Mathematics for Physicists: Complex Functions I

Exercise 1 of MATH281

Review

- Definition of differentiability, analytic functions
- Cauchy-Riemann equations: necessary and sufficient conditions
- Harmonic functions u, v and analytic functions $u + iv$

1. Simplify the following complex function into $u + iv$ form
$$f(z) = \frac{1+iz}{1-iz}.$$
2. Simplify the following complex function into $u + iv$ form
$$f(z) = z^3.$$
3. Using the definition, verify that the following function is differentiable
$$f(z) = z^2.$$
4. Use the Cauchy-Riemann conditions to determine if the following functions are analytic:
   (a) $f(z) = z^2$.
   (b) $f(z) = z\,|z|$.
5. Which, if any, of the following functions, $u(x, y)$, permit the construction of an harmonic function, $v(x, y)$, such that $f(z) = u + iv$ is an analytic function
   (a) $u(x, y) = y^2 \sin(xy)$.
   (b) $u(x, y) = 3x - 2xy$.
   (c) $u(x, y) = x^2 + y^2 - 2x - 9y + 2001$.

   In each case where the construction of $f(z)$ is possible, find the analytic function $f(z)$ that satisfies $f(0) = 0$.
6. * Which of the following functions are analytic?
   (a) $f(z) = x^2 + y^2 + 2ixy$.
   (b) $f(z) = x^2 + y^2 - 2ixy$.
   (c) $f(z) = x^2 + y^2 + 9ixy$.
   (d) $f(z) = x^2 - y^2 + 2ixy$.