DESY 09-129
SFB/CPP-09-74
LTH 839
August 2009

arXiv:0908.2746

Higher-order threshold resummation for semi-inclusive e^+e^- annihilation

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Abstract

The complete soft-enhanced and virtual-gluon contributions are derived for the quark coefficient functions in semi-inclusive e^+e^- annihilation to the third order in massless perturbative QCD. These terms enable us to extend the soft-gluon resummation for the fragmentation functions by two orders to the next-to-next-to-next-to-leading logarithmic (N^{3}LL) accuracy. The resummation exponent is found to be the same as for the structure functions in inclusive deep-inelastic scattering. This finding, together with known results on the higher-order quark form factor, facilitates the determination of all soft and virtual contributions of the fourth-order difference of the coefficient functions for these two processes. Unlike the previous (N^{2}LL) order in the exponentiation, the numerical effect of the N^{3}LL contributions turns out to be negligible at LEP energies.

Semi-inclusive e^+e^- annihilation (SIA) via a virtual photon or Z-boson, $e^+e^- \to \gamma/Z \to h+X$, is a classic process probing Quantum Chromodynamics (QCD), the theory of the strong interaction. A wealth of precise measurements have been performed, at various center-of-mass (CM) energies $\sqrt{s}$, of the total fragmentation function

$$\frac{1}{\sigma_{\rm tot}} \frac{d\sigma^h}{dx} \;=\; F^h(x,Q^2) \; , \tag{1}$$

where h stands for a specific hadron species or the sum over all (charged) light hadrons, see Ref. [1] for a general overview. In the CM frame the scaling variable x is the fraction of the beam energy carried by the hadron h, and $Q^2 = s$ is the square of the four-momentum q of the intermediate gauge boson. In perturbative QCD, the total (angle-integrated) fragmentation function $F_I^h \equiv F^h$, as well as the transverse (F_T), longitudinal (F_L) and asymmetric (F_A) fragmentation functions for the double-differential cross section $d\sigma^h/dx\, d\cos\theta_h$ [2], are given by

$$F_a^h(x,Q^2) \;=\; \sum_{\rm f=q,\,\bar{q},\,g} \int_x^1 \frac{dz}{z}\, C_{a,\rm f}\left(z,\alpha_{\rm s}(Q^2)\right) D_{\rm f}^h\left(\frac{x}{z},Q^2\right) \,+\, O\left(\frac{1}{Q}\right) \; . \tag{2}$$

Here $D_{\rm f}^h$ are the parton fragmentation functions, the final-state (timelike, $Q^2 = q^2$) analogue of the initial-state (spacelike, $Q^2 = -q^2$) parton distribution functions in deep-inelastic scattering (DIS). Without loss of information in the present context, the renormalization scale of $\alpha_{\rm s}$ and the factorization scale of $D_{\rm f}^h$ have been identified with the physical hard scale Q^2 in Eq. (2). The coefficient functions $C_{a,\rm f}$ are defined via expansions in the strong coupling $a_{\rm s} \equiv \alpha_{\rm s}/(4\pi)$.

Here we are interested in the dominant (anti-)quark contributions to F_I^h, F_T^h and F_A^h,

$$C_{a,q}(x,\alpha_{\rm s}) \;=\; \sigma_{\rm ew}(\delta(1-x) + a_{\rm s}\, c_{a,q}^{(1)}(x) + a_{\rm s}^2\, c_{a,q}^{(2)}(x) + a_{\rm s}^3\, c_{a,q}^{(3)}(x) + \ldots) \; . \tag{3}$$

The electroweak prefactors $\sigma_{\rm ew}$ can be found in Ref. [2]. The first- and second-order coefficient functions have been calculated long ago in Refs. [3] and [4], respectively. More recently the latter results have been confirmed (and some typos corrected) in two independent ways in Refs. [5, 6]. The three-loop corrections $c_a^{(3)}(x)$ have not been derived so far.

The coefficient functions in Eq. (3) include large-x (threshold) double-logarithmic enhancements of the form $a_{\rm s}^n\, (1-x)^{-1} \ln^k(1-x)$ with $k = 0, \ldots, 2n-1$. Such contributions, which spoil the convergence of the perturbation series at sufficiently large values of x, can be resummed by the soft-gluon exponentiation [7, 8]. For the process at hand this resummation has been worked out to next-to-leading logarithmic (NLL) accuracy in Ref. [9]. The inclusion of this resummation has led to improvements in a recent global fit of fragmentation functions [10]. Hence an extension of the soft-gluon exponentiation for $e^+e^- \to \gamma/Z \to h+X$ to a higher accuracy is not only of theoretical but also of phenomenological interest.

In this letter we employ the analytic continuation approach of Ref. [5] to derive the soft and virtual contributions to the third-order coefficient functions in Eq. (3). These results are then used to extend the results of Ref. [9] to the next-to-next-to-next-to-leading logarithmic (N^{3}LL) accuracy reached before for inclusive deep-inelastic scattering [11] and the total cross sections for lepton-pair and Higgs-boson production in proton–(anti-)proton collisions [12, 13]. A substantial intermediate step towards the present extension has been taken before in Ref. [14].

Up to small contributions from higher-order group invariants entering at the third and higher orders, the soft plus virtual contributions are the same for the DIS quark coefficient functions for F_1, F_2 and F_3 [15, 16]. The same holds for the corresponding (in this order) SIA coefficient functions for F_T, F_I and F_A. Hence we will drop the index a from now on, and refer to the former coefficient functions collectively as $c_{\mathrm{S}}^{(l)}(x)$, and the latter as $c_{\mathrm{T}}^{(l)}(x)$.

In this limit the bare (unrenormalized and unfactorized) partonic DIS (spacelike) structure function F_S^b is given by [17, 18]

$$F_{\mathrm{S}}^{\mathrm{b}}(\alpha_{\mathrm{s}}^{\mathrm{b}}, Q^2) \;=\; \delta(1-x) \,+\, \sum_{l=1} (\alpha_{\mathrm{s}}^{\mathrm{b}})^l \left(\frac{Q^2}{\mu^2}\right)^{-l\varepsilon} F_{\mathrm{S},l}^{\mathrm{b}} \tag{4}$$

with

$$\begin{aligned}
F_{\mathrm{S},1}^{\mathrm{b}} &= 2\mathcal{F}_1\,\delta(1-x) + \mathcal{S}_1 \\
F_{\mathrm{S},2}^{\mathrm{b}} &= (2\mathcal{F}_2 + (\mathcal{F}_1)^2)\,\delta(1-x) + 2\mathcal{F}_1\mathcal{S}_1 + \mathcal{S}_2 \\
F_{\mathrm{S},3}^{\mathrm{b}} &= (2\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2)\,\delta(1-x) + (\,2\mathcal{F}_2 + (\mathcal{F}_1)^2\,)\,\mathcal{S}_1 + 2\mathcal{F}_1\mathcal{S}_2 + \mathcal{S}_3\;.
\end{aligned} \tag{5}$$

Here μ is the scale of dimensional regularization with $D = 4-2\varepsilon$, and $a_{\mathrm{s}}^{\mathrm{b}}$ the bare strong coupling. $\mathcal{F}_l$ represents the l-loop quark form factor [17–22]. The x-dependence of the real-emission functions $\mathcal{S}_k$ is given by the D-dimensional +-distributions

$$f_{k\varepsilon}(x) \;=\; [\,(1-x)^{-1-k\varepsilon}\,]_+ \;=\; -\frac{1}{k\varepsilon}\,\delta(1-x) + \sum_{i=0} \frac{(-k\varepsilon)^i}{i\,!}\,\mathcal{D}_i \tag{6}$$

where we have introduced the abbreviation $\mathcal{D}_k \;=\; [(1-x)^{-1}\ln^k(1-x)]_+$.

The transition to the bare SIA (timelike) fragmentation functions $F_{\mathrm{S}}^{\mathrm{b}}$ is performed as follows: In Eq. (5) the factors $2\mathcal{F}_l$ are replaced everywhere by $2\,\mathrm{Re}\mathcal{F}_l^T$ and all products $\mathcal{F}_k\mathcal{F}_l$ by $|\mathcal{F}_k^T\,\mathcal{F}_l^T|$, where $\mathcal{F}_l^T$ is the complex l-loop timelike form factor which can be obtained from the spacelike $\mathcal{F}_l$ by Eq. (3.3) of Ref. [17]. The analytic continuation of the real-emission terms $\mathcal{S}_k$ is carried out as discussed in Ref. [5]. In fact, these functions turn out to be the same for the spacelike and timelike cases (this holds only in the present large-x limit, not for the full real emission contributions). Finally the standard renormalization and mass factorization is performed to the third order for the resulting timelike analogue of Eq. (5), yielding the $\mathcal{D}_k$ and $\delta(1-x)$ terms of $c_a^{(3)}(x)$ in Eq. (3).

For the convenience of the reader, we include also the large-x limits of the well-known first- and second-order $\overline{\mathrm{MS}}$ coefficient functions [3, 4]. As expected from the above discussion, these and the third-order coefficient function share all non-ζ_2 terms with their spacelike counterparts, hence we will present them via the corresponding differences $\delta_{\mathrm{TS}}\,c_n = c_{\mathrm{T}}^{(n)} - c_{\mathrm{S}}^{(n)}$. The results read

$$\delta_{\mathrm{TS}}\,c_1(x) \;=\; 12\,\zeta_2\,C_F\,\delta(1-x)\;, \tag{7}$$

$$\begin{aligned}
\delta_{\mathrm{TS}}\,c_2(x) \;=\;& 48\,\zeta_2\,C_F^2\,\mathcal{D}_1 - 36\,\zeta_2\,C_F^2\,\mathcal{D}_0 \\
&+ \left\{(-108 + 24\,\zeta_2)\,C_F^2 + \left(\frac{466}{3} - 24\,\zeta_2\right)C_A C_F - \frac{76}{3}\,C_F n_f\right\}\zeta_2\,\delta(1-x)\;,
\end{aligned} \tag{8}$$

$$
\begin{aligned}
\delta_{\rm TS}\, c_3(x) \;=\;& 96\,\zeta_2 C_F^3\, \mathcal{D}_3 - \{216 C_F^3 + 88 C_A C_F^2 - 16 C_F^2 n_f\}\,\zeta_2\, \mathcal{D}_2 \\
& - \Big\{(324 + 96\,\zeta_2) C_F^3 - \Big(\frac{3332}{3} - 192\,\zeta_2\Big) C_A C_F^2 + \frac{536}{3} C_F^2 n_f \Big\}\,\zeta_2\, \mathcal{D}_1 \\
& + \Big\{(306 + 216\,\zeta_2 - 96\,\zeta_3) C_F^3 - \Big(\frac{10504}{9} - 248\,\zeta_2 - 480\,\zeta_3\Big) C_A C_F^2 \\
& \quad + \Big(\frac{1672}{9} - 32\,\zeta_2\Big) C_F^2 n_f \Big\}\,\zeta_2\, \mathcal{D}_0 \\
& + \Big\{\Big(\frac{993}{2} + 180\,\zeta_2 - 936\,\zeta_3 + 72\,\zeta_2^2\Big) C_F^3 - \Big(\frac{13457}{6} + \frac{220}{3}\,\zeta_2 - 1616\,\zeta_3 \\
& \quad + \frac{108}{5}\,\zeta_2^2\Big) C_A C_F^2 + \Big(\frac{74728}{27} - 196\,\zeta_2 - 1056\,\zeta_3 + \frac{528}{5}\,\zeta_2^2\Big) C_A^2 C_F \\
& \quad + \Big(\frac{667}{3} + \frac{136}{3}\,\zeta_2 - 80\,\zeta_3\Big) C_F^2 n_f - \Big(\frac{23504}{27} + \frac{16}{3}\,\zeta_2 - 96\,\zeta_3\Big) C_A C_F n_f \\
& \quad + \Big(\frac{1624}{27} + \frac{16}{3}\,\zeta_2\Big) C_F n_f^2 \Big\}\,\zeta_2\, \delta(1-x)\ . \qquad (9)
\end{aligned}
$$

Here C_A and C_F are the standard group invariants, with $C_A = 3$ and $C_F = 4/3$ in QCD, and n_f the number of light flavours. ζ_k denotes Riemann's ζ-function. The third-order SIA coefficient functions can be obtained by adding the corresponding DIS results given in Eqs. (4.14) – (4.19) and Appendix B of Ref. [15], see also Eq. (3.8) of Ref. [16]. The first half of Eq. (9) agrees with the result of Ref. [14], the $\delta(1-x)$ contribution in the second half has not been presented before.

Below we will need the N-independent parts $\delta_{\rm TS}\, g_{0k} \equiv \delta_{\rm TS}\, c_k(N)|_{N^0}$ of the Mellin transforms of Eq. (7) – (9) obtained via

$$
a^N \;=\; \int_0^1 dx\, \left(x^{N-1} - 1\right) a(x)_+ \qquad (10)
$$

together with $\delta(1-x) \to 1$. These contributions are given by ($\gamma_{\rm e}$ is the Euler-Mascheroni constant)

$$
\zeta_2^{-1}\, \delta_{\rm TS}\, g_{01} \;=\; 12 C_F\ , \qquad (11)
$$

$$
\zeta_2^{-1}\, \delta_{\rm TS}\, g_{02} \;=\; C_A C_F \Big(\frac{466}{3} - 24\,\zeta_2\Big) - C_F^2\, \left(108 - 48\,\zeta_2 - 36\,\gamma_{\rm e} - 24\,\gamma_{\rm e}^2\right) - \frac{76}{3} C_F n_f\ , \qquad (12)
$$

$$
\begin{aligned}
\zeta_2^{-1}\, \delta_{\rm TS}\, g_{03} \;=\;& C_F^3 \Big(\frac{993}{2} + 18\,\zeta_2 - 792\,\zeta_3 + \frac{768}{5}\,\zeta_2^2 - 306\,\gamma_{\rm e} + 288\,\gamma_{\rm e}\zeta_3 - 162\,\gamma_{\rm e}^2 \\
& + 96\,\gamma_{\rm e}^2\zeta_2 + 72\,\gamma_{\rm e}^3 + 24\,\gamma_{\rm e}^4\Big) + C_A C_F^2 \Big(-\frac{13457}{6} + 482\,\zeta_2 + \frac{5024}{3}\,\zeta_3 \\
& - \frac{588}{5}\,\zeta_2^2 + \frac{10504}{9}\,\gamma_{\rm e} - 160\,\gamma_{\rm e}\zeta_2 - 480\,\gamma_{\rm e}\zeta_3 + \frac{1666}{3}\,\gamma_{\rm e}^2 - 96\,\gamma_{\rm e}^2\zeta_2 \\
& + \frac{88}{3}\,\gamma_{\rm e}^3\Big) + C_A^2 C_F \Big(\frac{74728}{27} - 196\,\zeta_2 - 1056\,\zeta_3 + \frac{528}{5}\,\zeta_2^2\Big) \\
& + C_F^2 n_f \Big(\frac{667}{3} - 44\,\zeta_2 - \frac{272}{3}\,\zeta_3 - \frac{1672}{9}\,\gamma_{\rm e} + 16\,\gamma_{\rm e}\zeta_2 - \frac{268}{3}\,\gamma_{\rm e}^2 - \frac{16}{3}\,\gamma_{\rm e}^3\Big) \\
& + C_A C_F n_f \Big(-\frac{23504}{27} - \frac{16}{3}\,\zeta_2 + 96\,\zeta_3\Big) + C_F n_f^2 \Big(\frac{1624}{27} + \frac{16}{3}\,\zeta_2\Big)\ . \qquad (13)
\end{aligned}
$$

The corresponding DIS coefficients can be found in Eqs. (4.6) – (4.8) of Ref. [11].

For processes such as DIS and SIA, the dominant large-$x/$ large-N contributions to the $\overline{\mathrm{MS}}$ coefficient functions C^N can be resummed by a single exponential in Mellin space [7]

$$C^N(Q^2) \;=\; g_0(Q^2)\cdot \exp\left[G^N(Q^2)\right] \,+\, O(N^{-1}\ln^n N) \; . \tag{14}$$

The prefactor g_0 collects, order by order in the strong coupling constant α_{s}, all N-independent contributions. The exponent G^N contains terms of the form $\ln^k N$ to all orders in α_{s}. Besides the physical hard scale Q^2 ($= \mp q^2$ in DIS/SIA, with q the four-momentum of the exchanged gauge boson), both functions depend on the renormalization scale μ_r and the mass-factorization scale μ_f.

The exponential in Eq. (14) is build up from universal radiative factors for each initial- and final-state parton p, Δ_{p} and J_{p}, together with a process-dependent contribution Δ^{int}. The resummation exponents for DIS and SIA [9] take the very similar form

$$\begin{aligned} G^N_{\mathrm{DIS}} \;&=\; \ln\Delta_{\mathrm{q}} + \ln J_{\mathrm{q}} + \ln\Delta^{\mathrm{int}}_{\mathrm{DIS}} \; , \\ G^N_{\mathrm{SIA}} \;&=\; \ln\Delta_{\mathrm{q}} + \ln J_{\mathrm{q}} + \ln\Delta^{\mathrm{int}}_{\mathrm{SIA}} \; . \end{aligned} \tag{15}$$

The radiation factors are given by integrals over functions of the running coupling. Specifically, the effects of collinear soft-gluon radiation off an initial-state or ‘observed’ final-state quark are collected by

$$\ln\Delta_{\mathrm{q}}(Q^2,\mu_f^2) \;=\; \int_0^1 \! dz\, \frac{z^{N-1}-1}{1-z} \int_{\mu_f^2}^{(1-z)^2Q^2} \frac{dq^2}{q^2} A(\alpha_{\mathrm{s}}(q^2)) \; . \tag{16}$$

Collinear emissions from an ‘unobserved’ final-state quark lead to the so-called jet function,

$$\ln J_{\mathrm{q}}(Q^2) \;=\; \int_0^1 \! dz\, \frac{z^{N-1}-1}{1-z} \left[\int_{(1-z)^2Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A(\alpha_{\mathrm{s}}(q^2)) + B(\alpha_{\mathrm{s}}([1-z]Q^2)) \right] . \tag{17}$$

Finally the process-dependent contributions from large-angle soft gluons are resummed by

$$\ln\Delta^{\mathrm{int}}(Q^2) \;=\; \int_0^1 \! dz\, \frac{z^{N-1}-1}{1-z} D(\alpha_{\mathrm{s}}([1-z]^2Q^2)) \; . \tag{18}$$

The functions g_0 in Eq. (14) and A, B and D in Eqs. (16) – (18) are given by the expansions

$$F(\alpha_{\mathrm{s}}) \;=\; \sum_{l=l_0} F_l\, \frac{\alpha_{\mathrm{s}}^l}{4\pi} \;\equiv\; \sum_{l=l_0} F_l\, a_{\mathrm{s}}^l \; , \tag{19}$$

where $l_0 = 0$ with $g_{00} = 1$ for $F = g_0$, and $l_0 = 1$ else.

The known expansion coefficients of the cusp anomalous dimension (the coefficients of $\mathcal{D}_0 \equiv 1/(1-x)_+$ in the $\overline{\mathrm{MS}}$ quark-quark splitting functions) read [23, 24]

$$\begin{aligned} A_1 \;&=\; 4C_F \\ A_2 \;&=\; 8C_F\left[\left(\frac{67}{18}-\zeta_2\right)C_A - \frac{5}{9}n_f\right] \\ A_3 \;&=\; 16C_F\left[C_A^2\left(\frac{245}{24}-\frac{67}{9}\zeta_2+\frac{11}{6}\zeta_3+\frac{11}{5}\zeta_2^2\right) + C_F n_f\left(-\frac{55}{24}+2\,\zeta_3\right)\right. \\ &\qquad \left. + C_A n_f\left(-\frac{209}{108}+\frac{10}{9}\zeta_2-\frac{7}{3}\zeta_3\right) + n_f^2\left(-\frac{1}{27}\right)\right] . \end{aligned} \tag{20}$$

The first three coefficients of the jet function (17) are given by [7, 11, 25]

$$B_1 \;=\; -3\,C_F\;, \tag{21}$$

$$B_2 \;=\; C_F^2\Big[-\frac{3}{2}+12\,\zeta_2-24\,\zeta_3\Big]+C_FC_A\Big[-\frac{3155}{54}+\frac{44}{3}\,\zeta_2+40\,\zeta_3\Big] \\ +C_Fn_f\Big[\frac{247}{27}-\frac{8}{3}\,\zeta_2\Big]\;, \tag{22}$$

$$B_3 \;=\; C_F^3\Big[-\frac{29}{2}-18\,\zeta_2-68\,\zeta_3-\frac{288}{5}\,\zeta_2^2+32\,\zeta_2\zeta_3+240\,\zeta_5\Big] \\ +C_AC_F^2\Big[-46+287\,\zeta_2-\frac{712}{3}\,\zeta_3-\frac{272}{5}\,\zeta_2^2-16\,\zeta_2\zeta_3-120\,\zeta_5\Big] \\ +C_A^2C_F\Big[-\frac{599375}{729}+\frac{32126}{81}\,\zeta_2+\frac{21032}{27}\,\zeta_3-\frac{652}{15}\,\zeta_2^2-\frac{176}{3}\,\zeta_2\zeta_3-232\,\zeta_5\Big] \\ +C_F^2n_f\Big[\frac{5501}{54}-50\,\zeta_2+\frac{32}{9}\,\zeta_3\Big]+C_Fn_f^2\Big[-\frac{8714}{729}+\frac{232}{27}\,\zeta_2-\frac{32}{27}\,\zeta_3\Big] \\ +C_AC_Fn_f\Big[\frac{160906}{729}-\frac{9920}{81}\,\zeta_2-\frac{776}{9}\,\zeta_3+\frac{208}{15}\,\zeta_2^2\Big]\;. \tag{23}$$

Together with Eqs. (11) – (13), all functions but D in Eqs. (16) – (18) are known to order α_s^3. Consequently the first three coefficients of $D^{\rm SIA}$ can by determined by comparing the α_s-expansion of Eq. (14) with the fixed-order results (7) – (9). This procedure yields

$$D_k^{\rm SIA} \;=\; 0 \tag{24}$$

for $k = 1, 2, 3$, hence $\Delta^{\rm int}_{\rm SIA} = 1$ to at least N^{3}LL accuracy. $D_1 = 0$ was, of course, included in the NLL resummation of Ref. [9]. However, B_2 was unknown at that time, and only $B_2 + D_2$ could be extracted from the two-loop results of Refs. [4] alone.

As expected from the identity of the DIS and SIA soft-emission functions $\mathcal{S}_k$ in Eq. (5), there is a strong similarity between the respective coefficient functions also in the framework of the soft-gluon exponentiation – recall that

$$D_k^{\rm DIS} \;=\; 0\;, \quad \Delta^{\rm int}_{\rm DIS} = 1 \tag{25}$$

was proven to all orders in α_s in Refs. [26, 27]. We expect that such a proof can also be derived for SIA. For the time being assuming the all-order validity of Eq. (24), the difference between the SIA (timelike, T) and DIS (spacelike, S) large-N coefficient functions exponentiates as

$$\delta_{\rm TS}\, C^N(Q^2) \;=\; \delta_{\rm TS}\, g_0(Q^2)\cdot \exp\left[G^N(Q^2)\right] + O(N^{-1}\ln^n N) \tag{26}$$

where, after performing the integrations in Eqs. (16) – (18), the function G^N takes the form

$$G^N(Q^2) \;=\; \ln N\cdot g_1(\lambda) + g_2(\lambda) + a_s\, g_3(\lambda) + a_s^2\, g_4(\lambda)\ldots \tag{27}$$

with $\lambda = \beta_0\, a_s \ln N$. The first three expansion coefficients of $\delta_{\rm TS}\, g_0$ for $\mu_r = \mu_f = Q$ have been given above in Eqs. (11) – (13). We will address the fourth-order coefficient below.

The functions g_1 to g_4 have been derived in Refs. [7, 11, 28, 29]. For completeness we include these functions, also here restricting ourselves to choice $\mu_r = \mu_f = Q$ of the scales:

$$g_1^{\rm DIS}(\lambda) = A_1(1-\ln(1-\lambda)+\lambda^{-1}\ln(1-\lambda)) \; , \quad (28)$$

$$\begin{aligned} g_2^{\rm DIS}(\lambda) &= (A_1\beta_1 - A_2)(\lambda+\ln(1-\lambda)) + \frac{1}{2}A_1\beta_1\ln^2(1-\lambda) \\ &\quad - (A_1\gamma_{\rm e} - B_1)\ln(1-\lambda) \; , \quad (29) \end{aligned}$$

$$\begin{aligned} g_3^{\rm DIS}(\lambda) &= \frac{1}{2}(A_1\beta_2 - A_1\beta_1^2 + A_2\beta_1 - A_3)\Big(1+\lambda-\frac{1}{1-\lambda}\Big) \\ &\quad + A_1\beta_1^2\Big(\frac{\ln(1-\lambda)}{1-\lambda} + \frac{1}{2}\frac{\ln^2(1-\lambda)}{1-\lambda}\Big) + \Big(A_1\beta_2 - A_1\beta_1^2\Big)\ln(1-\lambda) \\ &\quad + (A_1\beta_1\gamma_{\rm e} + A_2\beta_1 - B_1\beta_1)\Big(1-\frac{1}{1-\lambda}-\frac{\ln(1-\lambda)}{1-\lambda}\Big) \\ &\quad - \Big(A_1\beta_2 + \frac{1}{2}A_1(\gamma_{\rm e}^2+\zeta_2) + A_2\gamma_{\rm e} - B_1\gamma_{\rm e} - B_2\Big)\Big(1-\frac{1}{1-\lambda}\Big) \; , \quad (30) \end{aligned}$$

and

$$\begin{aligned} g_4^{\rm DIS}(\lambda) &= -\frac{1}{6}A_1\beta_1^3\frac{\ln^3(1-\lambda)}{(1-\lambda)^2} + \frac{1}{2}(A_1\beta_1^2\gamma_{\rm e} + A_2\beta_1^2 - B_1\beta_1^2)\frac{\ln^2(1-\lambda)}{(1-\lambda)^2} + \frac{1}{2}(A_1\beta_1^3 - A_1\beta_1\beta_2 \\ &\quad - A_1\beta_1(\gamma_{\rm e}^2+\zeta_2) + A_2\beta_1^2 - 2A_2\beta_1\gamma_{\rm e} - A_3\beta_1 + 2B_1\beta_1\gamma_{\rm e} + 2B_2\beta_1)\frac{\ln(1-\lambda)}{(1-\lambda)^2} \\ &\quad - (A_1\beta_1^3 - A_1\beta_1\beta_2)\frac{\ln(1-\lambda)}{1-\lambda} + \Big(\frac{1}{2}A_1\beta_1^3 - A_1\beta_1\beta_2 + \frac{1}{2}A_1\beta_3\Big)\ln(1-\lambda) + (A_1\beta_1^3 \\ &\quad - A_1\beta_1\beta_2 - A_1\beta_1^2\gamma_{\rm e} + A_1\beta_2\gamma_{\rm e} - A_2\beta_1^2 + A_2\beta_2 + B_1\beta_1^2 - B_1\beta_2)\Big(\frac{1}{2}-\frac{1}{1-\lambda} \\ &\quad + \frac{1}{2}\frac{1}{(1-\lambda)^2}\Big) + \frac{1}{2}\Big(\frac{1}{3}A_1\beta_1^3 - \frac{1}{6}A_1\beta_1\beta_2 - \frac{1}{6}A_1\beta_3 - \frac{1}{3}A_1(3\gamma_{\rm e}\zeta_2 + \gamma_{\rm e}^3 + 2\zeta_3) \\ &\quad + A_2\beta_1\gamma_{\rm e} - A_2(\gamma_{\rm e}^2+\zeta_2) - \frac{5}{6}A_2\beta_1^2 + \frac{1}{3}A_2\beta_2 + \frac{5}{6}A_3\beta_1 - A_3\gamma_{\rm e} - \frac{1}{3}A_4 - B_2\beta_1 \\ &\quad + B_1(\gamma_{\rm e}^2+\zeta_2) + 2B_2\gamma_{\rm e} + B_3\Big)\Big(1-\frac{1}{(1-\lambda)^2}\Big) + \frac{1}{3}\Big(A_1\beta_1^3 - 2A_1\beta_1\beta_2 + A_1\beta_3 \\ &\quad + A_2\beta_2 - A_2\beta_1^2 + A_3\beta_1 - A_4\Big)\lambda \; . \quad (31) \end{aligned}$$

Factors of $\beta_0 = 11/3\,C_A - 2/3\,n_f$ have been suppressed in Eqs. (28) – (31) for brevity. The dependence on β_0 is recovered by $A_k \to A_k/\beta_0^k$, $B_k \to B_k/\beta_0^k$, $\beta_k \to \beta_k/\beta_0^{k+1}$ and multiplication of g_3 and g_4 by β_0 and β_0^2, respectively. Note that Eq. (31) includes all known coefficient of the beta function of QCD, see Ref. [30] and references therein.

All parameters entering Eqs. (28) – (31) are known except for the four-loop cusp anomalous dimension A_4. The small (see below) impact of this quantity – which first occurs in the $\alpha_{\rm s}^5\ln^3 N$ contribution to $\delta_{\rm TS}C^N$ – can be included by a Padé estimate as in Ref. [11], backed up by a recent calculation of one Mellin moment of the fourth-order quark-quark splitting function [31], cf. also Ref. [32]. E.g., for $n_f = 5$ one may use $A_4 \approx 1550$ (recall our small expansion parameter $a_{\rm s} = \alpha_{\rm s}/(4\pi)$) and assign a conservative uncertainly of 50% to this value.

Due to the vanishing of $\delta_{\rm TS}\, g_{00}$ the two highest logarithms, $\alpha_s^l \ln^{2l} N$ and $\alpha_s^l \ln^{2l-1} N$, are the same for the SIA and DIS structure functions to all orders in α_s. The expansion of Eq. (26) with Eqs. (11) – (13) provides the six highest logarithms, cf. Ref. [11], of the coefficient-function difference $\delta_{\rm TS}\, C^N$, $\alpha_s^l \ln^{2l-a} N$ with $a = 2, \ldots, 7$, at all orders from the fourth. In particular, all $\ln N$ enhanced terms are thus fixed at order α_s^4. After transformation to x-space these contributions read

$$
\begin{aligned}
\delta_{\rm TS}\, c_4(x) \;=\;& 96\,\zeta_2 C_F^4\, \mathcal{D}_5 - \{360 C_F^4 + \frac{880}{3} C_A C_F^3 - \frac{160}{3} C_F^3 n_f\}\,\zeta_2\, \mathcal{D}_4 \\
& - \Big\{(432 + 576\,\zeta_2) C_F^4 - (3552 - 576\,\zeta_2\,) C_A C_F^3 + 576 C_F^3 n_f - \frac{1936}{9} C_A^2 C_F^2 \\
& + \frac{704}{9} C_A C_F^2 n_f - \frac{64}{9} C_F^2 n_f^2\Big\}\,\zeta_2\, \mathcal{D}_3 \;+\; \Big\{(1674 + 2160\,\zeta_2 + 192\,\zeta_3) C_F^4 \\
& - \Big(\frac{25238}{3} - 2800\,\zeta_2 - 2880\,\zeta_3\Big) C_A C_F^3 + \Big(\frac{4100}{3} - 352\,\zeta_2\Big) C_F^3 n_f \\
& - \Big(\frac{9616}{3} - 528\,\zeta_2\Big) C_A^2 C_F^2 + \Big(\frac{3248}{3} - 96\,\zeta_2\Big) C_A C_F^2 n_f - \frac{256}{3} C_F^2 n_f^2\Big\}\,\zeta_2\, \mathcal{D}_2 \\
& + \Big\{\Big(1122 + 936\,\zeta_2 - 4320\,\zeta_3 - \frac{1248}{5}\,\zeta_2^2\Big) C_F^4 - \Big(\frac{22916}{3} + \frac{23120}{3}\,\zeta_2 \\
& - 3584\,\zeta_3 - \frac{4368}{5}\,\zeta_2^2\Big) C_A C_F^3 + \Big(\frac{488}{3} + \frac{4592}{3}\,\zeta_2 + 64\,\zeta_3\Big) C_F^3 n_f \\
& + \Big(\frac{224230}{9} - \frac{17176}{3}\,\zeta_2 - 7392\,\zeta_3 + \frac{5184}{5}\,\zeta_2^2\Big) C_A^2 C_F^2 \\
& - \Big(\frac{69728}{9} - \frac{3056}{3}\,\zeta_2 - 576\,\zeta_3\Big) C_A C_F^2 n_f + \Big(\frac{4888}{9} - \frac{64}{3}\,\zeta_2\Big) C_F^2 n_f^2\Big\}\,\zeta_2\, \mathcal{D}_1 \\
& - \Big\{\Big(\frac{3003}{2} + 3312\,\zeta_2 - 3288\,\zeta_3 + 792\,\zeta_2^2 + 192\,\zeta_2\zeta_3 - 5184\,\zeta_5\Big) C_F^4 \\
& - \Big(\frac{24507}{2} + \frac{78428}{9}\,\zeta_2 - 8816\,\zeta_3 - 1452\,\zeta_2^2 - 1728\,\zeta_2\zeta_3 \\
& - 1440\,\zeta_5\Big) C_A C_F^3 + \Big(\frac{6620501}{243} - \frac{243752}{27}\,\zeta_2 - \frac{168560}{9}\,\zeta_3 + \frac{5952}{5}\,\zeta_2^2 \\
& + 1664\,\zeta_2\zeta_3 + 2784\,\zeta_5\Big) C_A^2 C_F^2 + \Big(\frac{3551}{9} + \frac{13568}{9}\,\zeta_2 + \frac{688}{3}\,\zeta_3\Big) C_F^3 n_f \\
& - \Big(\frac{1983208}{243} - \frac{66392}{27}\,\zeta_2 - 2336\,\zeta_3 + \frac{1152}{5}\,\zeta_2^2\Big) C_A C_F^2 n_f \\
& + \Big(\frac{135020}{243} - \frac{464}{3}\,\zeta_2 + \frac{128}{9}\,\zeta_3\Big) C_F^2 n_f^2\Big\}\,\zeta_2\, \mathcal{D}_0 \;+\; \ldots\;. \qquad (32)
\end{aligned}
$$

The first four terms correspond to a NNLO + NLL accuracy as first obtained for DIS in Ref. [33]. For the present case these terms have been presented, in a different notation, already in Ref. [14]. The coefficients of $\mathcal{D}_1$ and $\mathcal{D}_0$ (recall the definition below Eq. (6)) are new results of the present study. The latter coefficient depends on our assumption that Eq. (24) extends to $k = 4$.

The fourth-order result (32) can be verified, and extended to the $\delta(1-x)$ contribution, in the following manner. Eq. (5) is extended to the fourth order,

$$F^{\rm b}_{{\rm S},4} \;=\; (2\mathcal{F}_4 + 2\mathcal{F}_1\mathcal{F}_3 + (\mathcal{F}_2)^2)\,\delta(1-x) + (\,2\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2)\,\mathcal{S}_1 + (\,2\mathcal{F}_2 + (\mathcal{F}_1)^2\,)\,\mathcal{S}_2 + 2\mathcal{F}_1\mathcal{S}_3 + \mathcal{S}_4\;, \qquad (33)$$

and is subtracted from its timelike counterpart obtained as discussed above. Assuming that also $\mathcal{S}_4$ is identical in the two cases, the only unknown in $\delta_{\rm TS} F^{\rm b}_4$ to order ε^0 is the four-loop anomalous dimension A_4. All other unknown quantities, such as the ε^1 and ε^2 contributions to the spacelike three-loop form factor [18, 21, 22] (also the latter new result is not needed in the present context), drop out in this difference. Also the four-loop form factor is known from its exponentiation [34] to a sufficient accuracy in ε [18]. The soft and virtual contributions to $\delta_{\rm TS}\, c_4$ are then extracted from the fourth-order mass factorization formula (here given in terms of the bare coupling)

$$\delta_{\rm TS} F^{\rm b}_4 \;=\; \delta_{\rm TS}\, c_4 + \frac{1}{3}\,[\beta_2 - P_2]\,\delta_{\rm TS}\, a_1 + \left[\frac{4}{3}\,\beta_0\beta_1 - \frac{7}{6}\,P_1\beta_0 - \frac{2}{3}\,P_0\beta_1 + \frac{1}{2}\,P_0P_1\right]\delta_{\rm TS}\, b_1 + \left[\beta_0^3 - \frac{11}{6}\,P_0\beta_0^2 + P_0^2\beta_0 - \frac{1}{6}\,P_0^3\right]\delta_{\rm TS}\, d_1 + \left[\beta_1 - \frac{1}{2}\,P_1\right]\delta_{\rm TS}\, a_2 + \left[3\beta_0^2 - \frac{5}{2}\,P_0\beta_0 + \frac{1}{2}\,P_0^2\right]\delta_{\rm TS}\, b_2 + [3\beta_0 - P_0]\,\delta_{\rm TS}\, a_3 + \varepsilon\text{-terms}\;. \qquad (34)$$

For brevity we have suppressed the $\varepsilon^{-3}\ldots\varepsilon^{-1}$ terms which form a consistency check but do not provide new information. The functions a_n, b_n and d_n are the ε^1, ε^2 and ε^3 contributions, respectively, to the D-dimensional coefficient functions at order α_s^n, cf. Ref. [16], and P_n denotes the N^{n}LO quark-quark splitting functions. In x-space obviously all products of these functions in Eq. (34) have to be read as Mellin-convolutions.

The determination of $\delta_{\rm TS}\, c_4$ from Eqs. (33) and (34) reproduces the result in Eq. (32) — hence $D_k^{\rm SIA} = D_k^{\rm DIS}$ $(=0)$ in Eq. (18) corresponds to $\delta_{\rm TS}\mathcal{S}_k = 0$ in Eqs. (5), (33) and their higher-order generalizations — and includes the final large-x coefficient,

$$\zeta_2^{-1}\,\delta_{\rm TS}\, c_4\Big|_{\delta(1-x)} \;=\; \Big(-\frac{7255}{2} - 3779\,\zeta_2 - 3816\,\zeta_3 - \frac{13896}{5}\,\zeta_2^2 + 4080\,\zeta_2\zeta_3 + 14880\,\zeta_5 + \frac{31856}{105}\,\zeta_2^3 - 1216\,\zeta_3^2\Big)C_F^4 + \Big(\frac{191411}{12} + \frac{153802}{9}\,\zeta_2 - 42808\,\zeta_3 + \frac{62452}{9}\,\zeta_2^2 + \frac{8128}{3}\,\zeta_2\zeta_3 - \frac{67328}{3}\,\zeta_5 - \frac{102472}{105}\,\zeta_2^3 + 4064\,\zeta_3^2\Big)C_F^3C_A + \Big(-\frac{14817221}{324} - \frac{63347}{3}\,\zeta_2 + \frac{1856680}{27}\,\zeta_3 + \frac{5306}{45}\,\zeta_2^2 - 2032\,\zeta_2\zeta_3 + 6256\,\zeta_5 + \frac{2584}{21}\,\zeta_2^3 - 992\,\zeta_3^2\Big)C_F^2C_A^2 + \Big(\frac{13294462}{243} + \frac{206162}{27}\,\zeta_2 - \frac{416032}{9}\,\zeta_3 - 1100\,\zeta_2^2 + 1936\,\zeta_2\zeta_3 + 8976\,\zeta_5\Big)C_FC_A^3 + \Big(\frac{409}{6} - \frac{23350}{9}\,\zeta_2 + 6840\,\zeta_3 - \frac{55592}{45}\,\zeta_2^2 - \frac{2272}{3}\,\zeta_2\zeta_3 + \frac{6272}{3}\,\zeta_5\Big)C_F^3n_f + \Big(\frac{706405}{81} + \frac{187834}{27}\,\zeta_2 - \frac{416384}{27}\,\zeta_3 + \frac{6932}{45}\,\zeta_2^2 + 320\,\zeta_2\zeta_3$$

$$
\begin{aligned}
&-1408\,\zeta_5\Big)C_F^2C_An_f - \Big(\frac{2109553}{81} + \frac{106168}{27}\,\zeta_2 - \frac{127000}{9}\,\zeta_3 + 352\,\zeta_2\zeta_3 \\
&-\frac{1088}{5}\,\zeta_2^2 + 1632\,\zeta_5\Big)C_FC_A^2n_f - \Big(\frac{3233}{81} + \frac{14824}{27}\,\zeta_2 - \frac{20656}{27}\,\zeta_3 \\
&+\frac{2464}{45}\,\zeta_2^2\Big)C_F^2n_f^2 + \Big(\frac{305917}{81} + \frac{17504}{27}\,\zeta_2 - \frac{8336}{9}\,\zeta_3 - \frac{16}{5}\,\zeta_2^2\Big)C_FC_An_f^2 \\
-&\Big(\frac{39352}{243} + \frac{304}{9}\,\zeta_2 + \frac{64}{9}\,\zeta_3\Big)C_Fn_f^3 + \Big(768 + 1920\,\zeta_2 + 896\,\zeta_3 \\
&-\frac{384}{5}\,\zeta_2^2 - 5120\,\zeta_5\Big)\,fl_{11}\,C_F\,\frac{d^{abc}d_{abc}}{n_c} + 3A_4\;. \qquad (35)
\end{aligned}
$$

See Ref. [15] for the fl_{11} diagram class leading to the term with $d^{abc}d_{abc}/n_c = 5/18\,n_f$ in QCD. The numerical effect of this contribution is very small and will be disregarded in the following.

The Mellin transform of these equations provides the α_s^4 prefactor $\delta_{\rm TS}\,g_{04}$ in Eq. (26), and hence (up to the residual uncertainty due to A_4) the seventh tower of large-x logarithms from order α_s^5 for this difference. For $n_f = 5$ quark flavours, the numerical expansion of $\delta_{\rm TS}\,g_0$ is given by

$$
\delta_{\rm TS}\,g_0(\alpha_s) \;\simeq\; 2.094\,\alpha_s\left(1 + 1.463\,\alpha_s + 2.749\,\alpha_s^2 + \{6.659 + 0.094A_4/1000\}\,\alpha_s^3 + \ldots\right)\;. \qquad (36)
$$

Thus the two new terms form a correction of almost 5% at $\alpha_s = 0.12$, with a negligible uncertainty from the missing exact value of A_4, and the fourth-order contribution is less than half of the previous term for $\alpha_s < 0.2$. It is well-known that the coefficients in Eq. (36) are due to ζ_2-terms (i.e., powers of π^2) from the analytic continuation of the form factor which are subject to a separate exponentiation (see, e.g., Refs. [34]). The corresponding results for the SIA and DIS cases read

$$
\begin{aligned}
g_{\rm T,0}(\alpha_s) &= 1 + 1.045\,\alpha_s + 2.266\,\alpha_s^2 + 4.703\,\alpha_s^3 + \ldots\;, \\
g_{\rm S,0}(\alpha_s) &= 1 - 1.050\,\alpha_s - 0.797\,\alpha_s^2 - 1.056\,\alpha_s^3 + \ldots\;. \qquad (37)
\end{aligned}
$$

The pattern of the corrections in Eq. (37) and the size of the α_s^4-term in Eq. (36) strongly suggests that the fourth-order contribution to $g_{\rm T,0}$ amounts to less than 0.5% for $\alpha_s = 0.12$.

The coefficients of the known $\ln^k N$ terms are given in Table 1 to the tenth order in α_s, using the notation c_{ka} for the coefficient of $a_s^k \ln^{2k-a+1} N$ in $C^N_{\rm SIA}$. Hence, as in Ref. [11] for the DIS case, the coefficients of the leading (next-to-leading etc) logarithms are denoted by c_{k1} (c_{k2} etc). The qualitative pattern of these coefficients is similar to the DIS case (where all numbers $c_{k,a>2}$ are smaller). The higher-order coefficients rise very rapidly, by about an order of magnitude or more, with a until $a = k - \theta_{k4}$ without showing the larger-a turnover of the DIS coefficients, cf. Table 1 of Ref. [11]. Indeed, the coefficient for the two cases are very similar for $a \ll k$, but the SIA coefficient are more than double their DIS counterparts at $a > k$ where the numbers are large.

Consequently the higher-order soft plus virtual contributions are qualitatively similar, but larger in the timelike case. The numerical size of its resummed coefficient function (14) is illustrated in Fig. 1 for a value of α_s corresponding to LEP1, $s = M_Z^2$. Obviously the size of the coefficient function, as well as the relative impact of the new $\rm N^2LL$ and $\rm N^3LL$ corrections, increases towards lower CM energies. Nevertheless one can conclude from Fig. 1 that the accuracy now reached for the dominant large-x/ large-N contributions should be sufficient for the foreseeable future.

k	c_{k1}	c_{k2}	c_{k3}	c_{k4}	c_{k5}	c_{k6}	$c_{k7}/10$
1	2.66667	7.0785	—	—	—	—	—
2	3.55556	25.6908	105.621	104.34	—	—	—
3	3.16049	43.3408	309.335	1016.50	2306.0	2090	—
4	2.10700	46.6020	514.068	3125.96	11774.1	23741	4664
5	1.12373	36.4525	577.143	5393.82	32365.2	110255	29009
6	0.49944	22.3131	481.110	6314.54	55037.7	293931	119399
7	0.19026	11.1933	315.972	5515.83	65426.2	506294	294105
8	0.06342	4.7503	170.251	3808.07	58765.0	618949	487117
9	0.01879	1.7455	77.500	2160.26	41980.1	574684	589591
10	0.00501	0.5652	30.470	1035.7	24725.4	425171	551698

Table 1: Numerical values of the five-flavour coefficients c_{ka} of the $a_s^k \ln^{2k-a+1} N$ contributions to the coefficient function $C^N_{\rm SIA}$. The first six columns are exact up to the numerical truncation, and the same for F_I, F_T and F_A. The seventh column neglects the tiny (and non-universal) fl_{11} contributions, and uses the estimate $A_4 = 1550$ for the four-loop cusp anomalous dimension.

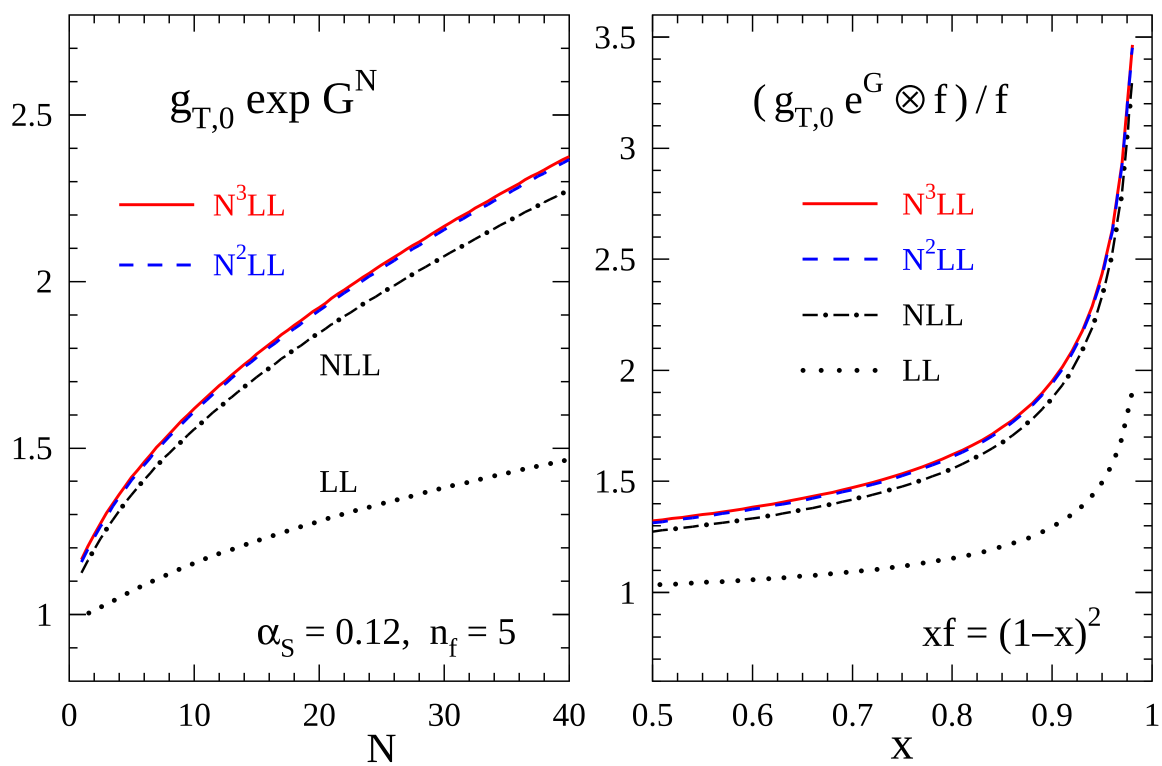

Figure 1: Left: the LL, NLL, N^{2}LL and N^{3}LL results for the threshold resummation (14) of the SIA coefficient functions (3) in N-space. Terms to order α_s^n are included in $g_{T,0}$ for the N^{n}LL curves. Right: the convolutions of these results with a schematic large-x shape for the quark fragmentation functions, using the standard 'minimal prescription' contour [8] for the Mellin inversion.

To summarize, we have first employed the close relation between the perturbative corrections to the structure functions in deep-inelastic scattering (DIS) and the fragmentation functions in semi-inclusive e^+e^- annihilation (SIA), see also Refs. [35], to derive the complete soft and virtual corrections to the third-order quark coefficient functions for the latter observables.

This result then made it possible to extend the soft-gluon exponentiation in SIA from the next-to-leading logarithmic (NLL) contributions [9] by two orders to N^3LL accuracy (we confirm the intermediate results in Ref. [14]). It turns out that the resummation exponents are the same, presumably to all orders, for the DIS and SIA coefficient functions. Hence the threshold enhancement is structurally identical in the two cases, and the same thus holds for the class of large-x $1/Q^2$ power corrections associated with the renormalon ambiguity of its perturbation series [27, 36].

The N^3LL exponentiation fixes the seven highest large-x logarithms at the fourth and all higher orders in α_{s}. The especially simple connection between the soft and virtual contributions to the DIS and SIA coefficient functions also facilitates a full N^3LL resummation of the SIA – DIS difference, including the next-to-next-to-next-to-leading order $\alpha_{\mathrm{s}}^4\,\delta(1-x)$ contribution to this difference.

Since the prefactor of the resummation exponential is larger in SIA than in DIS, the soft-gluon enhancement is numerically larger in the former case. However, while the N^2LL contributions are still significant at LEP energies, the N^3LL corrections are practically negligible, indicating that a sufficient perturbative accuracy in the large-x limit has been reached with the present results.

Acknowledgments: Some of the symbolic manipulations for this article have been performed in FORM [37]. S.M. acknowledges support by the Helmholtz Gemeinschaft under contract VH-NG-105 and in part by the Deutsche Forschungsgemeinschaft in Sonderforschungsbereich/Transregio 9. The research of A.V. has been supported by the UK Science & Technology Facilities Council (STFC) under grant numbers PP/E007414/1 and ST/G00062X/1.

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