is not versally unfolded.

Proof For case $a$, we have $\pi_1 = A'_0 - B'_0$; since we have a point $u \neq 0$, $B'_0 \neq 0$, so rotating the line is sufficient to yield a versal unfolding. For $a^*$ however, we have $A'_0 = B'_0 = 0$, so rotating the line together with deforming $C_t$ does not suffice.

For case $b$, we take $L$ to be the component labelled $\alpha$ to conform with the notation of §6.1, and have $\pi_1 = \beta_1 A'_0 + \beta'_1 B_0 - \beta_1 B'_0 - \beta'_1 C_0$, where $\beta_1$ and $\beta'_1$ are both non-zero. Again at a point other than $Y_t$ we have $A'_0 = u \neq 0$, so rotating the line gives a versal unfolding. However, for case $b^*$ we have (as before) $B_0 = B'_0 = C_0 = A'_0 = 0$. □
We next consider the cases \( \kappa(L, C_0) = 2 \) of type \( aa, ab \) or \( bb \).

**Lemma 6.10** In the cases \( aa, ab, bb \), the 2-point singularity is versally unfolded if and only if \((\text{Trot})\) holds.

**Proof** We have 2 singularities, each of type \( A_3 \) or \( D_4 \), and know by Lemma 6.9 that each is versally unfolded by rotating the line \( L \). Condition \((\text{Trot})\) states that the rotations corresponding to the two separate points are essentially independent, which is exactly what we require for versality. \( \square \)

### 6.4 Multi-jet transversality conditions

In this subsection we give explicit forms to the transversality conditions in condition (TMS), and obtain geometric equivalents for most of them. First, however, we obtain geometric characterisations of stalls.

**Lemma 6.11** (i) A point on \( \Gamma \) which is not a stall has a neighbourhood \( U \) such that no two tangents at points of \( U \) lie in a plane.

(ii) Near a stall there is an involution, given in local coordinates in the form 
\[
\tau(t) = -t + O(t^2),
\]

such that the tangents at \( t \) and \( t' \) are coplanar if and only if \( t' = \tau(t) \).

(iii) A point on \( \Gamma \) which is not a stall has a neighbourhood \( U \) such that an osculating plane at a point of \( U \) does not meet \( \Gamma \) again in \( U \).

(iv) In local coordinates at a stall there is a function \( \phi \) with \( \phi(t) = -3t + O(t^2) \) such that the osculating plane at \( t \) meets \( \Gamma \cap U \) again at \( t' \) only if \( t' = \phi(t) \).

**Proof** (i) Take local coordinates with respect to which the curve is given by 
\[
A(t) = (a(t), a'(t), a''(t)),
\]

with our usual notation. We may suppose \( a_0 = a_0' = 0 \); since the curve is embedded, that \( a_1 \neq 0 = a_1' \); and since the curvature is non-zero, that \( a_2' \neq 0 = a_2'' \). The condition for the tangents at \( t \) and \( t' \) to lie in a plane is that the vectors \( A(t') - A(t) \), \( A_1(t') \), and \( A_1(t) \) are dependent. The least order terms in the expansions give the determinant

\[
\begin{vmatrix}
    a_1(t' - t) & a_2'(t'^2 - t^2) & a_3''(t'^3 - t^3) \\
    a_1 & 2a_2't' & 3a_3''t'^2 \\
    a_1 & 2a_2't & 3a_3''t^2
\end{vmatrix},
\]

which reduces to \( 6a_1a_2'a_3''(t' - t)^4 \). Observe that \( A(t) - A(t') \) is divisible by \( t' - t \), with quotient \( Q \), say, and \( 2Q - A_1(t) - A_1(t') \) by \( (t' - t)^2 \), so the triple product is divisible by \( (t' - t)^4 \). The quotient is a unit and thus has no other zeros nearby.

(ii) If, however, \( a_2'' = 0 \) but \( a_1' \neq 0 \), a corresponding calculation yields the leading term \( 12a_1a_2'a_3''(t'-t)^4(t'+t) \). We have just seen that the factor \((t' - t)^4\) is to be removed, so we have uniquely

\[
t' = -t \text{ added to higher order terms.}
\]

(iii) For \( t' \) to lie on the osculating plane at \( t \) we need the vectors \( A(t') - A(t) \), \( A_1(t) \), and \( A_2(t) \) to be dependent. Here the determinant of leading terms gives \( 2a_2a_3'(t' - t)^3 \), but we see as above that the factor \((t' - t)^3\) must be removed, leaving a unit.

(iv) However in the case of a stall the corresponding determinant of leading terms reduces to \( 2a_2a_3'(t' - t)^3(t' + 3t) \), and our assertion follows as before. \( \square \)

Recall that the cases corresponding to a plane \( \Pi \) with \( \kappa(\Pi, \Gamma) = 3 \) are \( aa \) \((1 + 1 + 1)\), \( a_2 \), \( aa^* \) \((2 + 1)\) and \( a_2^3 \) \((3)\). These correspond to the following submanifolds of multijet spaces. Write \( W_{111} \) for the set of multijets in \( J^1(S^1, P^3) \) such that the 3 image points are distinct and not collinear, and the 3 image tangent vectors lie in
the plane $\Pi$ of the points. Let $W_{21}$ be the set of multijets in $J^2(S^1, P^3)$ such that
the image points $P$ and $Q$ are distinct, $Q \not\in T_P\Gamma$, and both $T_Q\Gamma$ and $O_P\Gamma$ lie in the
plane $\Pi$ defined by $T_P\Gamma$ and $Q$. Let $W_3$ be the set of jets in $J^3(S^1, P^3)$ such that
the curvature at $P$ is non-zero, and the third derivative lies in the plane $\Pi = O_P\Gamma$.

**Proposition 6.12** The multijet of $f$ fails to be transverse to $W_{111}$ if and only if $\Pi$
osculates $\Gamma$ at one of $P, Q, R$; to $W_{21}$ if and only if either $P$ is a stall or $\Pi$ osculates
$\Gamma$ at $Q$; to $W_3$ if and only if $k_{P}(\Gamma, \Pi) \geq 4$.

**Proof** (111) Take affine coordinates so that none of the points in question is at
infinity. The vector $X := P \wedge Q + Q \wedge R + R \wedge P$ is non-zero (since the points are not
collinear) and orthogonal to $\Pi$. The submanifold $W_{111}$ is defined by the vanishing of
the 3 expressions $X.P_1, X.Q_1$ and $X.R_1$. The condition for transversality is obtained by differentiating these expressions with respect to $t_p, t_q$ and $t_r$, and taking
the determinant of the resulting matrix.

Since $T_P\Gamma$ lies in $\Pi$, $\partial X/\partial t_p = P_1 \wedge (Q - R)$ is perpendicular to $\Pi$, hence is
orthogonal to $Q_1$ and $R_1$; similarly for $\partial X/\partial t_q$ and $\partial X/\partial t_r$. It follows that
the matrix is diagonal, and transversality fails if and only if one of the diagonal entries
vanishes. But $\partial (X.P_1)/\partial t_p = X.P_2$ vanishes only if $\Pi$ osculates $\Gamma$ at $P$.

(21) The vector $Y := P_1 \wedge (Q - P)$ orthogonal to $\Pi$, and $W_{21}$ is defined by the
vanishing of $Y.P_2$ and $Y.Q_1$. We have $\partial Y/\partial t_p = P_2 \wedge (Q - P)$ and $\partial Y/\partial t_q = P_1 \wedge Q_1$.
Thus
\[
\partial (Y.P_2)/\partial t_p = \partial Y/\partial t_p.P_2 + Y.P_3 = Y.P_3,
\partial (Y.P_2)/\partial t_q = \partial Y/\partial t_q.P_2 = [P_1, Q_1, P_2],
\partial (Y.Q_1)/\partial t_p = \partial Y/\partial t_p.Q_1 = [P_2, Q - P, Q_1],
\partial (Y.Q_1)/\partial t_q = \partial Y/\partial t_q.Q_1 + Y.Q_2 = Y.Q_2.
\]

The middle two expressions vanish since in each case all 3 of the vectors in the
scalar triple product are tangent to $\Pi$; thus for failure of transversality one of the
other two expressions must be zero, i.e. either $P_3$ or $Q_2$ tangent to $\Pi$.

(3) Here $P_1$ and $P_2$ are given to be non-parallel, and $W_3$ is defined by the
condition $[P_1, P_2, P_3] = 0$. Transversality fails if the derivative of this with respect
to $t_p$ vanishes. As this is $[P_1, P_2, P_3]$, the condition is that $P_3$ also is tangent to $\Pi$.

\[\square\]

We next deal with the ‘Greek letter cases’ $\alpha, \beta, \gamma$. The next result shows that in
these cases (TMS) is equivalent to the condition previously obtained in Lemma 6.8.

Define $W_\alpha$ to be the set of multijets in $J^2(S^1, P^3)$ such that the image points
$P$ and $Q$ are distinct and $Q \in T_P\Gamma$ (this corresponds to cases $\alpha, \delta$ of §4). Let $W_{\beta} \subset
J^3(S^1, P^3)$ be the set of jets such that $P, Q, R$ and $S$ are collinear and no plane
contains the tangents at three of them (this relates to case $\gamma$); $W_{\beta} \subset J^3(S^1, P^3)$
be the set of jets such that $P, Q, R$ are collinear and the tangents at $P$ and $Q$ (but
not that at $R$) lie in a plane $\Pi$ (this relates to cases $\beta, \beta_2$).

**Proposition 6.13** Transversality to $W_\alpha$ fails if and only if $T_Q\Gamma \subset O_P\Gamma$; to $W_\alpha$
if and only if the cross-ratio of the 4 collinear points $PQRS$ coincides with that of the
planes through the line containing the 4 respective tangents; to $W_\beta$ if and only if $R$
is not the T-centre of the T-secant $PQ$.

**Proof** (a) The condition $Q \in T_P\Gamma$ translates as $P_1 \wedge (Q - P) = 0$. Although
this appears to give 3 conditions, if we take coordinates so that $P$ and $Q$ lie on the
$x''$-axis, it suffices to take the first and second coordinates. Now
\[
\partial (P_1 \wedge (Q - P))/\partial t_p = P_2 \wedge (Q - P), \text{ and } \partial (P_1 \wedge (Q - P))/\partial t_q = P_1 \wedge Q_1.
\]
These vectors are proportional if and only if the plane $O_P\Gamma$ through $PQ$ contains
$T_Q\Gamma$.

(\gamma) Take the line $PQRS$ to be the $x''$-axis. Then $W_\gamma$ is defined by the vanishing of
the first and second coordinates of $X := P \wedge Q + Q \wedge R + R \wedge P$ and $Y :=
\[ P \cap Q + Q \cap S + S \cap P. \] We have \( \partial X / \partial t_p = P_1 \cap (Q - R) = (p_1, p_1', p_1'') \cap (0, 0, q_0 - r_0) = (q_0 - r_0)(p_1' - p_1, 0), \) and similarly for the others. The first two coordinates of the derivatives of \( X \) and \( Y \) with respect to the local co-ordinates \( t_p, t_q, t_r, t_s \) give the 4 \times 4 matrix

\[
\begin{pmatrix}
(q_0 - r_0)p_1' & -(q_0 - r_0)p_1 & (q_0 - s_0)p_1' & -(q_0 - s_0)p_1 \\
(r_0 - p_0)q_1' & -(r_0 - p_0)q_1 & (s_0 - p_0)q_1' & -(s_0 - p_0)q_1 \\
(p_0 - q_0)r_1' & -(p_0 - q_0)r_1 & (p_0 - q_0)s_1' & -(p_0 - q_0)s_1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

To simplify the determinant, choose coordinates so that \( p_1', q_1', r_1' \) and \( s_1' \) are all non-zero; divide the rows successively by these quantities, and write \( \theta_p := p_1'/p_1' \) etc. Remove the common factor \( (q_0 - r_0) \) from each entry in the two lower rows. Evaluate explicitly and collect terms to obtain

\[
(q_0 - r_0)(p_0 - q_0)(s_0 - r_0) + (q_0 - s_0)(p_0 - r_0)(r_0 - p_0) + (s_0 - p_0)(r_0 - q_0)(q_0 - s_0),
\]

But this same expression is obtained by multiplying the difference of cross ratios \( (p_0, q_0; r_0, s_0) - (\theta_p, \theta_q, \theta_r, \theta_s) \) by the denominator. Thus if transversality fails, either the two cross ratios are equal or the second is indeterminate, which occurs only when three of the \( \theta \)'s coincide; i.e. there is a plane containing the tangents at 3 of the collinear points \( P, Q, R, S \).

\( \beta \) We may take the line \( PQR \) as the \( x'' \)-axis, and the plane \( II \) as \( x' = 0 \). Then \( W_2 \) is defined by the vanishing of (the first and second coordinates of) \( X := P \cap Q + Q \cap R + R \cap P \), and of \( Z := [P_1, Q_1, Q - P] \). The first derivatives of \( X \) and \( Z \) are

\[
\begin{align*}
\frac{\partial X}{\partial t_p} &= (0, p_1(q_0'' - r_0''), 0), \\
\frac{\partial X}{\partial t_q} &= (p_0'' - q_0')(r_1' - r_1, 0), \\
\frac{\partial Z}{\partial t_p} &= [P_2, Q_1, Q - P] = 2q_2p_1(q_0' - q_0''), \\
\frac{\partial Z}{\partial t_q} &= [P_1, Q_2, Q - P] = -2q_2p_1(q_0' - q_0'').
\end{align*}
\]

We thus obtain the determinant

\[
\begin{vmatrix}
0 & p_1(q_0'' - r_0'') & 2q_2p_1(q_0'' - q_0'') \\
0 & q_1(r_0'' - p_0'') & -2q_2p_1(q_0' - q_0'') \\
(p_0'' - q_0'')r_1' & -(p_0'' - q_0'')r_1' & 0
\end{vmatrix}.
\]

In the determinant, we may divide the last row and column by \( (p_0'' - q_0'') \). The result reduces to \( 2q_1(p_0''q_2(q_0'' - r_0'') - q_1q^2_2(r_0'' - p_0'')) \). By hypothesis, \( r_1' \neq 0 \).

On the other hand, projecting the part of \( \Gamma \) near \( P \) from \( R \) to \( x'' = 0 \) gives

\[
\frac{r_0''}{r_0'' - p_0^2} (p_1t + O(t^2), p_2^2t^2 + O(t^3)).
\]

The radius of curvature of this is \( \frac{r_0''}{(r_0'' - p_0^2)^{3/2}} \). A similar result holds for \( Q \). The final term above thus vanishes if and only if the radii of curvature of the projections at \( P \) and at \( Q \) from \( R \) coincide, i.e. if and only if \( R \) is the T-centre of \( PQ \).

\[ \square \]

For case \( a^*b \) we have

**Proposition 6.14** In case \( a^*b \), \( (TMS) \) is equivalent to \( P \) not being a stall, so follows from \( (TK2) \).

**Proof** With our usual notation, the defining condition of this case is that we have a trisecant \( PQR \) lying in the osculating plane \( O_p\Gamma \). Choose co-ordinates with the trisecant along the \( x'' \)-axis and the osculating plane \( x' = 0 \). As the points are distinct, \( p_0'', q_0'', r_0'' \) are all distinct; as \( T_p\Gamma \) does not lie along the trisecant, \( p_1 \neq 0 \); as \( T_q\Gamma \) and \( T_r\Gamma \) do not lie in the osculating plane, \( q_1 \) and \( r_1' \) are non-zero.

Up to first order terms, \( X = P \cap Q + Q \cap R + R \cap P \) is equal to

\[
(q_1'(r_0'' - p_0'')t_q + r_1'(p_0'' - q_0'')t_r, p_1'(r_0'' - q_0'')t_p + q_1(p_0'' - r_0'')t_q + r_1(q_0'' - p_0'')t_r, 0),
\]

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and the determinant formed by $P - Q$, $P_1$ and $P_2$ is

$$
\begin{vmatrix}
0 & -q'_1 t_q & p''_0 - q''_0 \\
p_1 & 0 & p''_1 - q''_1 \\
2p_2 & 0 & 2p''_2
\end{vmatrix} = -2q'_1 (p_1 p''_2 - p'_1 p_2) t_q.
$$

Thus transversality is equivalent to non-vanishing of the coefficients of $t_r, t_p, t_q$ in the first, second and third of these expressions respectively. In view of the non-degeneracy conditions, this reduces to $p_1 p''_2 - p''_1 p_2 \neq 0$, i.e. to $\kappa_P(\Pi, \Gamma) \leq 2$, i.e. to $P$ not being a stall.

Finally we treat the more difficult cases $ab$ and $bb$.

**Proposition 6.15** In cases $ab$ and $bb$, (TMS) is equivalent to the family of self-projections of $\Gamma$ satisfying condition (Trot).

**Proof** In the projection from $P$ we have a line $L$ such that $\kappa(L, C_P) = 2$, attained at 2 points, each of type $A_3$ or $D_4$. An $A_3$ is the image of a point $Q \in \Gamma$ such that $T_Q \Gamma$ meets $T_P \Gamma$; a $D_4$ is the common image of 2 points $Q, R \in \Gamma$, so that $PQR$ is a trisecant.

The pre-image of $L$ is a plane $\Pi$ through $T_P \Gamma$: the pre-images of the 2 points are 2 lines through $P$ in $\Pi$, both distinct from $T_P \Gamma$. Choose co-ordinates so that $\Pi$ is the plane $x'' = 0$; $P$ is on the $x''$-axis, but not at the origin, so that we can project onto the plane $x'' = 0$, $T_P \Gamma$ is the $x''$-axis, and the other two lines through $P$ in $\Pi$ are given by $x'' + \epsilon x = p''_0$ for $\epsilon = \pm 1$.

First consider the $A_3$ case. We note the non-degeneracy conditions: $p''_1 \neq 0$ ($P$ is not a singular point), $q''_1 + \epsilon q_1 \neq 0$ ($P \notin T_Q \Gamma$), $p''_2, q''_2 \neq 0$ ($\Pi$ does not osculate $\Gamma$ at $P$ or $Q$). As we move $P$ we can deform $Q$ so as still to have a $T$-secant. We evaluate the first order term of this deformation. We require that $P - Q, \partial P/\partial t_p$ and $\partial Q/\partial t_q$ are coplanar. Expand $P$ and $Q$ as Taylor series in $t_p$ and $t_q$: this condition gives the vanishing of a determinant, in which all terms in the second column are already of first order. So to obtain first order terms in the determinant it suffices to ignore first order terms in the other columns. This gives

$$
\begin{vmatrix}
-q_0 & 0 & p''_0 - q''_0 \\
0 & 2p''_2 t_p & p''_1 \\
q_1 & 2q''_2 t_q & q''_1
\end{vmatrix},
$$

which, as $q''_0 + \epsilon q_0 = p''_0$, reduces to $2p''_2 t_p (-q_0 q''_1 - \epsilon q_1 q_0) + 2q''_2 t_q (p''_1 q_0)$. Taking out the non-zero factor $2q_0$, this gives $(q''_2 t_q) (p''_1 q_0) - 2p''_2 t_p (q''_1 q_0)$, and so the condition is to be preserved, we have to first order $t_q = -\frac{p''_2 (q''_1 + \epsilon q_1)}{2q''_2} t_p$.

In the $D_4$ case, we have non-degeneracy conditions $q_0 \neq r_0, q_0 \neq 0, r_0 \neq 0$ (the points $P, Q, R$ are distinct), $p''_1 \neq 0$ ($P$ is not a singular point), $q''_1, r''_1$ and $q'_1 (r''_1 + \epsilon r_1) - r'_1 (q''_1 + \epsilon q_1) \neq 0$ (we do not have a $T$-trisecant). We want to deform all of $P, Q$ and $R$ so as still to have a trisecant, so require the vanishing of $X = P \wedge Q \wedge R \wedge R \wedge P$. Up to first order, we have

\[
\begin{align*}
P &= (0, 0, p''_0 + p''_1 t_p) \\
Q &= (q_0 + q_1 t_q, q'_1 t_q, p''_0 - \epsilon q_0 + q''_1 t_q) \\
R &= (r_0 + r_1 t_r, r'_1 t_r, p''_0 - \epsilon r_0 + r''_1 t_r)
\end{align*}
\]

so $X$ has co-ordinates

\[
\begin{align*}
-(p''_0 + p''_1 t_p) q_1 t_q + q'_1 t_q (p''_0 - \epsilon r_0 + r''_1 t_r) - (p''_0 - \epsilon q_0 + q''_1 t_q) r'_1 t_r + r'_1 t_r (p''_0 + p''_1 t_p), \\
(p''_0 + p''_1 t_p) q_0 + q_1 t_q - (q_0 + q_1 t_q) (p''_0 - \epsilon r_0 + r''_1 t_r) + (p''_0 - \epsilon q_0 + q''_1 t_q) (r_0 + r_1 t_r) - (r_0 + r_1 t_r) (p''_0 + p''_1 t_p), \\
\end{align*}
\]

and $(q_0 + q_1 t_q) r'_1 t_r - (r_0 + r_1 t_r) q'_1 t_q$. The constant terms in these expressions cancel;

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the first order terms reduce to
\((\epsilon (q_0 r_1 t_q - q_1 r_0 t_q), \lambda_p r_1' (q_0 - r_0) + t_q r_0 (q_1'' + \epsilon q_1)) - \lambda_t q_0 (r_1'' + \epsilon r_1), q_0 r_1 t_r - r_0 q_1 t_q).\)

Thus to retain a trisecant we have, to first order,
\[
t_r = \frac{r_0 q_1'}{u_1''} t_q, \quad t_q = \frac{p_1'' (q_0 - r_0) r_1'}{r_0 (q_1'' + \epsilon r_1) - r_1' (q_1'' + \epsilon q_1)} t_p.
\]

Next consider condition (TMS) for \(ab\). Here we have points \(P, Q\) (corresponding to \(A_3\), with \(\epsilon = 1\)) and \(U, V\) (corresponding to \(D_4\), with \(\epsilon = -1\)). The defining conditions for the case are that \(P - Q, P_1, Q_1\) and \(P - U\) are coplanar, and that \(P, U, V\) are collinear. To check transversality, we apply the conditions to nearby points of \(\Gamma\), retaining only the first order terms. We have already done this for the first three and the final conditions, and verified that these yield independent conditions on the differentials \(t_p, t_q, t_u\) and \(t_v\). Coplanarity of \(P - Q, P_1, P - U\) gives a determinant where again we have only first order terms in the middle column, so can ignore them in the others, giving
\[
\begin{vmatrix}
-q_0 & 0 & p_0'' - q_0'' \\
0 & 2p_2' t_p & p_1'' \\
-u_0 & -u_1'' t_u & -u_0
\end{vmatrix},
\]
which reduces to \(2p_2' t_p (u_0 q_0 + u_0 (p_0'' - q_0'')) - q_0 p_1'' u_1'' t_u\), i.e. to \(4p_2' u_0 q_0 t_p - q_0 p_1'' u_1'' t_u\). Transversality is equivalent to the statement that this expression is linearly independent of the relation between \(t_p\) and \(t_u\) previously obtained, i.e. (writing that relation as \(t_u = \lambda_u t_p\)) that \(4p_2' u_0 \neq p_1'' u_1'' \lambda_u\), or in full
\[
\begin{vmatrix}
p_0'' (u_0 - v_0) v_1' \\
2p_2' u_0 (q_0 + p_0'' - q_0'') & v_0 (u_1' (v_1'' + v_1) - v_1' (u_1'' + u_1)) & q_0 p_1'' u_1'' \\
-u_0 & -u_1'' t_u & -u_0
\end{vmatrix} \neq 0.
\]

We must now compare this with the other condition. We have already seen how to deform the T-secant PQ and the 3-secant PUW with \(P\); we must now project from \(P\) and compare the angles made by the deformed lines.

For the projection of \(P\) itself we have \((p, p', p'') = \frac{p'}{p''} (p, p', p'')\). We only need the first component to \(0^\text{th}\) order, which is 0. The second component, to first order, is \(0 - \frac{p_0''}{p_0''} (2p_2' t_p)\). The point \(Q\) projects to \((\frac{p_0'' - q_0''}{p_0'' - q_0''}, \frac{p_0'' - q_0''}{p_0'' - q_0''}, \frac{p_0'' - q_0''}{p_0'' - q_0''})\); to first order, the denominator gives \(p_0'' + p_1' t_p - q_0'' - q_0'' t_q\); the respective numerators are \((p_0'' + p_1' t_p) (q_0 + q_1 t_q)\) and \((p_0'' + p_1' t_p) q_1 t_q\). To obtain the slope to first order, we only need the \(0^\text{th}\) terms of the first co-ordinate, viz. \(p_0'' q_0 / (p_0'' - q_0'') = p_0''\). The slope is thus \(q_1 t_q = 0\). Similarly, the projection of \(U\) has first co-coordinate \(-p_0''\) and second \((-p_0'' u_1'' / u_0) t_u\).

To first order, the 3 points are collinear iff \((-p_0'' u_1'' / u_0) t_u = -4p_2' (p_2' t_p)\). Our condition is equivalent to the failure of this, i.e. (substituting for \(t_u\)) to \(u_1'' / u_0 \lambda_u \neq 4p_2' / p_0''\). This coincides with the condition above.

Similar calculations apply to case \(bb\). Here we have points \(P, Q, R\) (corresponding to \(D_4\), with \(\epsilon = 1\)) and \(U, V\) (corresponding to \(D_4\), with \(\epsilon = -1\)). The defining conditions for the case are that \(P - Q, P_1, P - U\) and that each of \(P, Q, R, P, U, V\) are collinear. To check transversality, we apply the conditions to nearby points of \(\Gamma\), retaining only the first order terms. We have already done this for collinearity conditions, and verified that these yield independent conditions on the differentials \(t_p, t_q, t_r, t_u\) and \(t_v\). Coplanarity of \(P - Q, P_1, P - U\) gives a determinant where again we have only first order terms in the middle column, so can ignore them in the others, giving
\[
\begin{vmatrix}
-q_0 & -q_1' t_q & q_0 \\
0 & 2p_2' t_p & p_1' \\
-u_0 & -u_1'' t_u & -u_0
\end{vmatrix} = 4p_2' q_0 u_0 t_p + p_1' q_1' u_0 t_q - p_1' q_0 u_1' t_u.
\]
The transversality condition (TMS) is equivalent to non-vanishing of the coefficient of \( t_p \) in the result of substituting for \( t_q \) and \( t_u \) in this expression. Setting \( t_q = \lambda_q t_p \) and \( t_u = \lambda_u t_p \), this is \( 4p''_0 q_0 u_0 + p''_1 q'_1 u_0 \lambda_q - p''_1 q'_1 u_1 \lambda_u \neq 0 \).

The projections of \( P, Q \) and \( U \) again have first co-ordinates 0, \( p''_0 \) and \( -p''_0 \); their second co-ordinates, to first order, are \(-2(p''_0 p'_2 / p''_1) t_p, (-p''_0 q'_1 / q_0) t_q \) and \((p''_0 u'_1 / u_0) t_u \). The condition on relative rates of rotation is that these points should not be collinear (to first order), i.e. that \( 4(p''_0 p'_2 / p''_1) \neq (-p''_0 q'_1 / q_0) \lambda_q + (-p''_0 u'_1 / u_0) \lambda_u \). Again we see that the conditions coincide.

We make one final remark. Although the transversality arguments do not apply directly, the analysis of cases is essentially also valid in the complex case, so applies to algebraic curves in complex projective space. Our transversality conditions hold for curves which are general in a certain sense. For these, the number of 0-dimensional strata of each type is finite, and there are formulae (many of which are known) expressing these numbers in terms of the degree and genus of \( \Gamma \).

References