Density of Stable Interval Translation Maps

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Outline

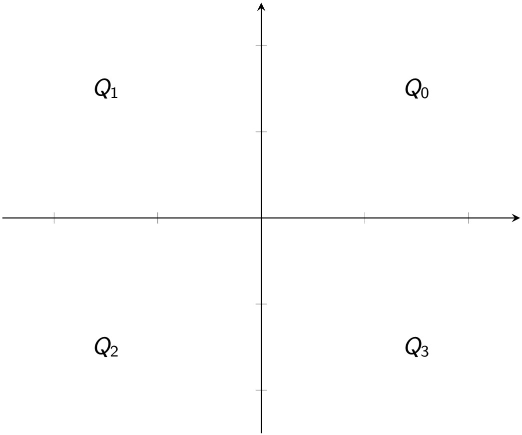
Piecewise Isometries

In recent decades the study of the dynamics of piecewise isometries has become increasingly popular.

In higher dimensions, their study is motivated by problems coming from engineering (error diffusion in digital printing, the three-capacitance mode, the buck converter, etc), machine learning and dynamics of games, where piecewise isometries arise as models.

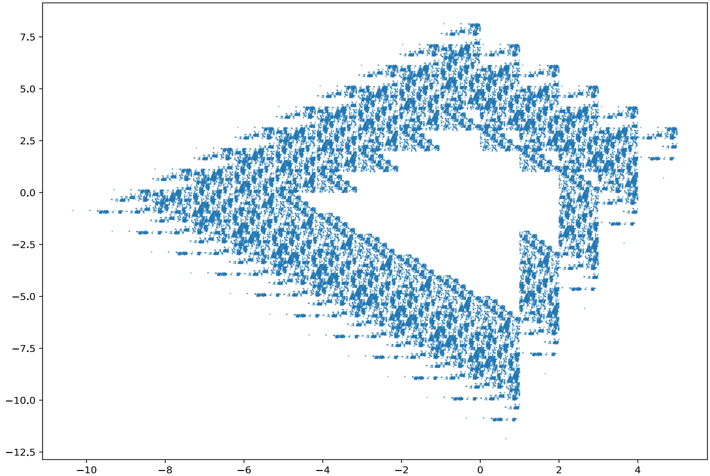
For example, if one studies population dynamics with two characteristics, a linear model would look like a piecewise translation on several regions in the plane.

A piecewise translation on 4 regions in $\mathbb{R}^2$



An attractor for a piecewise translation

Orbit (of length 95000) of translation map for a random initial value in Q0
translation = [-1. 1.] [-1.04719755 -1.   ] [1. -1.] [1.   3.14159265]



Dimension 1

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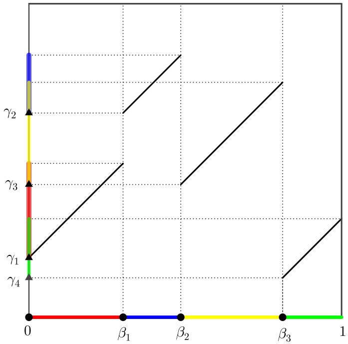
An ITM is a piecewise isometry $T : [0, 1) \to [0, 1)$ defined by two sets of parameters:

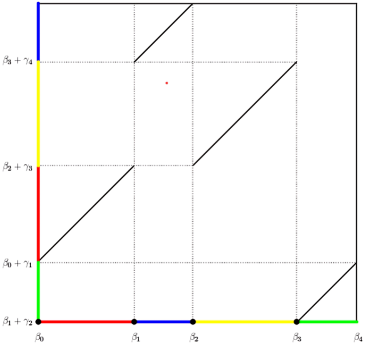
- **Critical points**: $0 < \beta_1 < \cdots < \beta_{r-1} < 1$;
- **Translation factors**: $\gamma_1, \ldots, \gamma_r$, so that we have:

$$T(x) = x + \gamma_i \text{ for } x \in [\beta_i, \beta_{i+1}).$$

ITM vs IET

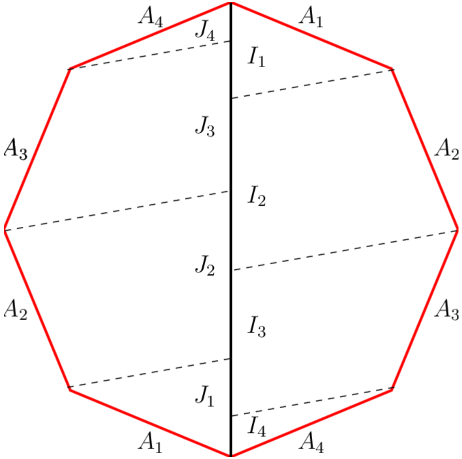


γ_2
γ_3
γ_1
γ_4
0
β_1
β_2
β_3
1

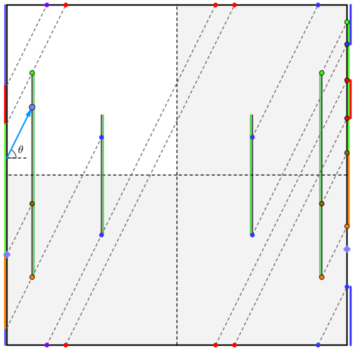


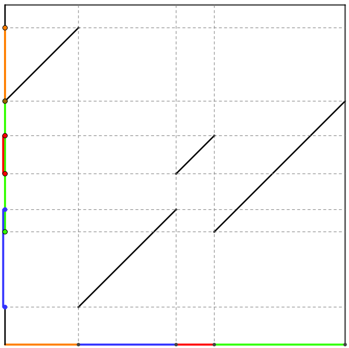
$\beta_3 + \gamma_4$
$\beta_2 + \gamma_3$
$\beta_0 + \gamma_1$
$\beta_1 + \gamma_2$
β_0
β_1
β_2
β_3
β_4

IET's $\iff$ Rational Polygonial Billiards



ITM's $\iff$ Spy Billiards





Basic Definitions

If T is an ITM, then let $X_n = T^n(I)$ for $n \geq 1$, and let

$$X = \bigcap_{n \geq 0} X_n.$$

An interval translation mapping T is said to be of:

- **Finite Type**: If $X_{n+1} = X_n$ for some n;
- **Infinite Type**: If $X_n \subsetneq X_{n+1}$ for all $n \geq 0$.

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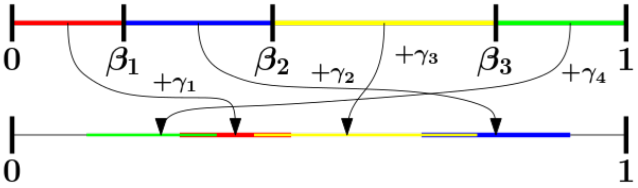
Classically, the set X has been called **the attractor** of T, but in fact the following result holds:

Lemma (Drach, S. and van Strien (2024))

$\overline{X}$ *is equal to the non-wandering set of* T.

ITM Dynamical Plane



Topological Classification

Both types of ITM's can be characterized in terms of the topology of their non-wandering sets. Here we assume that the map T is irreducible.

Theorem (Schmeling and Troubetzkoy (2000))

T is of finite type if and only if X is a finite union of intervals.

In this case, the return map to any interval J of X is an IET.

Theorem (Schmeling and Troubetzkoy (2000))

T is of infinite type if and only if X is a Cantor set.

The first example of an infinite type map was given in Boshernitzan and Kornfeld (1995), by constructing a 'self-similar' ITM.

Parameter space

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In lower dimensions, the best available result is:

Theorem (Volk (2012))

For $r = 3$, the set of finite type ITM's has full Lebesgue measure in the parameter space.

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In lower dimensions, the best available result is:

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For an arbitrary number of intervals, it is only known that:

Theorem (Schmeling and Troubetzkoy (2000))

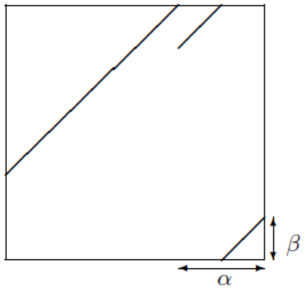
The set of finite type maps contains an open set.

There are stronger results only for some special parameter families, e.g. Bruin and Troubetzkoy (2002) and Bruin (2007).

A special class of ITM's

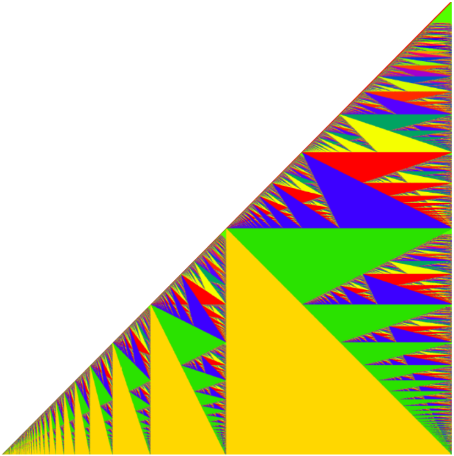
This class of maps has been first considered in Bruin and Troubetzkoy (2002). For the parameters set $\{0 \leq \beta \leq \alpha \leq 1\}$, let the map be defined as follows:

$$T_{\alpha,\beta}(x) = \begin{cases} x + \alpha & \text{for } x \in [0, 1-\alpha) \\ x + \beta & \text{for } x \in [1-\alpha, 1-\beta) \\ x + \beta - 1 & \text{for } x \in [1-\beta, 1) \end{cases}$$



Renormalization and Parameter space

This family is special because it comes with a nice renormalization operator acting on it. Because of this operator, we get the following picture for the parameter space:

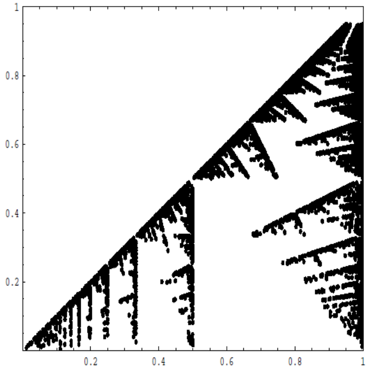


Infinite type in Parameter space

Moreover, this renormalization operator gives the following theorem:

Theorem (Bruin and Troubetzkoy (2002))

The set of infinite type maps for this special family has zero Lebesgue measure in parameter space.



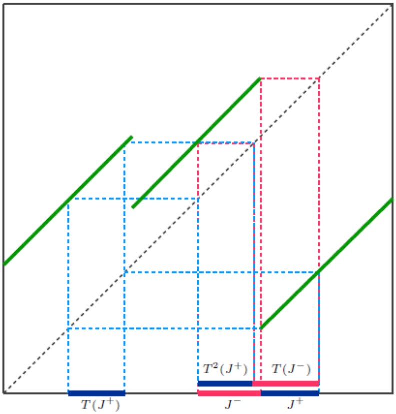
Stability

Definition (Stable maps)

We say that T is **stable** if there a neighbourhood $\mathcal{U}$ of T in the parameter space so that:

1. Each map in $\mathcal{U}$ is of finite type;
2. The non-wandering set $\mathcal{U} \ni \tilde{T} \mapsto \overline{X}(\tilde{T})$ moves continuously in the Hausdorff topology, and all these non-wandering sets are homeomorphic to $X(T)$;
3. The number of discontinuities in the non-wandering set is constant in $\mathcal{U}$.

Example of a stable map



The Main Conjectures

Conjecture (Stable maps in parameter space)

1. *The set of stable maps forms a dense subset of the parameter space.*
2. *Each stable map T is contained in an open simplex (minimal convex polytope) $[T]$ of maps which are all conjugate to it.*

Conjecture (Accumulation of infinite type maps)

Each simplex $[T]$ has infinite type maps arbitrarily close to its boundary.

Conjecture (Infinite type maps have measure zero)

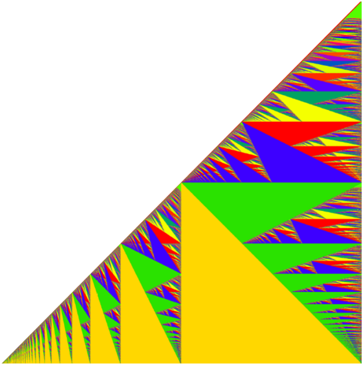
The set of all infinite type maps has measure zero in the parameter space.

Main Theorem

Our main result is the first part of the 'Stable maps in parameter space conjecture':

Theorem (Drach, S. and van Strien (2024))

The set of all stable maps is a dense subset of the parameter space $ITM(r)$.



The proof is in the following three steps:

Lemma (1 - Density of EP maps)

Eventually periodic maps are dense in the parameter space $ITM(r)$.

Theorem (2 - Characterization of Stability)

An ITM is stable if and only if it satisfies the Absence of Critical Connections (ACC) and Matching conditions.

Theorem (3 - Approximating EP maps with ACC + M)

Eventually periodic maps can be approximated by maps satisfying the ACC and Matching conditions.

Proof of Density of Stable Maps.

By the results above we have that:

$$
\begin{aligned}
&\overline{\{\text{Stable Maps}\}} \\
\overset{2}{=}\ &\overline{\{\text{Maps satisfying ACC and Matching}\}} \\
\overset{3}{\supseteq}\ &\overline{\{\text{Eventually periodic maps}\}} \\
\overset{1}{=}\ &ITM(r).
\end{aligned}
$$

$\square$

Absence of Critical Connections - ACC

Definition

We say that a finite-type T satisfies the **ACC** condition if the following holds:

1. (A1) For every interval J of X and each point $a \in J$, we have that the orbit of a up to and including the return time to J contains at most one critical point of T;
2. (A2) For every non-trivial interval J, we have that none of the boundary points of J land on discontinuities before returning to J;
3. (A3) The **ghost tree** $\mathcal{GT}(\beta)$ of every discontinuity β in the complement of X does not contain β.

Matching

Definition

We say that a finite-type T satisfies the **Matching** condition if for every non-trivial interval J of X, we have that at most one point a in J lands on a critical point before returning to J, and this point a is in the interior of J.

The reason we call this property Matching is that if a is the single point from above, then we must have that:

1. $J = [R_J(a^+), R_J(a^-))$;
2. $R_J^2(a^+) \sim R_J^2(a^-)$,

and thus the second iterates under R_J of a^+ and a^- must 'match'. The maps satisfying Matching are very rigid and our results show that this is the only possible type of rigidity.

Independence of critical itinerary vectors

The following Theorem is the main tool in the background of these results:

Theorem (Informal version)

The set of vectors corresponding to finite time itineraries of critical points form a linearly independent set. This means that the critical itineraries are 'independent', i.e. they can be individually controlled under perturbation.

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The main difficulty of this theorem is connecting dynamics to linear independence of vectors.

Thank you for your attention!

Bibliography

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