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GAFFNEY'S WORK ON EQUISINGULARITY

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To Terry Gaffney on his sixtieth birthday

Abstract

A survey of equisingularity theory focussed on Terry Gaffney's work.

The article begins with an account of the early history of equisingularity. Next I develop notation, particularly for polar varieties; recall the theory of integral closures of ideals, show how Gaffney generalised this to integral closures of modules, and list a variety of applications he has made.

The invariants available are classical and Buchsbaum-Rim multiplicities of modules, polar multiplicities and Segre numbers of ideals, and generalisations to modules. Some of the main theorems are of the form: the constancy of certain numerical invariants of a family imply equisingularity of the family (usually in the form of Whitney triviality). Many of the proofs use results showing that constancy of some invariants implies an integral dependence relation. One notable paper gives a sufficient condition for topological triviality of families of maps.

Introduction

The classification of singularities of plane curves was achieved in 1932 by Brauner [**2**], Burau [**6**],[**7**] and Zariski [**60**]: it yields an easily stated necessary and sufficient condition for topological equivalence, which clearly does not imply analytic equivalence. Probably the simplest example is the case of 4 concurrent lines $xy(x+y)(x+ty) = 0$ with t an invariant of analytic, but not of topological equivalence.

This situation presents the problem of creating a theory of equivalence of families of objects (e.g. algebraic varieties or morphisms) which will say when the members of the family are essentially the same. One needs a definition allowing some flexibility but with which calculations can be made. This is the problem of equisingularity, which lies at the heart of singularity theory.

Terry Gaffney has made major contributions to this, many of which appear in Proceedings of earlier São Carlos meetings. His philosophy is to seek invariants depending on members of the family whose constancy implies equisingularity.

In this article, I seek to describe Terry's work in this area. To put this in perspective, I also give an account of the earlier work which led up to

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it. This seems appropriate, as Terry has always sought to give full credit to others whose work or influence has contributed to his results. I am indebted to David Trotman, Andrew du Plessis and particularly to Terry himself for comments on earlier versions of this article.

I include a complete list of Gaffney's papers, which are cited with a G, e.g. as [G21], preceding the general bibliography.

1. *Early results on equisingularity*

The origins of the differential theory of equisingularity lie in attempts to classify singularities of differentiable mappings. This began with pioneering work of Whitney in the 1940s and 1950s [**51**], [**52**], [**53**], [**54**] classifying generic singularities in particular dimensions. A discussion in general was given by Réné Thom [**44**] in 1959, in particular conjecturing that topologically stable maps were $C^\infty-$dense in all dimensions.

Thom announced new ideas at a lecture in Zürich in 1960 (see [**45**]) (where the writer had the good fortune to be present). This contains definitions of stratifications, mention of regularity and a statement that "Whitney has proved that real algebraic sets admit regular stratifications", the apparatus (tubes, local retractions, carpeting functions), and went on with corresponding ideas for $C^\infty-$mappings. Thom had amazingly good geometric intuition; not only completely new ideas, but a good idea for what might be true and provable. It was often left to others to flesh out his ideas to obtain clear proofs. Gaffney is perhaps his true successor in having excellent geometric intuition, but he finds proofs with help from collaborators.

Details followed a few years later. In [**58**], Whitney studied the behaviour of the tangent plane T_xX at a smooth point $x \in X$ as x tends to a smooth point y_0 on a subvariety Y of X and formulated conditions on (X, Y) at y_0:

(A) if there is a sequence $x_i \in X$ such that $x_i \to y_0$ and $T_{x_i}X$ tends to a limit L, then $T_yY \subset L$,

(B) if there is also a sequence $y_i \in Y$ such that $y_i \to y_0$ and the unit vector in the direction y_ix_i (in the ambient space) tends to a limit v, then $v \in L$.

In fact, it was soon realised that (B) implies (A); however, (A) remains an important condition. Whitney defined stratifications, called a stratification regular if these conditions hold at all points, and proved that any complex analytic variety has a regular stratification. For the real semi-analytic case, a proof of existence of regular stratifications was given, also in 1965, in notes [**27**] by Łojasiewicz, which established basic facts about semi-analytic sets, including his famous inequalities.

Thom's 'first isotopy lemma' states that a regularly stratified set is locally topologically trivial along strata. Proofs were given by Thom [**47**] and in widely circulated lecture notes by Mather [**31**] in 1970. These involve the construction of controlled vector fields, and their integration.

A useful variant of the regularity conditions was given by Verdier [**49**], anticipated in part by Hironaka [**14**] and the c-cosecance of Teissier [**38**]. For linear subspaces $A, B \subset \mathbb{R}^N$, define

$$\delta(A, B) := sup_{u \in A^{\perp} \setminus \{O\},\ v \in B \setminus \{O\}} \frac{|(u, v)|}{\|u\|.\|v\|};$$

thus $\delta(A, B) = 0 \Leftrightarrow A \supseteq B$. We can re-state the Whitney (A) condition as $\delta(T_{x_i}X, T_{y_0}Y) \to 0$ as $x_i \to y_0$, i.e. as $\|x_i - y_0\| \to 0$. Now say that (X^o, Y) satisfies the Verdier condition W at y_0: if

(W) there exist a neighbourhood U of y_0 and $C > 0$ such that, for all $y \in U \cap Y$, $x \in U \cap X^o$, we have $\delta(T_xX, T_yY) \leq C\|x - y\|$.
Verdier established that subanalytic sets admit stratifications satisfying this condition, that it is stronger than Whitney's condition (B); also that when it holds, one obtains controlled vector fields satisfying a condition he terms 'rugose' (which is stronger than continuity but weaker than Lipschitz). This can be generalised to 'the strict Whitney condition A with exponent r' by replacing the right hand side of the inequality by $C\|x - y\|^r$ (see e.g. [**41**]).

In [**45**] Thom also enunciated a 'second isotopy lemma' giving a sufficient condition for topological triviality of a family of $C^\infty-$mappings. This was used in [**46**] to obtain deep results about singularities in general.

First he stated that any polynomial map is stratifiable, meaning that there are stratifications of source and target such that the map submerses each stratum of the source on a stratum of the target. He said that a map presents blowing-up if, writing for X a stratum of the source with image X', and $q(X)$ for $\dim X - \dim X'$, there exist strata $Y \subset \overline{X}$ with $q(Y) > q(X)$.

In [**47**], he defined a relative form of (A), now known as Thom regularity. If f is a stratified map, the Thom condition holds for (X, Y) relative to f at $y_0 \in Y$: if

(A_f) for any sequence $x_i \in X$ with $x_i \to y_0$ and $Ker(Tf|T_{x_i}X) \to L$ we have $Ker(Tf|T_yY) \subseteq L$.

Note that we can re-state this as: $\delta(Ker(Tf|T_xY), Ker(Tf|T_{x_i}X)) \to 0$ as $x_i \to y_0$. There is also a strict condition, first considered in [**12**],

(W_f) there exist a neighbourhood U of y_0 and $C > 0$ such that, for all $y \in U \cap Y$, $x \in U \cap X^o$, we have $\delta(Ker(Tf|T_yY), Ker(Tf|T_xX)) \leq C\|x - y\|$.

The second isotopy lemma now refers to a stratified map $f : X \to Y$ and a further $\pi : Y \to T$ such that, for each stratum S of Y, $\pi|S$ is a submersion. If also the maps are proper and A_f holds at all points, f is locally trivial, so the topological type of $f|(\pi \circ f)^{-1}(t)$ is independent of t. Proofs, similar to those of the first isotopy lemma, were sketched in [**47**] and in [**31**].

Clearly a necessary condition for f to possess a Thom regular stratification is that f does not exhibit blowing up. A useful construction of Thom regular maps is given in [**9**, §2]: here make the stronger hypothesis that the restriction $f|\Sigma(f)$ (where $\Sigma(f)$ denotes the critical set of f) is proper and finite-to-one.

Then a stratification of $f(\Sigma(f))$ is a critical value stratification c.v.s. (called partial stratification in [**9**]) if, for all strata U, $f^{-1}(U) \cap \Sigma(f)$ is smooth and f induces a local isomorphism of it on U, and for all pairs U, V of strata, $f^{-1}(V) \cap \Sigma(f)$ and $f^{-1}(V) \setminus \Sigma(f)$ are Whitney regular over $f^{-1}(U) \cap \Sigma(f)$. Whitney's arguments yield the existence of a c.v.s. provided the spaces and maps are semialgebraic. Given a c.v.s., we can stratify the target of f by the strata of the c.v.s. and their complement and stratify the source by the strata just listed: then f is a stratified map which satisfies Thom regularity.

A general proof that a proper, complex analytic map which does not exhibit blowing up admits a Thom regular stratification, and even one satisfying W_f, was finally given by Henry, Merle and Sabbah in [**12**]: the proof uses the technique of polar varieties. The real analytic case was discussed in [**15**], and can also be treated by taking real parts of the stratifications of [**12**]. The writer has been unable to find a reference for existence of a Thom stratification in the real semi-analytic case.

We turn to the algebro-geometric approach to equisingularity. From 1964, when Hironaka [**13**] established the resolution of singularities in characteristic zero, interest began to shift from resolving singularities to classifying them. Zariski created a theory of equisingularity for families of curves, and hence for a variety along a smooth subvariety of codimension 1, and proposed a general definition by induction: roughly speaking, he required equisingularity of the discriminant of a generic projection to a subspace one dimension lower. The simplest non-trivial case is when we have a smooth point of Y, which has codimension 2, so a transverse slice meets X in a plane curve. The theory for this case was developed in considerable detail in the series [**61**] in 1965-68. In this case, Zariski's definition of equisingularity is equivalent to Whitney's, as holds more generally when the curve is not required to be planar.

In 1971 in [**62**], Zariski compared his definition with others and posed a number of searching and motivating questions, notably the famous problem of topological invariance of multiplicity. What is the relation between different conditions? When do equivalent singularities lie in a 1-parameter family? Do equivalent varieties have the same multiplicity? More generally, is equisingularity preserved under taking generic hyperplane sections? or under taking the discriminant of a generic projection? We will see that the techniques of projection and of generic hyperplane sections are built in to the theory as it has been developed by Gaffney.

The rather simple example $z^3 + tx^4z + x^6 + y^6 = 0$ (due to Briançon and Speder [**3**]) is Whitney equisingular but not Zariski equisingular; moreover, as Zariski shows in [**64**] in 1977, the blowup along the $t-$axis fails to be equisingular at $t = 0$, which led Zariski to reject Whitney equisingularity as a good notion.

A general discussion of the equisingularity notions as known in 1974 was also given by Teissier in [**38**]. He starts with Zariski's work, which was his own inspiration, and compares equisingularity in Zariski's sense, in Whitney's sense, and topological local triviality, and he too formulated a number of questions and conjectures.

Finally in 1979 Zariski proposed [**65**] a modified version of his definition which, unlike earlier ones, is clearly invariant under local analytic equivalence; here he also constructs a stratification.

Zariski's work in [**61**] also led him to the notion of saturation, which he developed in [**63**] in 1971-75. Here he gives a more algebraic form to equisingularity for the plane curve case. In the introduction to Vol IV of Zariski's works, Teissier and Lipman record that later work inspired by equisaturation led to the study of Lipschitz equisingularity. Talks were presented on Lipschitz equivalence at this conference by Birbrair & Neumann, and by Valette giving new insight on this concept: it does not at present seem likely to lead to a workable theory of equisingularity in general.

A rather different equisingularity notion, of 'blow analytic equivalence', was proposed by Kuo [**25**]. While this has had some success, it too does not seem likely to lead to a general theory, as it is not clear that if a family of varieties over T is blow analytically trivial over the open subsets T_1, T_2 of T it must also be trivial over $T_1 \cup T_2$.

Milnor's 1968 book [**33**] was enormously influential, and focussed attention on isolated singularities of hypersurfaces. Let $\{f_t : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0) \,|\, t \in \mathbb{C}\}$ be a 1-parameter family of functions defining hypersurfaces X_t, each with an isolated singularity, with union $\mathbf{X} \subset \mathbb{C}^N \times \mathbb{C}$. There is an obvious numerical requirement for a family to be equisingular: constancy of the Milnor number $\mu(f_t)$.

For this case, major developments appeared at the Cargèse conference in 1972. Speder [**36**] proved that Zariski equisingularity implies the Whitney conditions. Lê [**22**] spoke on his result with Ramanujam that a $\mu-$constant deformation is topologically trivial (except possibly if $N = 3$).

There is also a major paper by Bernard Teissier [**37**], which is the real starting point of modern equisingularity theory. He discussed integral closures, introduced and studied the sequence $\mu^*(X_t)$ of Milnor numbers of generic linear sections of X_t, and showed that a μ^*-constant family is Whitney equisingular. This result was completed shortly afterwards by a proof [**4**] of Briançon and Speder that Whitney equisingularity implies μ^*-constant and their example [**3**] of a family with μ, but not μ^*, constant: for $z^5 + tzy^6 + xy^7 + x^{15} = 0$, $\mu^{(3)} = 15$ is constant but $\mu^{(2)}$ is not, and the family is topologically trivial.

In [G30] Gaffney and Massey described (with hindsight) a somewhat simplified version of Teissier's argument, summarised in three steps:

Whitney regularity of $F = \{f_t\}$ is implied by W-regularity, the condition that the derivative $\partial F/\partial t$ belong to the integral closure $\overline{\mathfrak{m}_N \cdot J_z F}$, where $J_z F = \langle \{\partial F/\partial z_i\} \rangle$;

this condition holds at a dense open set and ("the PSID"), provided the multiplicity $m(\mathfrak{m}_N \cdot J_z F)$ is constant, holds on a closed set; and

$m(\mathfrak{m}_N \cdot J_z F)$ is a linear combination with positive coefficients of the upper semicontinuous invariants $\mu^{(i)}(X_t)$.

With the success of Teissier's theory, one would like to extend it as far as possible. To achieve this, the following are needed: an extension of the theory of integral closure of ideals; invariants of equisingularity, corresponding to μ^* above; a calculus for working with these invariants; and a generalisation of the PSID. Gaffney has obtained many results of all of these types, and I will try to summarise them. The next sections are devoted respectively to integral closure; invariants and formulae for them; and criteria for equisingularity.

2. *Notations*

I now fix notation for the rest of this article, for simplicity of exposition; though some of Gaffney's results were obtained in greater generality than I give below. The reader should be warned that though this notation is based on Gaffney's, it differs from his in many cases. We have a complex analytic variety-germ $(X, 0) \subset (\mathbb{C}^N, 0)$ (for brevity, I will restrict almost entirely to the complex analytic case, though Gaffney also has many results in the real case); we assume X equidimensional, of dimension d, and generically reduced. Write ΣX for the singular set and $X^o := X \setminus \Sigma X$. We may suppose X given as $F^{-1}(0)$ for $F : \mathbb{C}^N \to \mathbb{C}^p$; take co-ordinates $\{z_i\}$ on $\mathbb{C}^N$.

When we wish to study families we take $T = \mathbb{C}^s$ as parameter space, with co-ordinates $\{t_j\}$, let $\mathbf{X} \subset \mathbb{C}^N \times T$ be given as $F^{-1}(0)$ for $F : \mathbb{C}^N \times T \to \mathbb{C}^p$, write $\pi : \mathbb{C}^N \times T \to T$ for the projection (and its restriction to $\mathbf{X} \to T$), $\mathbf{X}^o$ for the set of points where $\mathbf{X}$ is smooth and π submersive. Write also $X_t := \mathbf{X} \cap \pi^{-1}(t)$, and suppose each X_t as in the preceding paragraph. We write T for $\{0\} \times T \subset \mathbf{X}$ and study Whitney equisingularity of $\mathbf{X}$ over T along T.

For any k, $\mathcal{O}_k$ denotes the ring of germs of holomorphic functions at $0 \in \mathbb{C}^k$; $\mathfrak{m}_k$ denotes its maximal ideal. We write $\mathcal{O}_X$ for the sheaf of holomorphic functions on X, $\mathcal{O}_{X,x}$ for the sheaf of germs at $x \in X$, and $\mathfrak{m}_{X,x}$ for its maximal ideal. I will normally use roman letters to denote rings and modules and calligraphic ones for sheaves.

First we suppose each X_t has an isolated singular point at O; later we relax this. Also we first have the hypersurface case $p = 1$, then the complete intersection case where F is a submersion at a generic point, then the general case.

Sometimes regularity conditions A_f and W_f are considered relative to a further map $f : (X, O) \to (\mathbb{C}, 0)$ or $f : (\mathbf{X}, T) \to (\mathbb{C}, 0)$: these are non-trivial even if $p = 0$ so $\mathbf{X} = \mathbb{C}^N \times T$. We denote the zero locus of f by Z (or $\mathbf{Z}$).

Whitney's and other related conditions are defined in terms of the limiting behaviour of tangent spaces to X. Thus we are led to the study of the Nash blowup $N(X)$, the closure of the set of pairs $(x, T_x X)$ where $x \in X^o$, and the conormal space $C(X)$, the closure of the set of pairs (x, H) where $x \in X^o$ and $H \in P^{N-1}$ is a hyperplane containing the tangent space to X at x. If X has codimension 1 these coincide, but for a subset of higher codimension, while early work used the Nash blowup, it was shown by Henry and Merle [**11**] that the conormal space was more convenient and gave better results. For example, polar varieties and polar multiplicities are defined below by pulling back from linear subspaces of projective space; to use the Nash blowup, we would have to study instead subvarieties of Grassmannians. In the case of a family, we have the relative conormal, which can be defined as the closure of the set of pairs (x, H) with $x \in \mathbf{X}^o$ and H a hyperplane tangent to $\mathbf{X}$ at x and containing (a parallel of) T. We denote this by $C_T\mathbf{X}$ and can regard it as a subspace of $\mathbf{X} \times P^{N-1}$.

Given an ideal $\mathcal{I} = \langle g_1, \ldots, g_q \rangle \lhd \mathcal{O}_X$, the blowup $B_\mathcal{I}(X)$ is defined to be the closure in $X \times P^{q-1}$ of the graph of $X \setminus V(\mathcal{I}) \to P^{q-1}$ defined by $z \mapsto (g_1(z), \ldots, g_q(z))$, with projection $b_I : B_\mathcal{I}(X) \to X$; we write $D_\mathcal{I}$ for the exceptional divisor. Equivalently, we can form the Rees algebra (see [**35**]) $R(\mathcal{I}) := \bigoplus_{n \geq 0} \mathcal{I}^n$; then its graded ideals define $B_\mathcal{I}(X) = Proj(R(\mathcal{I}))$ (where $Proj$ denotes the analytic homogeneous spectrum). If X has codimension 1, the Jacobian ideal is $J(F) = \langle \partial F/\partial z_1, \ldots, \partial F/\partial z_N \rangle$ and $C(X)$ coincides with its blowup $B_{J(F)}(X)$.

If X has codimension greater than 1, $C(X)$ is not a blowup of X: it has dimension $N - 1$. To extend the techniques to this case, Gaffney introduced the following. Write $\mathcal{E} := \mathcal{O}_X^p$ for the free module, $J_M F \subset \mathcal{E}$ for the (Jacobian) submodule generated by the columns of the Jacobian matrix $JF = (\partial F_i/\partial z_j)$, which has generic rank $N - d$; write $S\mathcal{E}$ for the symmetric algebra on $\mathcal{E}$ and $\mathcal{R}J_M F$ for the (Rees) subalgebra generated by $J_M F$. Then $P = Proj(S\mathcal{E})$ has dimension $d+p-1$, and the image of $Proj(\mathcal{R}J_M F)$ in P can be identified with $C(X)$.

In general, for $L \subset \mathbb{C}^N \times T$ a linear subspace, write $J_M F_L$ for the submodule of $J_M F$ generated by the $\partial F/\partial v$ for $\partial/\partial v$ tangent to L (in the case $p = 1$, JF_L for the subideal of JF). Thus in the case of a family $F : \mathbb{C}^N \times T \to \mathbb{C}^p$, $J_M F_T$ is generated by the columns $\partial F_i/\partial z_j$ for z_j the co-ordinates in $\mathbb{C}^N$. The relative conormal $C_T\mathbf{X}$ is now the image of $Proj(\mathcal{R}J_M F_T)$ in P.

Gaffney [G28] comments that the exceptional divisors of these blowups record behaviour of the limiting tangent hyperplanes, which are relevant for the Whitney conditions: more precisely, the fibre over O of the exceptional divisor of $J_M F_T$ records the limits as $(x, t) \to (0, 0)$ of limiting tangent hyperplanes to X_t; in the module case, the fibre of the conormal need not be a divisor, but does still record the limits.

3. *Integral closures*

Let I be an ideal in a ring R (we write $I \lhd R$). We say that x is integral over I if there exist elements $a_r \in I^r$ with $x^k + \sum_1^k a_r x^{k-r} = 0$. The set of such elements x is called the integral closure of I and denoted $\overline{I}$: it is an ideal in R. The proof uses the fact that x is integral over I if and only if there is a faithful finitely generated $R-$module M with $xM \subseteq I \cdot M$.

The integral closure has marvellous properties. An excellent reference is the beautiful set of lecture notes [**24**] of Monique Lejeune-Jalabert and Bernard Teissier. These were, it seems, motivated by a study of a section of Hironaka's big paper [**13**].

For R a complete local ring, and I a proper ideal, I defines a function $\overline{\nu}_I$ on R by setting $\nu_I(x) := \sup\{n \in \mathbb{N} \mid x \in I^n\}$ and $\overline{\nu}_I(x) := \lim_{k\to\infty} \nu_I(x^k)/k$. This is an order function, i.e. $\overline{\nu}_I(x+y) \geq \inf(\overline{\nu}_I(x), \overline{\nu}_I(y))$, $\overline{\nu}_I(xy) \geq \overline{\nu}_I(x) + \overline{\nu}_I(y)$, $\overline{\nu}_I(0) = \infty$ and $\overline{\nu}_I(1) = 0$. Then x is integral over I if and only if $\overline{\nu}_I(x) \geq 1$.

If $\overline{\nu}_I(x) > 1$, we say that f is *strictly dependent* on $\mathcal{I}$, and write $f \in \mathcal{I}^\dagger$: $\mathcal{I}^\dagger$ also is an integrally closed ideal. Although strict dependence had been used previously, the definition of $\mathcal{I}^\dagger$, and of a corresponding notion for modules, are due to Gaffney [G27].

We can regard $\overline{I}$ as the largest ideal equivalent to I: sometimes we would prefer a smallest. If $J \subset I$ and $\overline{J} = \overline{I}$, then J is said to be a *reduction* of I; a *minimal reduction* is one with the minimal number of generators. For I of finite codimension in $\mathcal{O}_k$, we can take the ideal generated by k general elements of I.

Now let X be a reduced complex analytic space, $\mathcal{I} \lhd \mathcal{O}_X$ the coherent sheaf of ideals defining a nowhere dense analytic subspace Y, $x \in Y$, $\mathcal{I}_x$ the germ of $\mathcal{I}$ at x; $f \in \mathcal{O}_X$. The following are equivalent ([**24**], see also [**41**]):

(i) (algebraic condition) $f \in \overline{\mathcal{I}_x}$;

(ii) (evaluation) for some f.g. faithful $\mathcal{O}_{X,x}-$module $\mathcal{M}_x$, $f.\mathcal{M}_x \subset \mathcal{I}_x.\mathcal{M}_x$;

(iii) (valuative criterion) for every arc, i.e. map-germ $\phi : (\mathbb{C}, 0) \to (X, x)$, we have $f \circ \phi \in \phi^*\mathcal{I}_x.\mathcal{O}_1$;

(iv) (growth condition) for V a neighbourhood of x in X, and $\{g_i\}$ generators of $\Gamma(V, \mathcal{I})$, there exist $C \in \mathbb{R}^+$ and a neighbourhood of x on which $|f(y)| \leq C \sup_i |g_i(y)|$.

Moreover, the $\overline{\mathcal{I}_x}$ are the stalks of a coherent sheaf $\overline{\mathcal{I}}$.

If $\mathcal{I} \subset \mathcal{J}$ there is a natural map $B_{\mathcal{J}}(X) \to B_{\mathcal{I}}(X)$. This map is finite if and only if $\overline{\mathcal{I}} = \overline{\mathcal{J}}$, i.e. I is a reduction of J.

We also need the normalised blowup, which we denote by $\tilde{b}_{\mathcal{I}} : \tilde{B}_{\mathcal{I}}(X) \to X$, with exceptional divisor $\tilde{D}_{\mathcal{I}}$; write $\mathcal{I}^*$ for the pullback of $\overline{\mathcal{I}}$. To construct it, take the normalisation $\tilde{X}$ of X, the pullback $\mathcal{I}'$ to it of $\mathcal{I}$, the integral closure $\overline{\mathcal{I}'}$ of $\mathcal{I}'$, and then $\tilde{B}_{\mathcal{I}}(X) = B_{\overline{\mathcal{I}'}}(\tilde{X})$.

Suppose X compact, or more generally that we consider germs along some compact subset $K \subset X$. Then $\tilde{D}_{\mathcal{I}}$ has only finitely many irreducible components D_α. The ideal $\mathcal{I}^*$ is supported on $\bigcup_\alpha D_\alpha$; in the neighbourhood of a smooth point of D_α, it can only be a power $\mathcal{I}_\alpha^{n_\alpha}$ of the ideal $\mathcal{I}_\alpha$ defining D_α.

Thus if $f \in \mathcal{I}$, the lift of f to $\tilde{B}_{\mathcal{I}}(X)$ must belong to each $\mathcal{I}_\alpha^{n_\alpha}$. But much more is true. Suppose X normal, $\mathcal{I} \lhd \mathcal{O}_X$ an invertible ideal defining a Cartier divisor $D = \bigcup_\alpha D_\alpha$, $f \in \Gamma(X, \mathcal{O}_X)$. Then

$f_x \in \overline{\mathcal{I}_x}$ for all $x \in X \Leftrightarrow$ for each α we can find $x_\alpha \in D_\alpha$ with $f_{x_\alpha} \in \mathcal{I}_{\alpha,x}^{n_\alpha}$.

It follows that in (iii) above, we only need one arc ϕ_α for each component D_α of D. This has numerous consequences: for example we can define fractional powers by letting $f \in \mathcal{I}^{[p/q]}$ if, for some N, $f^{qN} \in \overline{\mathcal{I}}^{pN}$: it follows that we can take all fractions to have denominator the least common multiple of the n_α; this is also a denominator for the Łojasiewicz exponent.

We can now give a global version of the above set of equivalent conditions to $f \in \overline{\mathcal{I}}$, referring to germs at K:

(i) $f \in \overline{\mathcal{I}_x}$ $(\forall x \in K)$;

(ii) for every proper $\pi : X' \to X$ with image containing K and $\mathcal{I}.\mathcal{O}_{X'}$ invertible, there is an open neighbourhood U' of $\pi^{-1}(K)$ with $f.\mathcal{O}_{U'} \in \mathcal{I}.\mathcal{O}_{U'}$;

(iii) for each α, $f \circ \phi_\alpha \in \phi_\alpha^* \mathcal{I}_x.\mathcal{O}_1$;

(iv) there exist an open neighbourhood U of K in X, generators $\{g_i\}$ of $\Gamma(U, \mathcal{O}_X)$ and $C \in \mathbb{R}^+$ such that $|f(x)| \leq C \cdot \sup\{|g_i(x)|\}$ for all $x \in U$.

Integral closures of ideals are the key to the study of equisingularity for families of hypersurfaces. To generalise to subvarieties of larger codimension, Gaffney begins [G21] by introducing the integral closure of a module or, more accurately, of a module given as a submodule of a free module. An earlier, purely algebraic treatment of an equivalent concept had been given by Rees [**35**], generalising results in [**66**]; but Gaffney takes the valuative criterion as definition. He notes that this also gives a useful notion in the real case, but we will restrict to the complex case.

For $h \in \mathcal{O}^p_{X,x}$ and $\mathcal{M}_x \subset \mathcal{E}_x = \mathcal{O}^p_{X,x}$ a submodule, the following are equivalent:

(i) $h \in \overline{\mathcal{M}_x}$ in the sense of Rees,

(ii) for some faithful $\mathcal{I}_x \lhd \mathcal{O}_{X,x}$ we have $\mathcal{I}_x \cdot h \subseteq \mathcal{I}_x \cdot \mathcal{M}_x$.

(iii) for all germs $\phi : (\mathbb{C}, 0) \to (X, x)$, we have $h \circ \phi \in \phi^*(\mathcal{M}_x) \cdot \mathcal{O}_1$ (as above, it suffices to check for rather few arcs ϕ),

(iv) For each choice $\{s_i\}$ of a set of generators for $\mathcal{M}_x$ there is a neighbourhood U of x such that for each $\phi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$ there exists $C > 0$ such that, for all $z \in U$ we have $|\phi(z)h(z)| \leq C \sup_i |\phi(z)s_i(z)|$.

Moreover, if $\mathcal{M}$ is a coherent sheaf of submodules of $\mathcal{O}_X^p$, there is a unique coherent sheaf $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}}_x = \overline{\mathcal{M}_x}$ for all $x \in X$.

Integral closures of modules can be reduced, to some extent, to those of ideals. Suppose X irreducible, and write $\mathcal{H} = \mathcal{M} + \mathcal{O}_X \cdot h$. Then [G21, 1.7] $h \in \overline{\mathcal{M}}$ if and only if $\wedge^k \mathcal{H} \subseteq \overline{\wedge^k \mathcal{M}}$, where k is the largest integer with $\wedge^k \mathcal{H} \neq 0$. Here $\wedge^k$ refers to exterior powers, or rather to their images in $\wedge^k \mathcal{E}$.

We can now define $\mathcal{M}$ to be a reduction of $\mathcal{N}$ if $\mathcal{M} \subset \mathcal{N}$ and $\overline{\mathcal{M}} = \overline{\mathcal{N}}$. According to [**21**, (2.6)], this is equivalent to the natural map $Proj(\mathcal{RM}) \to Proj(\mathcal{RN})$ being a finite map. Again a minimal reduction is one with the minimum number of generators; if $\mathcal{M}$ has finite codimension in $\mathcal{O}_{X,x}^p$, a general set of $d + p - 1$ elements generates a minimal reduction.

Strict dependence in the module case was introduced in [G27], see also [G29, §3]. We say that h is strictly dependent on $\mathcal{M}$, and write $h \in \mathcal{M}^\dagger$ if, for all germs $\phi : (\mathbb{C}, 0) \to (X, x)$, we have $h \circ \phi \in \phi^*(\mathcal{M}_x) \cdot \mathfrak{m}_1$. Then $\mathcal{M}^\dagger$ is a module, and if $\mathcal{M}$ is a reduction of $\mathcal{N}$, $\mathcal{M}^\dagger = \mathcal{N}^\dagger$.

Gaffney has shown the flexibility of the concept of integral closure of modules by applying it to obtain simplified and strengthened versions of a variety of interesting results. Most relevant to equisingularity theory is a recasting of Whitney regularity conditions using integral closures. Observe that regularity gives a condition on tangent hyperplanes of $\mathbf{X}$ at a series of points converging to T. In the hypersurface case, the tangent hyperplane to $\mathbf{X}$ is given by $t\partial F/\partial t + \sum_i z_i \partial F/\partial z_i = const$. The Whitney A condition requires $\partial F/\partial t$ to be 'smaller' than the $\partial F/\partial z_i$. Recall that JF_T denotes the ideal $\langle \partial F/\partial z_i \rangle \lhd \mathcal{O}_{N+s}$ generated by the $\partial F/\partial z_i$. For $\dim(T) = 1$, we have the following characterisations:

$\partial F/\partial t \in \mathfrak{m}_N \cdot JF_T$ is Mather's criterion for triviality under right equivalence.

$\partial F/\partial t \in (JF_T)^\dagger$ is Gaffney's [G27] criterion for A-regularity.

$\partial F/\partial t \in \overline{\mathfrak{m}_N \cdot JF_T}$ is equivalent [**41**] to the W condition of Verdier,

but care is needed: for Mather's condition, these are ideals in $\mathcal{O}_{N+1}$, for the others, in $\mathcal{O}_X$.

For the general case $F : \mathbb{C}^N \times T \to \mathbb{C}^p$, we have:

A-regularity holds if and only if for all tangent vectors $\partial/\partial t$ to T, $\partial F/\partial t \in J_M F^\dagger$, i.e. $J_M F_T \subseteq J_M F^\dagger$ [G29];

W-regularity holds if and only if for all tangent vectors $\partial/\partial t$ to T, $\partial F/\partial t \in \overline{\mathfrak{m}_N \cdot J_M F}$, i.e. $J_M F_T \subseteq \overline{\mathfrak{m}_N \cdot J_M F}$ [G21],[G25];

There are corresponding assertions for A_f and for W_f with F replaced by (F, f) ([G33, 2.1], [G40, 2.8]). Simpler versions for the case when F is absent are given in [G27] for A_f and in [G28] for W_f.

We now mention briefly several other papers of Gaffney giving applications of integral closure of modules.

In [G27], Gaffney shows that a hyperplane H is a limiting tangent plane to X if and only if $J_M F_H$ is *not* a reduction of $J_M F$. In [G29] he shows that $\mathbf{X}$ is A-regular over T if and only if every limiting tangent hyperplane contains T (which gives the above condition for A-regularity); and that $J_M F_T$ is a reduction of $J_M F$ if and only if *no* hyperplane containing T is a limiting tangent hyperplane. Relativisations of these equivalences are given in [G29,§5].

In [G31], Gaffney obtains a necessary and sufficient condition for a polynomial map $F : \mathbb{C}^N \to \mathbb{C}^p$ to be non-characteristic over t_0 at infinity. He gives an inequality which generalises Malgrange's condition, and shows that this is equivalent to the non-characteristic condition, and to an inequality $\partial F'/\partial t_i \in \overline{J_M F'}$ at t_0, where F' is essentially F referred to affine co-ordinates with t_0 at the origin, and that this implies local topological triviality.

In [G34] he studies finite determinacy with respect to left equivalence. He recalls that if F is an injective immersion outside T, the condition for a rugose trivialisation is W regularity, i.e. $J_M F_T \subseteq \overline{\mathfrak{m}_N \cdot J_M F}$. For left equivalence, one must consider double points, so for any ideal $I \lhd \mathcal{O}_n$, define $I_D \lhd \mathcal{O}_{2n}$ to be the ideal generated by the $h(z) - h(w)$ for $h \in I$. Then he obtains the necessary and sufficient condition $(\mathfrak{m}_n^{k+1})_D \subset (f^*\mathfrak{m}_p)_D^\dagger$ for $f : \mathbb{C}^n \to \mathbb{C}^p$ to be $k-$determined with respect to rugose trivialisation.

In [G43] (with Trotman and Wilson, and generalising [G24]) he studies the t^r condition, due originally to Thom [**46**] in the real case, and refined by Kuo and Trotman [**26**] and Trotman and Wilson [**48**]. One says that $\mathbf{X}$ is (t^r) regular over T at O if every C^r-submanifold Q of dimension N, transverse to T at O, is transverse to $\mathbf{X}$ near O. We say $\mathbf{X}$ is (t^r) regular for P over T at O if this holds for all Q with the same $r-$jet as P (here we may regard Q as the graph of a map $\mathbb{R}^N \to T$). With this form of the condition it is the condition on jets that is crucial, not the degree of differentiability, so the condition is also non-vacuous in the complex case. There are also variants depending on the degree of differentiability of Q; P only enters via its $r-$jet. Then provided $r > 0$ (the results for $r = 0$ are slightly different),

$X \setminus \Sigma_F$ is t^r for P if and only if $\mathfrak{m}_N^r \cdot JM_t F \subseteq \overline{\mathfrak{m}_N \cdot J_M F_P + I(P) JM_t(F)}$,
and

X is t^r for P if and only if $X \setminus \Sigma_F$ is so and $\mathfrak{m}_N^r \cdot \mathcal{O}_\Sigma \subseteq \overline{I(P)\mathcal{O}_\Sigma}$.
A genericity theorem, which states that the multiplicity $m(J_M F_P; \mathbf{X} \cap P)$ takes its generic value among all transversals with a given $(r-1)-$jet if and only if t^r holds for P, ties this to other results.

4. *Invariants*

The main theorems about equisingularity are stated in terms of numerical invariants, mostly multiplicities. The development of the theory of these invariants is an important part of the story. In fact, few of the definitions are solely due to Gaffney: his role has been that of someone with ideas and suggestions, provoking others (most notably, David Rees, Steve Kleiman and Anders Thorup), to make his ideas precise and to prove his conjectures.

Algebraic definitions of multiplicity start with an ideal I of finite colength in a sufficiently nice ring R of dimension d (e.g. a noetherian local ring $\mathcal{O}_X$ with $\dim X = d$); then for k large, the length (in our case, dimension over $\mathbb{C}$) of R/I^k is a polynomial in k with leading term $m\frac{k^d}{d!}$; and m is defined to be the multiplicity $m(I)$. If $I \subseteq J$, then [**34**] $m(I) \leq m(J)$, and $m(I) = m(J) \Leftrightarrow \overline{I} = \overline{J}$. Gaffney points out that this shows that multiplicities 'control' integral closures; it follows that they control Whitney conditions. Moreover, I admits a reduction with d generators, and if I itself has d generators, $m(I) = \dim(R/I)$.

This is easily generalised to the mixed multiplicities $m_{i,j}(I,J)$ $(i + j = d)$ of two ideals. For k and ℓ large enough, it is shown in [**37**, Chap I,§2] that the length of R/I^kJ^ℓ is a polynomial of degree d in k and ℓ, whose leading term we denote by $\sum_{i=0}^{d} m_{i,d-i}(I,J)\frac{k^i}{i!}\frac{\ell^{d-i}}{(d-i)!}$. It can be shown that $m_{i,j}(I,J)$ is equal to the multiplicity of the ideal generated by i general elements of I and j elements of J. The multiplicity of the product ideal is given [**37**] by $m(I \cdot J) = \sum_{i=0}^{d} \binom{d}{i} m_{i,d-i}(I,J)$.

The concept generalises to the Buchsbaum-Rim multiplicity [**5**] of a submodule M of finite colength of a free module $E := R^p$. Here we take the (graded) symmetric algebra $S(E)$ over R generated by E, the Rees subalgebra $R(M)$ generated by M, and consider its colength $\dim(S_n(E)/R_n(M))$ in grade n. For n large enough, this is polynomial of degree $d+p-1$ in n; if the leading term is $m(M)\frac{n^{d+p-1}}{(d+p-1)!}$, then $m(M)$ is an integer defined to be the multiplicity.

The following were conjectured by Gaffney and proved by Rees and Kirby [**19**]: if $M \subseteq N$, then $m(M) \leq m(N)$, and $m(M) = m(N) \Leftrightarrow \overline{M} = \overline{N}$: in this latter case, M is said to be a reduction of N. Moreover, by [**35**] or [G25], M admits a reduction with $d + p - 1$ generators (it suffices to take $d + p - 1$ generic elements of M), and if M admits $d + p - 1$ generators, we have ([**5**], [G25]) the simple formula $m(M) = \dim(E/M) = \dim(R/\wedge^p(M))$. In our usual setup, the function $m(J_M F_t)$ is upper semicontinuous [G29, Prop 1.1].

One can define mixed multiplicities for two modules M, N each of finite colength as in case of ideals: take i generic elements from M and j generic elements of N where $i + j = d + p - 1$, and compute the Buchsbaum-Rim multiplicity of the module they generate. There is a product formula [**19**] which, in the geometric case, is

$$m(I \cdot M) = \binom{m}{d} m(I) + \sum_{j=0}^{d-1} \binom{m}{j} m(M|S_j),$$

where S_j is the quotient of $\mathcal{O}_X$ by j generic elements of I.

We can also consider the relative multiplicity of a submodule $N \subset M \subset E$ of finite colength in M: not surprisingly, with increased technicalities.

Geometrical invariants are obtained from polar varieties and their generalisations. Polar curves of plane curves were used in the mid 19^{th} century, for example in the proof of Plücker's formulae counting singularities of plane curves. The use of polar varieties in equisingularity theory begins with [**38**], where Teissier states that their use was advocated by Thom. For X as above, and $0 \leq k < d$, take a linear subspace $L \subset \mathbb{C}^N$ of codimension $d-k+1$, defining $p_L : \mathbb{C}^N \to \mathbb{C}^{d-k+1}$. Then the polar variety is defined to be the closure $P_k = P_k(X, L)$ of the set of $x \in X^o$ critical for $p_L|X$ (the notation is chosen so that P_k has codimension k). It can be shown that for a dense Z-open[†] set of subspaces L of codimension $d-k+1$, the multiplicity at O of $P_k(X, L)$ is independent of L: this is defined to be the polar multiplicity $m_k(X)$. The extreme cases are $k = 0$, with $P_0 = X$, so $m_0(X)$ is the multiplicity of X at 0, and $k = d$, with P_d discrete, so $m_d(X) = 0$.

When we have a family $\mathbf{X}$, we still choose a generic $L \subset \mathbb{C}^N$, now defining $p_L : \mathbb{C}^N \times T \to \mathbb{C}^{d-k+1} \times T$, and define the relative polar variety to be the closure $P_{T,k}$ of the set of $x \in \mathbf{X}^o$ critical for p_L, and $m_{T,k}(\mathbf{X})_t$ to be the multiplicity of $P_{T,k} \cap (\mathbb{C}^N \times \{t\})$ at $0 \times \{t\}$. We may not assume that this is equal to $m_k(X_t)$ since the notion of genericity for the subspace L is different in the two cases. In the extreme case $k = d$, $P_{T,d}$ is finite over d, so its closure meets T at points with $m_{T,d}(\mathbf{X})_t > 0$. We see that the absence of such points is important for equisingularity.

More generally, if $\mathcal{I} \lhd \mathcal{O}_X$ is an ideal with q generators and $\Lambda \subset P^{q-1}$ is a linear subspace of codimension k, the polar variety $P_k(X, \Lambda)$ is the image in X of $B_{\mathcal{I}}X \cap (X \times \Lambda)$. Again, for a dense Z-open set of subspaces Λ of codimension k, the multiplicity at O of $P_k(X, \Lambda)$ is independent of Λ, and is defined to be the polar multiplicity $m_k(X, \mathcal{I})$. In the case when X is the zero set of $F : \mathbb{C}^N \to \mathbb{C}$ and $\mathcal{I}$ is the Jacobian ideal JF, we see (taking Λ as the annihilator of L) that $P_k(X, JF)$ is the same as $P_k(X)$. As above, if $\mathcal{I} \lhd \mathcal{O}_{\mathbf{X}}$ there is a relative version $m_{T,k}(\mathcal{I})_t$.

If $\mathcal{I} \subset \mathcal{J}$ and $\overline{\mathcal{I}} = \overline{\mathcal{J}}$, the natural map $B_{\mathcal{J}}(X) \to B_{\mathcal{I}}(X)$ is finite, so we can take the same Λ for both $\mathcal{I}$ and $\mathcal{J}$, and use the projection formula to see that $m_k(X, \mathcal{I})$ depends only on the integral closure of $\mathcal{I}$.

If the polar variety is a curve, then so is its strict transform under the simple blowup of the origin in $\mathbb{C}^N$, and $m_1(X)$ is the intersection number of the preimage with the exceptional divisor. In general we can intersect with generic hyperplanes to cut down to a curve, then do the same. Equivalently, take the preimage in the blowup, and intersect with $d-k-1$ hyperplanes and with the exceptional divisor. This leads to an interpretation of polar multiplicities as intersection numbers, which is due to Kleiman and Thorup [**21**].

The projection $B_{\mathfrak{m}\cdot\mathcal{I}}(X) \to X$ factors through both $B_{\mathfrak{m}}(X)$ and $B_{\mathcal{I}}(X)$. Over $B_{\mathcal{I}}(X)$ we have the line bundle induced from the universal bundle on P^{q-1}: write ℓ_I for its first Chern class and D_I for the corresponding

[†]Here and below we write Z-open to mean open in the Zariski topology.

exceptional divisor; and use the same notations for their pullbacks to $B_{\mathfrak{m}\cdot\mathcal{I}}(X)$. Over $B_{\mathfrak{m}}(X)$ we have the line bundle coming from the universal bundle on $P(\mathbb{C}^N)$, with first Chern class $\ell_{\mathfrak{m}}$ and divisor $D_{\mathfrak{m}}$. Then $P(X,L)$ corresponds to the intersection ℓ_I^k and $m_k(X,\mathcal{I})$ is given by $\ell_I^k \cdot \ell_{\mathfrak{m}}^{d-k-1} \cdot D_{\mathfrak{m}}$ or, in the notation of intersection theory,

$m_k(X,\mathcal{I}) = \int \ell_I^k \cdot \ell_{\mathfrak{m}}^{d-k-1} \cdot D_{\mathfrak{m}}$.

For the application to equisingularity for hypersurfaces, we take $\mathcal{I}$ to be the Jacobian ideal. If X has non-isolated singularity, this does not have finite codimension. To allow for this case, in [G28] Gaffney & Gassler define further invariants. For $L \subset P^q$ of codimension k, the polar variety of $\mathcal{I}$ is the projection to X of $P(X,\mathcal{I}) := B_{\mathcal{I}}(X) \cap (X \times L)$ and we now define the Segre cycle as the image of $Q(X,\mathcal{I}) := D_I \cap B_{\mathcal{I}}(X) \cap (X \times L)$. Its multiplicity is constant for L in a dense Z-open set, defining the Segre number $se_k(X,\mathcal{I})$. This too can be defined as an intersection number

$se_k(X,\mathcal{I}) = \int \ell_I^{k-1} \ell_{\mathfrak{m}}^{d-k-1} D_I \cdot D_{\mathfrak{m}}$ $(1 \le k \le d-1)$,
$se_d(X,\mathcal{I}) = \int \ell_I^{d-1} D_I$.

Thus

$se_k(X,\mathfrak{m}_N \cdot \mathcal{I}) = \int h^{k-1} \ell_{\mathfrak{m}}^{d-k-1} D \cdot D_{\mathfrak{m}}$ $(1 \le k \le d-1)$,
$se_d(X,\mathfrak{m}_N \cdot \mathcal{I}) = \int h^{d-1} D$,

so substituting $h = \ell_I + \ell_{\mathfrak{m}}$ leads to the product rules

$se_k(X,\mathfrak{m}_N \cdot \mathcal{I}) = \sum_{i=1}^{k} \binom{k-1}{i-1} se_i(X,\mathcal{I})$ $(1 \le k \le d-1)$,
$se_d(X,\mathfrak{m}_N \cdot \mathcal{I}) = \sum_{i=0}^{d-1} \binom{d}{i} m_i(X,\mathcal{I}) + \sum_{i=1}^{d} \binom{d-1}{i-1} se_i(X,\mathcal{I})$.

These generalise the Lê numbers of Massey [**29**], which are defined in the case $p = 1$ as $\lambda_k(F_t) := se_k(X_t, JF_T)$. Massey showed that the reduced Euler characteristic $\tilde{\chi}^k(t)$ of the Milnor fibre of $F_t|L^k$ (for L^k a generic linear subspace of dimension k) is given by $\tilde{\chi}^k(t) = m_k(F_t) + \sum_{i=0}^{k}(-1)^i \lambda_{k-i}(F_t)$. It is also shown in [G28, Cor 4.5] that if $\mathcal{I} \lhd \mathcal{O}_{\mathbf{X}}$ induces $\mathcal{I}_t$ on X_t, while the $se_i(X_t,\mathcal{I}_t)$ individually need not be upper semicontinuous in t, the sequence $(se_1,\ldots,se_d)$ is, provided sequences are ordered lexicographically.

Somewhat similar definitions for the case of modules are made in [G29], following [**21**]. Suppose $\mathcal{M} \subset \mathcal{E} = \mathcal{O}_X^p$ has finite colength (or, in the relative case, that $Supp(\mathcal{E}/\mathcal{M})$ is finite over T: in fact all the definitions are as easily made over T, this is what is needed for the application). Set $P := Proj(\mathcal{SE})$, and $P' := Proj(\mathcal{RM})$, where $\mathcal{RM}$ is the Rees algebra. Let B be the blowup of P by the sheaf of ideals on $\mathcal{SE}$ generated by $\mathcal{M}$, with exceptional divisor D. Let ℓ_E, ℓ_M denote the classes on B induced from the first Chern classes of the tautological sheaves on P and P'. Now define Segre classes by $se_i := \int \ell_M^{i-1} \ell_E^{r-i}[D]$, where $r = d+p-1$. In fact, $se_i = 0$ for $i < d$.

Now set $e^j := \sum_{i=1}^{r-j} se_i$ (so that $e^{p-1} = se_d$, $e^0 = \sum_i se_i$). Then the e^j are all upper semicontinuous (the se_i need not be). With this definition, the Kleiman-Thorup multiplicity is e^0.

More generally, suppose given two submodules $\mathcal{M} \subset \mathcal{N} \subset \mathcal{E}$, with $\overline{\mathcal{N}}/\overline{\mathcal{M}}$ of finite length, or equivalently, $\overline{\mathcal{M}} = \overline{\mathcal{N}}$ in a punctured neighbourhood of x. Write $\rho(\mathcal{M})$ for the ideal on $Proj(\mathcal{R}\mathcal{N})$ generated by $\mathcal{M}$, $B_{\mathcal{M}}$ for the normalised blowup of $Proj(\mathcal{R}\mathcal{N})$ along $\rho(\mathcal{M})$, and $D_{\mathcal{M},\mathcal{N}}$ for the exceptional divisor. Over $B_{\mathcal{M}}$ there are two canonical line bundles, one coming from $\mathcal{M}$ one from $\mathcal{N}$: denote their first Chern classes by ℓ_M, ℓ_N. Then, following Kleiman and Thorup [**21**], Gaffney defines the multiplicity of the pair $\mathcal{M} \subset \mathcal{N}$ as

$$m(\mathcal{M},\mathcal{N}) := \sum_{j=0}^{d+r-2} \int \ell_M^{d+r-2-j} \ell_N^j D_{\mathcal{M},\mathcal{N}}.$$

This generalises the Buchsbaum-Rim multiplicity $m(\mathcal{M}) = m(\mathcal{M},\mathcal{E})$, and satisfies additivity: if $\mathcal{L} \subset \mathcal{M} \subset \mathcal{N}$ then $m(\mathcal{L},\mathcal{N}) = m(\mathcal{L},\mathcal{M}) + m(\mathcal{M},\mathcal{N})$.

However, the analogues of the polar and Segre multiplicities do not have such good properties as in the ideal case, as is shown by a counterexample in [G37,§4].

5. *Criteria for equisingularity*

We now discuss the main developments in chronological order, and begin in 1973 with Teissier's work [**37**] on isolated hypersurface singularities. The plan of his proof is as follows. Since the Whitney conditions hold generically, the inclusion $\partial F/\partial t \in \overline{\mathfrak{m}_N \cdot JF_T}$ which characterises them holds at an open set of points in T. The key step is now the "Principle of Specialisation of Integral Dependence" (PSID) which shows that for any g and any ideal $\mathcal{I} \lhd \mathcal{O}_{\mathbf{X}}$ such that the $\mathcal{I}_t \lhd \mathcal{O}_{X,t}$ have finite codimension and the multiplicity $m(\mathcal{I}_t)$ is constant along T, the set of $t \in T$ at which the germ $h \in \overline{\mathcal{I}_t}$ is closed.

The rough idea is as follows (for a fuller, but still very short and geometric account see [G28, pp 700-701]). Let $\mathcal{I} \lhd \mathcal{O}_{\mathbf{X}}$; blow up along $\mathcal{I}$; let D be the exceptional divisor of the blowup. If $m(\mathcal{I}_t)$ is constant, the projection $D \to T$ is equidimensional. Now whether or not $g \in \overline{\mathcal{I}}$ depends on the valuations $v_i(g)$ corresponding to the components D_i of D. Since the projection is equidimensional, the list of v_i is independent of t; since also the $v_i(g)$ are semicontinuous the result follows.

The invariants μ^* appear as follows. First, the relation of multiplicities to Milnor numbers for icis (due to Lê [**23**] and Greuel [**10**]) gives $m(JF_T) = \mu^{(N)}(X_t) + \mu^{(N-1)}(X_t)$. Next the formula for the multiplicity of a product of ideals gives $m(\mathfrak{m}_N \cdot J) = \sum_{i=0}^{N} \binom{N}{i} m_{i,N-i}(\mathfrak{m}_N, J)$. But the mixed multiplicity $m_{i,N-i}(\mathfrak{m}_N, J)$ is equal to the multiplicity of the restriction of J to a generic codimension i subspace, and hence to $\mu^{(i+1)}(X_t) + \mu^{(i)}(X_t)$. Thus $m(\mathfrak{m}_N \cdot JF_T) = \sum_{i=0}^{N} \binom{N}{i}(\mu^{(i+1)}(X_t) + \mu^{(i)}(X_t))$. It follows, since the $\mu^{(i)}$ are semicontinuous, that $\mu^*(X_t)$ is independent of t if and only if $m(\mathfrak{m}_N \cdot JF_T)$ is.

By 1980, Teissier had obtained a general criterion for regularity, which appeared in [**41**]. He developed the theory of (relative) polar multiplicities, in essentially our standard situation. Then his main theorem states that Whitney regularity is equivalent to constancy of polar multiplicities. More precisely, for $\mathbf{X}$ reduced complex analytic of pure dimension d and T a smooth subspace, the following are equivalent:

(i) $(m_0\mathbf{X}, m_1\mathbf{X}, \dots, m_{d-1}\mathbf{X})_t$ is constant along T;

(ii) (if $\dim(T) = 1$) $\mathbb{C}^N \times \{0\}$ is transverse to all limits of tangent spaces of $\mathbf{X}^o$ and $(m(\mathbf{X})_t, m_{T,1}(\mathbf{X})_t, \dots, m_{T,d}(\mathbf{X})_t)$ is constant along T;

(iii) $(\mathbf{X}^o, T)$ satisfies at t the Whitney A and B conditions;

(iv) $(\mathbf{X}^o, T)$ satisfies at t the strict Whitney A condition with exponent 1 and strict Whitney B with exponent > 0.

It follows that for any stratification of a complex analytic space, constancy along strata of polar multiplicities is necessary and sufficient for Whitney regularity, or for Verdier regularity. Teissier deduced that a complex analytic variety has a unique minimal regular stratification.

In Terry's great 1993 paper [G22] these results are refined and extended to obtain sufficient conditions for equisingularity of a family of maps. Jim Damon, using largely algebraic arguments, had previously proved in [**8**], unifying several earlier results:

If $F_0 : \mathbb{C}^N \to \mathbb{C}^p$ is $\mathcal{A}-$finite, i.e. finitely determined with respect to right-left-equivalence, any polynomial unfolding of non-negative weight is topologically trivial.

Gaffney aimed to improve this by replacing the weight condition by geometrical conditions on the unfolding. Now if F_0 is $\mathcal{A}-$finite, any (multi-)germ of F_0 outside the origin is stable. We require the same to hold for any (multi-)germ outside T of the unfolding $F : \mathbb{C}^N \times T \to \mathbb{C}^p \times T$ of F_0: a criterion on F for this to hold follows from Damon's work. Hence differentiable triviality is à priori guaranteed at points outside T. Adding a simplicity hypothesis, we may suppose that only finitely many $\mathcal{K}-$classes (strata) occur. It thus remains to prove, for each stratum in source or target, Whitney regularity over T: we then have a c.v.s. and the desired result will follow from the second isotopy lemma.

For this we have the above criterion of Teissier for regularity in terms of polar multiplicities. However, these are defined in terms of the whole variety $\mathbf{X} \subset \mathbb{C}^N \times T$ (or $\mathbf{Y} \subset \mathbb{C}^p \times T$). Gaffney's aim was to find a condition depending only on the individual fibres X_t. To replace relative polar multiplicities of $\mathbf{X}$ by polar multiplicities of X_t, two main issues arise. First, one needs to show that each $\mathbb{C}^N \times \{t\}$ is transverse to limiting tangent planes of strata. Gaffney succeeds here by using the fact that the stratification is regular in the complement of T and $m_{T,d}(\mathbf{X})$ vanishes along T. He then uses delicate arguments to show that $m_i(P_i(F,t))$ is constant if and only if $m_i(X_t)$ is so for $0 \leq i < d$ and, again, $m_{T,d}(\mathbf{X}) = 0$ along T.

Secondly, since $d = \dim X_t$, $m_d(X_t)$ is not defined, so we need a replacement to control $m_{T,d}(\mathbf{X})$: this is the main difficulaty. Let $\mathcal{Q}$ be a $\mathcal{K}-$equivalence class and $\mathcal{S}(\mathcal{Q})$ the corresponding stratum: write $d := \dim(\mathcal{S}(\mathcal{Q})) - \dim T$. To define the $d-$stable multiplicity $m_d(F_t, \mathcal{Q})$, take a versal unfolding $G : \mathbb{C}^n \times T \to \mathbb{C}^p \times T$, denote projection on T by π, and pick a general hyperplane $L \subset \mathbb{C}^p$ defining a polar variety $P := P_d(\overline{\mathcal{Q}(G)}, \pi)$. Define $m_d(F, \mathcal{Q})$ as the multiplicity of the ideal $\mathfrak{m}_s \cdot \mathcal{O}_P \lhd \mathcal{O}_P$: this can be proved independent of all choices. Now to avoid 'coalescing', i.e. an arc in $\mathcal{S}(\mathcal{Q})$ converging to a point of T, it is enough to require $m_0(F, \mathcal{Q})$ constant.

This paper has led to a whole industry of obtaining more explicit, and much shorter lists of invariants in low dimensions whose constancy guarantees equisingularity. Many such results have been reported at São Carlos. In [G22], Gaffney gave a detailed study of the cases $\mathbb{C}^2 \to \mathbb{C}^2$ and $\mathbb{C}^2 \to \mathbb{C}^3$; his results were improved by Houston [**16**]. Jorge Peréz treated map-germs $\mathbb{C}^3 \to \mathbb{C}^3$ in his thesis [**18**]. In [G38], Gaffney and Vohra dealt with maps $\mathbb{C}^n \to \mathbb{C}^2$. Corank 1 maps $\mathbb{C}^n \to \mathbb{C}^{n+1}$ were treated by Houston [**17**]. In addition, in several of the equisingularity results (for spaces) below, constancy of the invariants controls some strata below the top dimension.

In successive papers generalising Teissier's first equisingularity theorem, certain themes reappear. In the original argument, the fact that Whitney conditions hold generically, which Whitney proved geometrically using his wing lemma, was used. Later Teissier replaced this by his 'idealistic Bertini theorem' [**40**]. In later papers, Gaffney and collaborators used a transversality theorem of Kleiman [**20**] for this purpose. There are also successive versions of the PSID. In each case, the general outline of the argument is the same: first we blow up, then examine the geometry. In each case we have $\mathbf{X}$ as usual, $h \in \overline{\mathcal{I}}$ (for ideals) or $h \in \overline{\mathcal{M}}$ (for modules) on X_t for a dense set of points $t \in T$, and can conclude the same at all points provided certain multiplicities are constant.

In 1996 Gaffney [G25] extends Teissier's results on isolated hypersurface singularities to the icis case. In fact in [G22] he had already shown that, in this case, Whitney regularity was equivalent to constancy of the polar invariants $m_k(X_t)$, so it follows from properties of the invariants discussed in §4 that, for icis germs, the following are equivalent:

(i) (X, T) is Whitney regular,
(ii) $m_k(X_t)$ is constant $(0 \leq k \leq d)$
(iii) for all tangent vectors $\partial/\partial t$ to T, $\partial F/\partial t \in \overline{\mathfrak{m}_N \cdot JF_T}$
(iv) $m(\mathfrak{m}_N \cdot JF_T)_t$ is constant.

He also proves $m(\mathfrak{m}_N \cdot J_M F) = \sum_{j=0}^{d} \binom{N-1}{j} m_{d-j}(X)$. The argument in this paper does not refer directly to a PSID, but such a result applicable to this case was later obtained in [G29]. Again the Lê-Greuel formula allows restatement in terms of $\mu^{(i)}$.

Three of Gaffney's major papers appeared in 1999. We referred above to the introduction and development by Gaffney & Gassler [G28] of the Segre numbers of an ideal. Next they obtain a PSID. Given a family in our usual notation and ideal $\mathcal{I} \lhd \mathcal{O}_X$, if h has germ at t in $\overline{\mathcal{I}_t}$ for a dense open set of t, and the Segre numbers $se_k(\mathcal{I}_t)$ are constant for $1 \leq k \leq d$, then $h \in \overline{\mathcal{I}}$. They apply this version of the PSID to the general hypersurface $\mathbf{Z}$ in $\mathbf{X} = \mathbb{C}^N \times T$, and show that the W_f condition holds, and hence the smooth part of $\mathbf{Z}$ is Whitney regular over T, provided the polar multiplicities $m_*(f_t)$ and Segre numbers $se_*(f_t)$ are constant, or equivalently, m_* and the reduced Euler characteristics $\overline{\chi}^*$ are constant. Also in this situation, the codimension 1 strata of $\Sigma(\mathbf{Z})$ are Whitney regular over T. Even more surprisingly, the converse holds: if $\mathbf{Z}$ admits a Whitney stratification with T a stratum, then W_f is satisfied.

Gaffney & Kleiman [G29] prove a version of PSID for modules $\mathcal{M}$ of finite codimension: more precisely, the support S of $\mathcal{E}/\mathcal{M}$ is finite over T. The required condition is that $m(\mathcal{M})$ is constant. It is also shown in the icis case in [G29] that A-regularity holds if the Segre numbers $se_*(J_M F)$ are constant: in fact they state the result in terms of the partial sums $e^j = \sum_1^{r-j} se_i$. This condition is not necessary for A-regularity, and they offer an interesting example to show that no condition depending only on the members of the family can be both necessary and sufficient: the families $z_1^2 - z_2^3 + z_2^2 t^b = 0$ are A-regular if $b \geq 2$ but not if $b = 1$, but have the same sets of members. They also obtain a number of results on A_f, which lead them to conjecture that A_f holds when the multiplicity or the Milnor numbers are defined and constant.

Gaffney and Massey [G30] prove the equivalence (in the complete intersection case) of:

(i) W_f holds,

(ii) both $\mathbf{X}$ and $\mathbf{Z}$ are Whitney regular over T, and

(iii) the X_t and Z_t all have only isolated singularities and the sequences $\mu^*(X_t)$ and $\mu^*(Z_t)$ are both constant in t.

If we just know that the X_t and Z_t have isolated singularities and $\mu(X_t)$ and $\mu(Z_t)$ are both constant in t, then A_f holds. In [G33] the relation between Milnor numbers and multiplicities is used to reduce the condition to constancy of $m(\mathfrak{m} \cdot J_M(F, f))$.

It is also shown that the conditions W_f and A_f usually imply local analytic triviality if the target dimension of f exceeds 1, so are not of interest in this case.

6. *Recent work*

The pattern of nearly all the equisingularity theorems is to relate failure of regularity to jumps in dimension of exceptional sets arising in blowups and thence to non-constancy of invariants such as polar multiplicities. In [G37] Gaffney gives examples where the invariants previously studied (the Segre numbers) are constant and yet the dimension jumps. He argues that this is because strata of different dimensions are making contributions, and so he is led to a new approach.

The basic idea is as follows. Suppose X as usual and $\mathcal{M} \subset \mathcal{E} = \mathcal{O}_X^p$ a submodule. Decompose the support of $\mathcal{E}/\mathcal{M}$ in X into its irreducible components V_α and denote by $\mathcal{F}_k$ the union of the V_α of codimension $\geq k$. We think of these as defining a stratification, though the V_α do not need to be disjoint, or even to intersect nicely. A further complication is that rather than using the components of the support S of $\mathcal{E}/\mathcal{M}$ in X, we must form the preimage of S under π, take its components, and define the V_α as the projections of these back down to X.

Define the hull $\mathcal{H}_i(\mathcal{M})$ of $\mathcal{M}$ as the set of elements integrally dependent on $\mathcal{M}$ in codimension i: $h \in \mathcal{H}_i(\mathcal{M}) \iff \forall z \notin \mathcal{F}_{i+1}$, $h \in \overline{M}_z$. This can be refined by considering germs at $x \in X$ of both $\mathcal{H}_i(\mathcal{M})$ and $\mathcal{F}_{i+1}$.

Then $\mathcal{F}_{i+1}$ is the projection to X of the cosupport of $\rho(\overline{\mathcal{M}})\mathcal{R}(\mathcal{H}_i(\mathcal{M}))$, thought of a sheaf of modules on $Proj(\mathcal{R}(\mathcal{H}_i(\mathcal{M})))$.

For each α, we choose a smooth point z_α of V_α and a slice S_α at that point. Then if V_α has codimension $i+1$, the multiplicity $m(\mathcal{M}|S_\alpha, \mathcal{H}_i(\mathcal{M})|S_\alpha, z_\alpha)$ is defined, and depends only on α. Denote it by $m_\alpha(\mathcal{M})$.

The above extends naturally to families $p : \mathcal{X} \to T$ and modules $\mathcal{M} \subset \mathcal{E}$ over $\mathcal{X}$, but we cannot necessarily identify the fibre of the cosupport of $\mathcal{M}$ over $t \in T$ with the cosupport of the restriction of $\mathcal{M}$ to X_t. This point makes for technical problems, and forces the proofs to go by induction on the codimension of the V_α.

The formal treatment appears in [G36]. Start with $\mathcal{M} \subset \mathcal{N} \subset \mathcal{E} = \mathcal{O}_X^p$, each of generic rank r: remark that though the cosupports of $\mathcal{M}$ and $\overline{\mathcal{M}}$ in $\mathcal{E}$ are the same, this is not the case for their cosupports in $\mathcal{N}$. Denote by π_X the projection to X of $Proj(\mathcal{R}(\mathcal{M}))$ or similars. Gaffney constructs a sequence

$$\mathcal{M} \subset \overline{\mathcal{M}} = \mathcal{H}_d(\mathcal{M}) \subset \mathcal{H}_{d-1}(\mathcal{M}) \dots \mathcal{H}_0(\mathcal{M}) \subset \mathcal{E},$$

with each $\mathcal{H}_i(\mathcal{M})$ integrally closed, the components of the cosupport of $\rho(\overline{\mathcal{M}})\mathcal{R}(\mathcal{H}_i(\mathcal{M}))$ project to sets of codimension at least $i+1$, and $\mathcal{H}_i(\mathcal{M})$ is as small as possible subject to this.

Proceed by induction on i (I omit numerous details) with induction basis:

$$e_0(\mathcal{M}) := p - r, \quad \mathcal{H}_0(\mathcal{M}) := \{h \in \mathcal{E} \mid e_0(\mathcal{M} + h \cdot \mathcal{O}_X) = e_0(\mathcal{M})\}.$$

Now assume $e_{k-1}(\mathcal{M})$ and a coherent sheaf $\mathcal{H}_{k-1}(\mathcal{M})$ defined; consider the cosupport CS_k of $\rho(\mathcal{M})$ on $Proj(\mathcal{R}(\mathcal{H}_{k-1}(\mathcal{M})))$ induced by the inclusion $\mathcal{R}(\mathcal{M}) \subset \mathcal{R}(\mathcal{H}_{k-1}(\mathcal{M}))$; let A_k index the components C_α of codimension k of

$\pi_X(CS_k)$. As part of the inductive construction, there are none of lower codimension; we can also expect these C_α to coincide with the V_α of codimension k. Now set

$$e_k(\mathcal{M}) := \sum_{\alpha \in A_k} m(C_\alpha) m(\mathcal{M}|S_\alpha, \mathcal{H}_{k-1}(\mathcal{M})|S_\alpha),$$
$$\mathcal{H}_k(\mathcal{M}) := \{h \in \mathcal{E} \mid e_k(\mathcal{M} + h \cdot \mathcal{O}_X) = e_k(\mathcal{M})\},$$

where S_α is a slice transverse to C_α at a smooth point.

As well as the hulls, this yields new invariants $e_i(\mathcal{M})$. An important first result: if $\mathcal{M} \subset \mathcal{N}$ and $e_i(\mathcal{M}) = e_i(\mathcal{N})$ for $0 \leq i \leq d$ then $\mathcal{N} \subset \overline{\mathcal{M}}$.

The central result of this work to date is called by Gaffney the 'multiplicity-polar' theorem. Suppose $\mathcal{M} \subset \mathcal{N} \subset \mathcal{E}$ such that the support C of $\overline{\mathcal{N}}/\overline{\mathcal{M}}$ is finite over T. Then for each $t \in T$, $C \cap \pi^{-1}(t)$ is a finite set of points x_i, at each of which $m(\mathcal{M}_t, \mathcal{N}_t, x_i)$ is defined: define

$$m(\mathcal{M}, \mathcal{N})_t := \sum_{x \in C \cap \pi^{-1}(t)} m(\mathcal{M}, \mathcal{N}, x).$$

Recall that the polar variety $P_k(\mathcal{M})$ is defined as $\pi_X(Proj(\mathcal{R}\mathcal{M}) \cap (X \times L))$, where L is a generic plane of codimension $r + k - 1$. Then the multiplicity polar theorem states

$$m(\mathcal{M}, \mathcal{N})_t - m(\mathcal{M}, \mathcal{N})_{gen} = mult_t P_d(\mathcal{M}) - mult_t P_d(\mathcal{N}),$$

where gen is a generic point of T. A proof in the special case of ideals is given in [G37]; the general proof appears in [G46]. The proof is motivated by a review of the definition of Segre numbers of a module in terms of a sequence of polars of different codimensions.

In his São Carlos paper [G37], Gaffney considers the case of isolated singularities, and deduces the following version of the PSID: for $\mathcal{M} \subset \mathcal{N}$ as above, if $h \in \mathcal{N}$ and $h_t \in \overline{\mathcal{M}_t}$ for a dense open set of t, then provided $m(\mathcal{M}, \mathcal{N})_t$ is constant, we have $h \in \overline{\mathcal{M}}$. The key point of the proof is to study the dimensions of the fibres over T of the preimage in $Proj(\mathcal{R}\mathcal{M})$ of the locus of points where $\mathcal{M}$ is not free.

He deduces a criterion for Whitney regularity for the general case when the X_t have isolated singularities and $\Sigma\mathbf{X} = T$: W-regularity holds if and only if $m(\mathcal{M}, \mathcal{N})_t + mult_t P_d(\mathcal{N})$, with $\mathcal{M} = \mathfrak{m}_N \cdot J_M(F_t)$ and $\mathcal{N} = \mathcal{H}_0(J_{M_z}(F))$, is independent of t. In the relative case when also the Z_t have isolated singularities, the condition for A_f to hold is obtained from this by taking $\mathcal{M} = J_M(F, f)$ and $\mathcal{N} = \mathcal{H}_0(J_M(F, f))$.

Terry has also used the multiplicity polar theorem to obtain a version of the PSID in which the multiplicities required to be constant are the $m_\alpha(M_t) + mult_{z_\alpha} P_{i+1}(H_i(M))$.

As well as its theoretical value, Gaffney shows how to use the theorem to obtain numerous effective calculations of numerical invariants; in [G39] for a family of hypersurfaces whose singular locus has dimension 1, or other constant dimension; also for map-germs $\mathbb{C}^2 \to \mathbb{C}^3$; and in [G44] for isolated singularities; in particular a calculation of MacPherson's Euler obstruction.

Bibliography of Terry Gaffney's papers

[G 1] *Properties of finitely determined germs*, thesis, Brandeis, 1976.

[G 2] On the order of determination of a finitely determined germ, Invent. Math. **37** (1976) 83–92.

[G 3] A note on the order of determination of a finitely determined germ, Invent. Math. **52** (1979) 127–130.

[G 4] (with R. Lazarsfeld) On the ramification of branched coverings of P^n, Invent. Math. **59** (1980) 53–58.

[G 5] (with A. A. du Plessis) More on the determinacy of smooth map-germs, Invent. Math. **66** (1982) 137–163.

[G 6] (with J. W. Bruce) Simple singularities of mappings $(C, 0) \to (C^2, 0)$, J. London Math. Soc. **26** (1982) 465–474.

[G 7] (with T. Banchoff and C. McCrory) *Cusps of Gauss mappings*, Research Notes in Math. **55**, Pitman, 1982.

[G 8] (with J. N. Damon) Topological triviality of deformations of functions and Newton filtrations, Invent. Math. **72** (1983) 335–358.

[G 9] The structure of $T\mathcal{A}(f)$, classification and an application to differential geometry, pp 409–427 in *Singularities* (ed. P. Orlik), Proc. Symp. Pure Math. **40(1)**, Amer. Math. Soc., 1983.

[G 10] (with L. C. Wilson) Equivalence theorems in global singularity theory, pp 439–447 in *Singularities* (ed. P. Orlik), Proc. Symp. Pure Math. **40(1)**, Amer. Math. Soc., 1983.

[G 11] Multiple points and associated ramification loci, pp 429–437 in *Singularities* (ed. P. Orlik), Proc. Symp. Pure Math. **40(1)**, Amer. Math. Soc., 1983.

[G 12] The Thom polynomial of $\overline{\sum^{1111}}$, pp 399–408 in *Singularities* (ed. P. Orlik), Proc. Symp. Pure Math. **40(1)**, Amer. Math. Soc., 1983.

[G 13] (with L. C. Wilson) Equivalence of generic mappings and C^∞ normalization, Compositio Math. **49** (1983) 291–308.

[G 14] (with J. W. Bruce and A. A. du Plessis) On left equivalence of map germs, Bull. London Math. Soc. **16** (1984) 303–306.

[G 15] (with H. Hauser) Characterizing singularities of varieties and of mappings, Invent. Math. **81** (1985) 427–447.

[G 16] (with T. Banchoff and C. McCrory) Counting tritangent planes of space curves, Topology **24** (1985) 15–23.

[G 17] New methods in the classification theory of bifurcation problems, pp 97–116 in *Multiparameter bifurcation theory (Arcata, 1985)*, Contemp. Math. **56**, Amer. Math. Soc., 1986.

[G 18] Multiple points, chaining and Hilbert schemes, Amer. J. Math. **110** (1988), 595–628.

[G 19] (with D. Mond) Cusps and double folds of germs of analytic maps $C^2 \to C^2$, J. London Math. Soc. **43** (1991) 185–192.

[G 20] (with D. Mond) Weighted homogeneous maps from the plane to the plane, Math. Proc. Camb. Phil. Soc. **109** (1991) 451–470.

[G 21] Integral closure of modules and Whitney equisingularity, Invent. Math. **107** (1992) 301–322.

[G 22] Polar multiplicities and equisingularity of map germs, Topology **32** (1993) 185–223.

[G 23] Punctual Hilbert schemes and resolutions of multiple point singularities, Math. Ann. **295** (1993) 269–289.

[G 24] Equisingularity of plane sections, t_1 condition and the integral closure of modules, pp 95–111 in *Real and complex singularities (São Carlos III, 1994)* (ed. W. L. Marar), Pitman Res. Notes Math. **333**, 1995.

[G 25] Multiplicities and equisingularity of ICIS germs, Invent. Math. **123** (1996) 209–220.

[G 26] (with A. A. du Plessis and L. C. Wilson) Map-germs determined by their discriminants, pp 1–40 in *Stratifications, singularities and differential equations, I (Marseille, 1990; Honolulu, 1990)* (eds. D. Trotman and L. C. Wilson), Travaux en Cours **54**, Hermann, 1997.

[G 27] Aureoles and integral closure of modules, pp 55–62 in *Stratifications, singularities and differential equations, II (Marseille, 1990; Honolulu, 1990)* (eds. D. Trotman and L. C. Wilson), Travaux en Cours **54**, Hermann, 1997.

[G 28] (with R. Gassler) Segre numbers and hypersurface singularities, J. Alg. Geom. **8** (1999) 695–736.

[G 29] (with S. Kleiman) Specialization of integral dependence for modules, Invent. Math. **137** (1999) 541–574.

[G 30] (with D. Massey) Trends in equisingularity theory, pp 207–248 in *Singularity theory (Liverpool, 1996)* (eds. J. W. Bruce and D. Mond), London Math. Soc. Lecture Notes **263**, Cambridge Univ. Press, 1999.

[G 31] Fibers of polynomial mappings at infinity and a generalized Malgrange condition, Compositio Math. **119** (1999) 157–167.

[G 32] Plane sections, W_f and A_f, pp 16–32 in *Real and complex singularities (São Carlos V, 1998)* (eds. J. W. Bruce and F. Tari), Res. Notes Math. **412**, Chapman & Hall/CRC 2000.

[G 33] (with S. Kleiman) W_f and integral dependence, pp 33–45 in *Real and complex singularities (São Carlos V, 1998)* (eds. J. W. Bruce and F. Tari), Res. Notes Math. **412**, Chapman & Hall/CRC 2000.

[G 34] $\mathcal{L}^0$-equivalence of maps, Math. Proc. Camb. Phil. Soc. **128** (2000) 479–496.

[G 35] The theory of integral closure of ideals and modules: applications and new developments. With an appendix by Steven Kleiman and Anders Thorup, pp 379–404 in *New developments in singularity theory (Cambridge, 2000)* (eds. D. Siersma, C. T. C. Wall and V. Zakalyukin), NATO Sci. Ser. II Math. Phys. Chem. **21**, Kluwer, 2001.

[G 36] Generalized Buchsbaum-Rim multiplicities and a theorem of Rees, Comm. Algebra **31** (2003), no. 8 (Special issue in honor of Steven L. Kleiman), 3811–3827.

[G 37] Polar methods, invariants of pairs of modules and equisingularity, pp 113–135 in *Real and complex singularities (São Carlos VII)* (eds. T. Gaffney and M. A. S. Ruas), Contemp. Math. **354**, Amer. Math. Soc., 2004.

[G 38] (with R. Vohra) A numerical characterization of equisingularity for map germs from n-space, $(n \geq 3)$, to the plane, J. Dyn. Syst. Geom. Theor. **2** (2004), 43–55.

[G 39] The multiplicity of pairs of modules and hypersurface singularities, pp 143–168 in *Real and complex singularities (São Carlos VIII)* (eds. J.-P. Brasselet and M. A. S. Ruas), Trends Math., Birkhäuser, Basel, 2007.

[G 40] Nilpotents, Integral Closure and Equisingularity conditions, pp 23–33 in *Real and Complex Singularities, (São Carlos IX)* (eds. M. J. Saia and J. Seade), Contemp. Math. **459**, Amer. Math. Soc., 2008.

[G 41] Invariants of $D(q,p)$ singularities, pp 13–22 in *Real and Complex Singularities, (São Carlos IX)* (eds. M. J. Saia and J. Seade), Contemp. Math. **459**, Amer. Math. Soc., 2008.

[G 42] (with J. Fernandez de Bobadilla) The Lê numbers of the square of a function and their applications, J. London Math. Soc. **77** (2008) 545–567.

[G 43] (with D. Trotman and L. C. Wilson.) Equisingularity of sections, (t^r) condition, and the integral closure of modules, J. Alg. Geom. **18** (2009) 651–689.

[G 44] The Multiplicity Polar Theorem and Isolated Singularities, J. Alg. Geom. **18** (2009) 547–574.

[G 45] (with M. Vitulli) Weak subintegral closure of ideals. Math arXiv:0708.3105 25 pp. To appear in Advances in Math.

[G 46] The Multiplicity-Polar Theorem. Math.CV/0703650 21 pp.

[G 47] Complete Intersections with non-isolated singularities and the $\mathcal{A}_f$ condition, pp 85–93 in *Singularities I: Algebraic and Analytic Aspects, in honor of the 60th Birthday of Lê Dung Tráng.* (eds. J.-P. Brasselet, J. L. Cisneros-Molina, D. Massey, J. Seade, and B. Teissier), Contemp. Math. **474**, Amer. Math. Soc., 2008.

References

1. V. I. Arnold, Singularities of smooth mappings, Russian Math. Surveys **23i** 1-43 (1968), Reprinted as pp 3–45 in *Singularity Theory*, London Math. Soc Lecture Notes **53** Cambridge Univ. Press, 1981.

2. K. Brauner, Zur Geometrie der Funktionen zweier komplexen Veränderlichen II–IV, Abh. Math. Sem. Hamburg **6** (1928) 1–54.

3. J. Briançon and J.-P. Speder, La trivialité topologique n'implique pas les conditions de Whitney, Comptes Rendus Acad. Sci. Paris **280**A (1975) 365.

4. J. Briançon and J.-P. Speder, Les conditions de Whitney impliquent μ constant, Ann. Inst. Fourier **26** (1976) 153–163.

5. D. Buchsbaum and D. S. Rim, A generalized Koszul complex II: depth and multiplicity, Trans. Amer. Math. Soc. **111**(1963) 197–224.

6. W. Burau, Kennzeichnung der Schlauchknoten, Abh. Math. Sem. Hamburg **9** (1932) 125–133.

7. W. Burau, Kennzeichnung der Schlauchverkettungen, Abh. Math. Sem. Hamburg **10** (1934) 285–297.

8. J. N. Damon, Finite determinacy and topological triviality I, Invent. Math. **62** (1980) 299–324.

9. C. G. Gibson, K. Wirthmüller, A. A. du Plessis and E. J. N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in math. **552**, Springer-Verlag, 1976.

10. G.-M. Greuel, Der Gauss-Manin Zusammenhang isolierter Singularitäten vollständiger Durchschnitten, Math. Ann. **214** (1975) 235–266.

11. J. P. G. Henry and M. Merle, Limites de normales, conditions de Whitney et éclatement d'Hironaka, pp 575–584 in *Singularities* (ed. P. Orlik), Proc. Symp. in Pure Math. **40(1)**, Amer. Math. Soc., 1983.

12. J. P. G. Henry, M. Merle and C. Sabbah, Sur la condition de Thom stricte pour un morphisme analytique complexe, Ann. Sci. Ec. Norm. Sup. **17** (1984) 227–268.

13. H. Hironaka, Resolution of singularities of an algebraic variety ove a field of characteristic zero, Ann. of Math. **79** (1964) 109–326.

14. H. Hironaka, Normal cones in analytic Whitney stratifications, Publ. Math. IHES **36** (1969) 27–138.

15. H. Hironaka, Stratification and flatness, pp 199–265 in *Real and complex singularities* (Proc. Ninth Nordic Summer School, Oslo, 1976) (ed. Per Holm) Sijthoff and Noordhoff, 1977.

16. K. Houston, Disentanglements and Whitney equisingularity, Houston J. Math. **33** (2007) 663–681.

17. K. Houston, On equisingularity of images of corank 1 maps, pp 201–208 in *Real and complex singularities (São Carlos VIII)* (eds. J.-P. Brasselet and M. A. S. Ruas), Trends Math., Birkhäuser, Basel, 2007.

18. V. H. Jorge Peréz, Polar multiplicities and equisingularity of map-germs from $\mathbb{C}^3$ to $\mathbb{C}^3$, Houston J. Math. **29** (2003) 901–923.

19. D. Kirby and D. Rees, Multiplicities in graded rings I: the general theory, in *Commutative algebra: syzygies, multiplicities and birational algebra* (eds. William J. Heinzer, Craig L. Hunecke and Judith D. Sally), Contemp. Math. **159**, Amer. Math. Soc., 1994.

20. S. L. Kleiman, On the transversality of a general translate, Compositio Math. **28** (1974) 287–297.

21. S. L. Kleiman and A. Thorup, A geometric theory of the Buchsbaum-Rim multiplicity, J. Algebra **167** (1994) 168–231.

22. D. T. Lê, Un critère d'équisingularité, pp 183–192 in *Singularités à Cargèse*, Astérisque **7-8**, Soc. Math. France, 1973.

23. D. T. Lê, Calcul du nombre de Milnor d'une singularité isolée d'intersection complète, Funct. Anal. Appl. **8** (1974) 45–52.

24. M. Lejeune-Jalabert and B. Teissier, *Clôture intégrale des idéaux et équisingularité*, Notes issued by Institut Fourier, Grenoble, 1974.

25. T.-C. Kuo, On classification of real singularities, Invent. Math. **82** (1985) 257–262.

26. T.-C. Kuo and D. Trotman, On (w) and (t^s)-regular stratifications, Invent. Math. **92** (1988) 633–643.

27. S. Łojasiewicz, Ensembles semi-analytiques, Notes issued by IHES, Bures-sur-Yvette, 1965.

28. B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, 1966.

29. D. Massey, *Lê cycles and hypersurface singularities*, Lecture Notes in Math. **1615**, Springer-Verlag, 1995.

30. J. N. Mather, Stability of $C^\infty-$mappings I: The division theorem, Ann. of Math. **87** (1968) 89-104; II Infinitesimal stability implies stability, Ann. of Math. **89** (1969) 254-291; III Finitely determined map-germs, Publ. Math. IHES **35** (1969) 127-156; IV Classification of stable germs by $\mathbb{R}-$algebras, Publ. Math. IHES **37** (1970) 223-248; V Transversality, Advances in Math. **4** (1970) 301-335; VI The nice dimensions, Lecture Notes in math. **192**, Springer-Verlag 1971, 207-253.

31. J. N. Mather, *Notes on topological stability*, Notes issued by Harvard Univ., 1970.

32. J. N. Mather, Stratifications and mappings, pp 195-232 in *Proc. conference on dynamical systems* (ed. M. M. Peixoto) Academic Press, 1973.

33. J. W. Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, 1968.

34. D. Rees, A-transforms of ideals, and a theorem on multiplicities of ideals, Proc. Camb. Phil. Soc. **57** (1961) 8–17.

35. D. Rees, Reduction of modules, Math. Proc. Camb. Phil. Soc. **101** (1987) 431–449.

36. J.-P. Speder, L'équisingularité à la Zariski implique les conditions de Whitney, pp 41–46 in *Singularités à Cargèse*, Astérisque **7-8**, Soc. Math. France, 1973.

37. B. Teissier, Cycles évanescents, sections planes, et conditions de Whitney, pp 285–362 in *Singularités à Cargèse*, Astérisque **7-8**, Soc. Math. France, 1973.

38. B. Teissier, Introduction to equisingularity problems, pp 593–632 in *Algebraic geometry (Arcata 1974)* (ed. R. Hartshorne), Proc. Symp. in Pure Math. **29**, Amer. Math. Soc., 1975.

39. B. Teissier, Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces. Invent. Math. 40 (1977), no. 3, 267–292.

40. B. Teissier, The hunting of invariants in the geometry of discriminants, pp 565–678 in *Real and complex singularities* (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, 1977.

41. B. Teissier, Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney. *Algebraic geometry (La Ràbida, 1981)*, Lecture Notes in Math. **961**, Springer-Verlag 1982, 314–491.

42. R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier **6** (1955-56) 43-87. (see also Sém. Bourbaki 134).

43. R. Thom, 1956, Un lemme sur les applications différentiables, Bol. Soc. Mat. Mex. **1** 59-71.

44. R. Thom and H. I. Levine, *Singularities of differentiable mappings*, Bonn Math. Schrift **6**, 1959. Reprinted in Lecture Notes in Math. **192**, Springer-Verlag 1971, 1–89.

45. R. Thom, La stabilité topologique des applications polynomiales, L'Ens. Math. **8** (1962) 24–33.

46. R. Thom, 1964, Local topological properties of differentiable mappings, pp 191-202 in *Differential Analysis*, Oxford Univ. Press.
47. R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. **75** (1969) 240–284.
48. D. Trotman and L. C. Wilson, Stratifications and finite determinacy, Proc. London Math. Soc. **78** (1999) 334–368.
49. J.-L. Verdier, Stratifications de Whitney et théorème de Bertini-Sard, Invent. Math. **36** (1976) 295–312.
50. J. Wahl, Equisingular deformations of plane algebroid curves, Trans. Amer. Math. Soc. **193** (1974) 143–170.
51. H. Whitney, The general type of singularity of a set of $(2n-1)$ smooth functions of n variables, Duke Math. J. **10** (1943) 161-172.
52. H. Whitney, The self-intersections of a smooth n-manifold in $2n$-space, Ann. of Math. **45** (1944) 220-246.
53. H. Whitney, The self-intersections of a smooth n-manifold in $(2n-1)$-space, Ann. of Math. **45** (1944) 247-293.
54. H. Whitney, On singularities of mappings of euclidean spaces I: mappings of the plane into the plane, Ann. of Math. **62** (1955) 374-410.
55. H. Whitney, Elementary structure of real algebraic varieties, Ann. of Math. **66** (1957) 545–556.
56. H. Whitney, Singularities of mappings of euclidean spaces, pp 285-301 in *Symposium Internacional de Topología algebraica*, UNAM & UNESCO, 1958.
57. H. Whitney, Local properties of analytic varieties, pp 205–244 in *Differential and combinatorial topology* (ed. S. S. Cairns) Princeton, 1965.
58. H. Whitney, Tangents to an analytic variety, Ann. of Math. **81** (1965) 496–549.
59. H. Whitney and F. Bruhat, Quelques propriétés fondamentales des ensembles analytiques-réels, Comm. Math. Helv. **33** (1959) 132–160.
60. O. Zariski, On the topology of algebroid singularities, Amer. J. Math. **54** (1932) 453–465. (in Works, vol III)
61. O. Zariski, Studies in equisingularity I-III, Amer. J. Math. **87** (1965) 507–536, 972–1006, **90** (1968) 961–1023.
62. O. Zariski, Some open questions in the theory of singularities, Bull. Amer. Math. Soc. **77** (1971) 481–491.
63. O. Zariski, General theory of saturation and of saturated local rings I-III, Amer. J. Math. **93** (1971) 573–648, 872–964, **97** (1975) 415–502, see also Astérisque **7-8** (1973) 21–29.
64. O. Zariski, The elusive concept of equisingularity and related questions, pp 9–22 in *Algebraic geometry: the Johns Hopkins centennial lectures* (supplement to Amer. J. Math.), Johns Hopkins Press, 1977.
65. O. Zariski, Foundations of a general theory of equisingularity on $r-$dimensional algebraic and algebroid varieties, of embedding dimension $r+1$, Amer. J. Math. **101** (1979) 453–514.
66. O. Zariski and P. Samuel, *Commutative algebra, vol. II*, van Nostrand, Princeton, 1960.

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