Investigating the Foundations of Physical Law
7 Nilpotent quantum field theory

A perturbation calculation

We are now ready to extend the discussion to a yet higher level of complexity to show that fundamental considerations still apply there, and lead us to results not so far achieved by any other method. Some of the calculations will be more involved than the ones we have done previously, but they are necessary to show how far the fundamental ideas can be influential, even at a level of relative complexity, and can solve problems for which there are no current answers. These will be followed by a less technical consideration of some more general aspects of the nilpotent quantum theory.

A significant amount of quantum field theory is already present in nilpotent quantum mechanics, where the nilpotent operator already provides interaction over the entire quantum field. Nilpotent 'wavefunctions' are the result of creation and annihilation operators acting on the vacuum state and come already second quantized. The mathematical proof of this is given in *Zero to Infinity*, in addition to results in QED, including an electron scattering calculation and a derivation of renormalization for interacting particles. Renormalization, in the sense of rescaling, as the interacting electric charges are screened by the vacuum field, should still apply as in conventional QED. What should no longer apply, if the nilpotent formalism is a more fundamentally symmetric, and in this sense more 'natural', version of quantum mechanics, is the need for the infinite self-energy term that has caused such problems in the past. If the nilpotent formalisms derives from the symmetries which are most fundamental in nature, then something of this kind should only arise as an artefact of alternative mathematical structures in which these symmetries are not fully preserved.

In the last lecture, we saw that an exact supersymmetry appears as a consequence of the nilpotent formalism and its representation of vacuum. In this case, we should expect a free fermion in vacuum to produce its own loop cancellations and its energy to acquire a finite value without renormalization. Free fermion plus boson loops should cancel at all levels of calculation, and there should be no hierarchy problem. We can examine this possibility by performing a basic perturbation calculation for first order coupling in QED, and seeing if it leads to zero in the case of a free fermion. Let us suppose a fermion acted on by the electromagnetic potentials ϕ, $\mathbf{A}$. Then, using only the lead terms of the spinors for simplicity, we have the standard equation

$$\left(-\left(\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}\boldsymbol{k}e\phi\right)-\left(\boldsymbol{i}\boldsymbol{i}\nabla-\boldsymbol{i}\boldsymbol{i}e\mathbf{A}\right)+\boldsymbol{j}m\right)\psi=0\,,$$

which can be rearranged as

$$\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)\psi=-e\left(+i\boldsymbol{k}\phi+i\boldsymbol{i}\mathbf{A}\right)\psi$$

We now apply a perturbation expansion to ψ, so that

$$\psi = \psi_0 + \psi_1 + \psi_2 + \dots ,$$

with

$$\psi_0 = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ e^{-i(Et - \mathbf{p.r})}$$

as the solution of the unperturbed equation:

$$\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)\psi=0\,,$$

which represents zeroth-order coupling, or a free fermion of momentum **p**.

Using the perturbation expansion, we can write

$$\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)(\psi_0+\psi_1+\psi_2+\ldots)=-e(\boldsymbol{k}\phi+i\boldsymbol{i}\mathbf{A})(\psi_0+\psi_1+\psi_2+\ldots)\,,$$

from which we can extract the first-order coupling, from the first iteration of the perturbation expansion, as

$$\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)\psi_1=-e(\boldsymbol{k}\phi+i\boldsymbol{i}\mathbf{A})\psi_0\ .$$

If, using a standard technique, we expand $(\boldsymbol{k}\phi + i\boldsymbol{i}\mathbf{A})$ as a Fourier series, and sum over momentum **k**, we obtain

$$(\boldsymbol{k}\phi + i\boldsymbol{i}\mathbf{A}) = \psi\,(\boldsymbol{k}\phi(\mathbf{k}) + i\boldsymbol{i}\mathbf{A}\ (\mathbf{k}))\ e^{i\mathbf{k.r}},$$

so that

$$\begin{aligned}\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)\psi_1&=-e\sum\left(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k})\right)e^{i\mathbf{k.r}}\psi_0\\&=-e\sum\left(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k})\right)e^{i\mathbf{k.r}}\left(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\right)e^{-i(Et-\mathbf{p.r})}\\&=-e\sum\left(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k})\right)\left(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\right)e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}\end{aligned}$$

If we now expand ψ_1 as

$$\psi_1=\sum\nu_1(E,\mathbf{p}+\mathbf{k})e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$

then

$$\sum\left(-\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}i\nabla+\boldsymbol{j}m\right)v_1(E,\mathbf{p}+\mathbf{k})e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$
$$=-e\sum(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k}))(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$

and

$$\sum(i\boldsymbol{k}E+\boldsymbol{i}(\mathbf{p}+\mathbf{k})+\boldsymbol{j}m)v_1(E,\mathbf{p}+\mathbf{k})e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$
$$=-e\sum(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k}))(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$

and, equating individual terms,

$$(i\boldsymbol{k}E+\boldsymbol{i}(\mathbf{p}+\mathbf{k})+\boldsymbol{j}m)\ v_1(E,\mathbf{p}+\mathbf{k})=-e\ (\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}\ (\mathbf{k}))(\ i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m).$$

We can write this in the form

$$v_1(E,\mathbf{p}+\mathbf{k})=-e[i\boldsymbol{k}E+\boldsymbol{i}(\mathbf{p}+\mathbf{k})+i\boldsymbol{j}m]^{-1}\ (\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}\ (\mathbf{k}))(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$$

which means that

$$\psi_1=-e\sum[i\boldsymbol{k}E+\boldsymbol{i}(\mathbf{p}+\mathbf{k})+\boldsymbol{j}m]^{-1}(\boldsymbol{k}\phi(\mathbf{k})+i\boldsymbol{i}\mathbf{A}(\mathbf{k}))(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)e^{-i(Et-(\mathbf{p}+\mathbf{k}).\mathbf{r})}$$

This is the wavefunction for first-order coupling, with a fermion absorbing or emitting a photon of momentum **k**.

But, if we observe the process in the rest frame of the fermion and eliminate any *external* source of potential, then $\mathbf{k}=0$, and the only possible potential $(\boldsymbol{k}\phi+i\boldsymbol{i}\mathbf{A})$ that could apply is the internal, self-interacting one, not dependent on **k**, which, in the rest frame, will reduce to the static value, $\boldsymbol{k}\phi$, with ϕ as a self-potential. In this case, ψ_1 becomes

$$\psi_1=-e[i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m]^{-1}(\boldsymbol{k}\phi)(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})}\ ,$$

as the summation is no longer strictly required for a single order of the pure self-interaction. Since we can also write this as

$$\psi_1=e(-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)(-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)(\boldsymbol{k}\phi)e^{-i(Et-\mathbf{p}.\mathbf{r})}\ ,$$

we see that $\psi_1=0$, for any fixed value of ψ. Clearly, this will also apply to higher orders of self-interaction. In other words, we have a first indication that a *non-interacting* nilpotent fermion requires no renormalization as a result of its self-energy.

The process could also be adapted for interacting particles, subject to external potentials. Here we can imagine redefining the E and **p** operators to incorporate external potentials to make them 'internal', while simultaneously changing the structure of the phase factor to accommodate this. The change of phase factor would, of course, require a corresponding change in the amplitude, which could be taken as redetermining the value of the coupling constant, e, as required. Ultimately, however, it is the structure of $(i\mathbf{k}E + \mathbf{i}\mathbf{p} + \mathbf{j}m)$ as a *nilpotent* which seemingly eliminates the infinite self-interaction terms in the perturbation expansion at the same time as showing that they are merely an expression of the nature of the nilpotent vacuum as a reflection of the exactly supersymmetric nature of the original particle state.

Cancellation of loops

If the previous argument is correct, then we should also be able to achieve the same result using the supersymmetric properties of the nilpotent operator to cancel fermion and boson loops directly. This is precisely what we would expect from a nilpotent system, where the total energy is zero, and one way of realising this would be to combine negative energy fermions with positive energy bosons. In the nilpotent formulation, as we have seen, every fermionic state has an intrinsic supersymmetric spin 1 bosonic vacuum partner with the same energy, momentum and mass. If we represent a spin ½ fermion by, say, $(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)$, and a spin –½ fermion by $(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)$, then each of these is unchanged by postmultiplication any number of times by the vacuum operator $\mathbf{k}\,(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)$ or $\mathbf{k}\,(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)$. However

$$(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\ldots$$

and

$$(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\mathbf{k}\,(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\ldots$$

are indistinguishable from

$$(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\pm i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp i\mathbf{k}E \pm \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\ldots$$

and

$$(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\pm i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,(\mp i\mathbf{k}E \mp \mathbf{i}\mathbf{p} + \mathbf{j}m)\,\ldots$$

which alternate spin ½ and spin –½ fermions with spin 1 and spin –1 bosons. In effect the fermion generates its own vacuum boson partner, with the same E, $\mathbf{p}$ and m. Since the nilpotent structure is founded on zero totality, with the vacuum and fermion being in both zero superposition and zero combination, we may assume that this is an indication that the total energy made by positive boson and negative fermion loops is zero.

The calculation is surprisingly easy, if we use results obtained from conventional QED. In fact, we can reduce it to simple arithmetic! Using a result from Peter West, *Introduction to Supersymmetry and Supergravity* (World Scientific, 1986), p. 15, we find that the vacuum energy for a particle of mass m and spin j is given by:

$$\frac{1}{2}(-1)^{2j}(2j+1)\int d^3k\sqrt{k^2+m_j^2} = \frac{1}{2}(-1)^{2j}(2j+1)\int d^3k\sqrt{k^2}\left(1+\frac{1}{2}\frac{m_j^2}{k^2}-\left(\frac{m_j^2}{k^2}\right)^2+...\right)$$

Here we see quartic, quadratic and logarithmic divergences. To remove these, we need to ensure that

$$\sum_j(-1)^{2j}(2j+1)=0$$
$$\sum_j(-1)^{2j}(2j+1)m_j^2=0$$
$$\sum_j(-1)^{2j}(2j+1)m_j^4=0$$

The first condition requires equal numbers of fermionic and bosonic degrees of freedom. If we have $j = \pm ½$ for the fermionic loops and $j = \pm 1$ for the bosonic loops, then

$$(-)^{2j}\,(2j+1) = -2 \qquad \text{for} \qquad j = ½$$
$$(-)^{2j}\,(2j+1) = 3 \qquad \text{for} \qquad j = 1$$
$$(-)^{2j}\,(2j+1) = 0 \qquad \text{for} \qquad j = -½$$
$$(-)^{2j}\,(2j+1) = -1 \qquad \text{for} \qquad j = -1$$

giving a total of

$$\sum_j(-1)^{2j}(2j+1) = -2+3+0-1=0$$

as required.

The other two conditions additionally require the fermions and bosons to have equal masses, which is true if the supersymmetry is intrinsic. Since all three conditions are fulfilled in the nilpotent formalism, it would appear that the intrinsic supersymmetry automatically removes the ultraviolet divergence.

The same hierarchy problem of divergence at each level of calculation also applies to bosons, most famously in the case of the spin 0 Higgs boson, but the same reasoning should also apply here. For a spin 0 boson, we have a fundamental structure of either

$$(\pm\, i\mathbf{k}E \pm i\mathbf{p} + \mathbf{j}m)\;(\mp\; i\mathbf{k}E \mp\; i\mathbf{p} + \mathbf{j}m)$$

or

$$(\pm\, i\mathbf{k}E \mp\; i\mathbf{p} + \mathbf{j}m)\;(\mp\; i\mathbf{k}E \pm i\mathbf{p} + \mathbf{j}m)$$

with a combination of spin ½ and spin –½ fermions / antifermions (to which we can again apply vacuum operators). (The application of vacuum operators to the two partners in the combination would leave alternate creations of fermion and boson as before.) Since

$$(-)^{2j}\,(2j+1)^{2j} = 1 \qquad \text{for} \qquad j = 0$$

we can find a combination of spin ½ and spin 0, together with spin –½ and spin 0, which will lead to

$$\sum_j (-1)^{2j}(2j+1) = -2+1+0+1 = 0$$

again as required, and, with *m* common to fermions and bosons, also fulfilling the second and third conditions. It would appear from this argument that the divergence is again removed and, in particular, that there is no reason to expect a hierarchy problem for the Higgs boson.

An additional, related problem is the matter / antimatter asymmetry between fermions and antifermions. Answers to this long-standing problem have been generally sought for in cosmology. It is assumed that almost equal amounts of fermions and antifermions were created in the big bang, with the fermions *slightly* in excess. Following this, the mysterious process of baryogenesis led to the annihilation of all the antifermions. But, there are very good reasons for seeing the asymmetry as generic, and, in fact, not an asymmetry at all. According to our foundational ideas, we have two vector spaces, characterised in the nilpotent representation by positive and negative energies. Between these two spaces, there are the same number of fermions and antifermions. Just as fermions, with *E*, can be seen as the characteristic particles defining real (observable) space, antifermions, with –*E*, can be seen as the characteristic particles defining vacuum space. In this description, there will not be a symmetry between the two particle types in either of the spaces. In the nilpotent structure, there are two energies, two directions of time, two directions of spin, two fermions and two antifermions. There are even two causalities: forward causality for the local state, the thing we observe; and *backward* or reverse causality for the unobservable nonlocal vacuum, which contains all the future causes of everything that will happen. The first corresponds with *E* and *t*, the second with –*E* and –*t*. Every fundamental concept provides us with a totality zero. Only rest mass has a purely positive value, but this is not fundamental, being a concept whose main purpose is to separate the observational part of the structure from that which is not observed.

Propagators

An aspect of quantum field theory which benefits massively from being cast in the nilpotent, or, we could say, *fundamental*, formalism is the use of *propagators* (which are, in principle, Green's functions). Though this is a rather technical subject, it does show the power of the nilpotent concepts in a particularly direct way, and also shows the significance of the dual spaces in creating fermions as point singularities. Here, again, there is a divergence which is not fundamental, but which we can show to be a result of using a less symmetric mathematical formalism. In addition, the concept of boson propagator has not been fully worked out, as there are in fact three boson propagators corresponding to the three different types of boson, and not a generic one which doesn't quite correspond to any of them.

We have seen that a physical singularity (perhaps the only one that can exist) emerges from the combination of two dual vector spaces, at the same time as a *zitterbewegung* is generated through the switching between them, which is equivalent to the switching between $+iE$ and $-iE$. Now, the conventional Feynman formalism for the particle propagator produces a 'pole' or singularity at exactly the division or 'switchover' between these two energy states, or between fermion and antifermion. It is a problem because it leads to a divergence at that point, which can only be dealt with by a mathematical subterfuge. In the nilpotent theory, however, the pole is no longer a 'naked' singularity, causing an infinite divergence, but one accommodated within the dual spaces on which the theory is founded. The nilpotent formalism incorporates the pole automatically without divergence because of its direct inclusion of vacuum states. Conventional theory assumes that a fermion propagator takes the form

$$S_F(p) = \frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2},$$

where $\not{p}$ represents $\gamma^\mu \partial_\mu$, or its eigenvalue, and that there is a singularity or 'pole' (p_0) where $p^2 - m^2 = 0$, the 'pole' being the origin of positron states. On either side of the pole there are positive energy states moving forwards in time, and negative energy states moving backwards in time, the terms ($\not{p} + m$) and ($-\not{p} + m$) being used to project out, respectively, the positive and negative energy states. The normal solution is to add an infinitesimal term $i\varepsilon$ to $p^2 - m^2$, so that $iS_F(p)$ becomes

$$iS_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} = \frac{(\not{p} + m)}{2p_0}\left(\frac{1}{p_0 - \sqrt{p^2 + m^2} + i\varepsilon} + \frac{1}{p_0 + \sqrt{p^2 + m^2} - i\varepsilon}\right)$$

and take a contour integral over the complex variable to give the solution

$$S_F(x - x') = \int d^3p \frac{1}{(2\pi)^3} \frac{m}{2E}\left[-i\theta(t - t')\sum_{r=1}^{2} \Psi(x)\overline{\Psi}(x') + i\theta(t' - t)\sum_{r=3}^{4} \Psi(x)\overline{\Psi}(x')\right]$$

with summations over the up and down spin states.

This mathematical subterfuge is unnecessary in the nilpotent formalism because the denominator of the propagator term is always a nonzero scalar. We write

$$S_F(p) = \frac{1}{(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)},$$

and choose our usual interpretation of the reciprocal of a nilpotent to give:

$$\frac{1}{(\pm ikE \pm \mathbf{ip} + jm)} = \frac{(\pm ikE \mp \mathbf{ip} - jm)}{(\pm ikE \pm \mathbf{ip} + jm)(\pm ikE \mp \mathbf{ip} - jm)} = \frac{(\pm ikE \mp \mathbf{ip} - jm)}{4(E^2 + p^2 + m^2)},$$

which is finite at all values. The integral is now simply

$$S_F(x - x') = \int d^3p \frac{1}{(2\pi)^3} \frac{m}{2E} \theta(t - t')\, \Psi(x)\, \bar{\Psi}(x'),$$

in which $\Psi(x)$ is the usual

$$\Psi(x) = (\pm ikE \pm \mathbf{ip} + jm) \exp(ipx),$$

with the phase factor written as a 4-vector, and the adjoint term becomes

$$\bar{\Psi}(x') = (\mp ikE \pm \mathbf{ip} + jm)(ik) \exp(-ipx').$$

Since the nilpotent formalism comes as a complete package with a single phase term, automatic second quantization, and the negative energy states matched with reverse time states, there is no averaging over spin states or separation of positive and negative energy states on opposite sides of a pole. The particle structure is itself the singularity. There is no division between the particle and antiparticle because the two come as a single unit incorporating real space and vacuum space on an equal footing.

The fermion propagator can also be used to define boson propagators. In conventional theory, we derive the boson propagator directly from the Klein-Gordon equation, while recognizing that its mathematical form depends on the choice of gauge:

$$\Delta_F(x - x') = \frac{\not{p} + m}{p^2 - m^2}.$$

This is because the Klein-Gordon operator

$$\left(\gamma^0 \frac{\partial}{\partial t} + \boldsymbol{\gamma}.\nabla + im\right)\left(\gamma^0 \frac{\partial}{\partial t} + \boldsymbol{\gamma}.\nabla - im\right) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)$$

is the only scalar product which can emerge from a linear differential operator defined as in the two bracketed terms on the left. The Klein-Gordon equation, however, is not specific to boson states or an identifier of them. It merely defines a universal zero condition which is true for all states, whether bosonic or fermionic. And, the propagator defined by conventional theory does not correspond to any known bosonic state. Instead, we have *three* boson propagators.

Spin 1: $$\Delta_F(x-x') = \frac{1}{(\pm i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)(\mp i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)},$$

Spin 0: $$\Delta_F(x-x') = \frac{1}{(\pm i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)(\mp i\boldsymbol{k}E \mp \mathbf{ip} + \boldsymbol{j}m)},$$

Paired Fermion: $$\Delta_F(x-x') = \frac{1}{(\pm i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \mp \mathbf{ip} + \boldsymbol{j}m)}.$$

Where the spin 1 bosons are massless (as in QED), we will have expressions like:

$$\Delta_F(x-x') = \frac{1}{(\pm i\boldsymbol{k}E \pm \mathbf{ip})(\mp i\boldsymbol{k}E \pm \mathbf{ip})}.$$

Clearly, the relationship of the fermion and boson propagators is of the form

$$S_F(x-x') = (i\gamma^\mu \partial_\mu + m)\,\Delta_F(x-x'),$$

or, in our notation,

$$S_F(x-x') = (\pm i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m)\,\Delta_F(x-x').$$

which is exactly the same relationship as is defined between fermion and boson in the nilpotent formalism. Now, using

$$iS_F(p) = \frac{1}{2p_0}\left(\frac{1}{p_0 - \sqrt{p^2+m^2} + i\varepsilon} + \frac{1}{p_0 + \sqrt{p^2+m^2} - i\varepsilon}\right),$$

which is the same as the conventional fermion propagator up to a factor ($\not{p} + m$), we can perform a contour integral which is similar to that for the fermion to produce

$$i\Delta_F(x-x') = \int d^3p\,\frac{1}{(2\pi)^3}\frac{1}{2\omega}\,\theta(t-t')\,\phi(x)\phi^*(x').$$

Here, ω takes the place of E / m, while $\phi(x)$ and $\phi(x')$ are now scalar wavefunctions. However, in our notation, they will be scalar products of ($\pm i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m$) exp ($ipx$) and ($\mp i\boldsymbol{k}E \pm \mathbf{ip} + \boldsymbol{j}m$) exp ($ipx'$) and $\phi(x)\phi^*(x')$ reduces to a product of a scalar term, which can be removed by normalization, and exp $ip(x-x')$.

In off-mass-shell conditions, where $E^2 \neq p^2 + m^2$, poles in the propagator are a mathematical, rather than physical, problem, and removed by the use of $i\varepsilon$ and the contour integral, which is *ad hoc* but effective. But, in the specific case of massless bosons, such as the photon or gluon, conventional theory cannot prevent 'infrared' divergences appearing in the expression for the propagator when such bosons are emitted from an initial or final stage which is on the mass shell. Such divergences, however, do not occur where there is no naked pole, as in the nilpotent expression.

The nilpotent definition of the boson propagator not only shows that one of the principal divergences in quantum electrodynamics is, as the procedure used to remove it would suggest, merely an artefact of the mathematical structure we have imposed, and not of a fundamentally physical nature, but also suggests that the formalism which removes it is a more exact representation of the fundamental physics. Ultimately, this is because it allows an exact representation of the vacuum simultaneously with the fermionic state, in line with the dual spaces needed to generate a fermion singularity.

A weak interaction calculation

To complete the more technical aspects of nilpotent quantum field theory, we can show how it would be used in a weak interaction calculation in the usual four-point Fermi interaction approximation. Though there is nothing here that can't be done by conventional methods, it is interesting to see how a different approach could possibly be more fertile in more complex problems. Conventionally, in describing a weak interaction, such as muon decay, we calculate the traces of the tensors using the trace theorem:

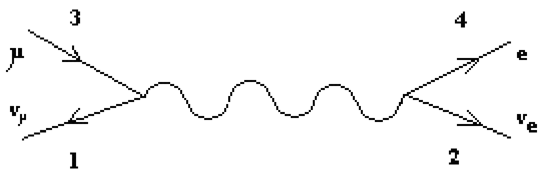
$$Tr\,[\gamma^\mu (1-\gamma^5)\, \not{p}_1 \gamma^\nu (1-\gamma^5)\, \not{p}_2]\; Tr\,[\gamma_\mu (1-\gamma^5)\, \not{p}_3 \gamma_\nu (1-\gamma^5)\, \not{p}_4]$$
$$= 256\, (p_1 \cdot p_2)\, (p_3 \cdot p_4)$$

This is because, for an invariant amplitude $\mathcal{M}$, for muon decay,

$$|\mathcal{M}|^2 = \frac{G^2}{2} Tr\left[\gamma_\mu \left(1-\gamma^5\right)\bar{\nu}_\mu (p_1) \gamma^\nu \left(1-\gamma^5\right) \mu (p_2)\right] Tr\left[\bar{\nu}_e \left(1-\gamma^5\right)(p_3) e \left(1-\gamma^5\right)(p_4)\right]$$

and the spin-averaged probability, $|\mathcal{M}|^2$, is given by

$$\tfrac{1}{2} \sum_{spins} Tr[\gamma^\mu \left(1-\gamma^5\right)\bar{\nu}_\mu (p_1) \gamma^\nu \left(1-\gamma^5\right) \mu (p_2)] Tr[\gamma_\mu \left(1-\gamma^5\right)\bar{\nu}_e (p_3) \gamma_\nu \left(1-\gamma^5\right) e (p_4)]$$
$$= 64 G^2 \left(p_1 \cdot p_2\right)\left(p_3 \cdot p_4\right)$$



Using nilpotents we can take a different approach, by directly investigating the 'bosonic' states at the two vertices. First we take:

$$\gamma^\mu = \gamma^0 + \gamma^1 + \gamma^2 + \gamma^3 = i\boldsymbol{k} + \mathbf{i}\boldsymbol{i} + \mathbf{j}\boldsymbol{i} + \mathbf{k}\boldsymbol{i} = i\boldsymbol{k} + \mathbf{1}\boldsymbol{i},$$

combining the vectors **i**, **j**, **k**, for convenience, into the single unit **1**. Then:

$$(1 - \gamma^5) = (1 - ij)$$

$$\gamma^\mu (1 - \gamma^5) = (ik + 1i)(1 - ij) = ik + i + 1i - ik1$$

Here, we assume:

$$\not{p}_1 = (\pm ikE_1 \pm i\,\mathbf{p}_1),$$

but, by directly incorporating a mass term, we could use:

$$(\pm ikE_1 \pm i\,\mathbf{p}_1 + jm_1).$$

So
$$\gamma^\mu (1 - \gamma^5)\, \not{p}_1 \gamma_\nu (1 - \gamma^5)\, \not{p}_3$$
$$= (ik + 1i)(1 - ij)(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(ik + 1i)(1 - ij)(\pm ikE_2 \pm i\,\mathbf{p}_2)$$
$$= (ik + 1i - i - i1k)\,(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(ik + 1i - i - i1k)(\pm ikE_2 \pm i\,\mathbf{p}_2).$$

Using

$$(ik)(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(ik) = (\pm ikE_1 \mp i\,\mathbf{p}_1)$$
$$(1i)(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(1i) = (\pm ikE_1 \mp i\,\mathbf{p}_1)$$
$$(-i)(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(-i) = (\pm ikE_1 \mp i\,\mathbf{p}_1)$$
$$(i1k)\,(\pm ikE_1 \pm i\,\mathbf{p}_1)\,(i1k) = (\pm ikE_1 \mp i\,\mathbf{p}_1)$$

we obtain a total of 4 $(\pm ikE_1 \mp i\,\mathbf{p}_1)$ for this scalar product, or, for a state vector representing an antifermion (where $E_1 \to -E_1$), this would become 4 $(\mp ikE_1 \mp i\,\mathbf{p}_1)$.

For a vertex involving a fermion, with state vector $(\pm ikE_3 \pm i\,\mathbf{p}_3)$, taking over all four terms in the Dirac spinor, each

$$4\,(\mp ikE_1 \mp i\,\mathbf{p}_1)\,(\pm ikE_3 \pm i\,\mathbf{p}_3) = 4 \times 4\,(E_1E_3 - \mathbf{p}_1\,\mathbf{p}_3) = -\,16\,(p_1 \,.\, p_3),$$

and the equivalent of $Tr\,[\gamma^\mu (1 - \gamma^5)\, \not{p}_1 \gamma^\nu (1 - \gamma^5)\, \not{p}_2]\; Tr\,[\gamma_\mu (1 - \gamma^5)\, \not{p}_3 \gamma_\nu (1 - \gamma^5)\, \not{p}_4]$ becomes 256 $(p_1 \,.\, p_3)\,(p_2 \,.\, p_4)$, leading once again to a spin-averaged probability:

$$|\mathcal{M}|^2 = 64G^2 (p_1 . p_2)(p_3 . p_4).$$

This approach is only valid for antifermion-fermion vertices, where the $V - A$ term $\gamma^\mu(1 - \gamma^5)$ is included – that is, where the interaction is dipolar and single-handed. Otherwise, the product of the two scalar products does not correspond with the product of the two traces. In this method, however, the terms $p_1 \,.\, p_3$ and $p_2 \,.\, p_4$ can be easily extended to become scalar products of nilpotent operators where mass is to be taken into account.

BRST quantization

Though the Dirac nilpotent operator is automatically second quantized, and so already incorporates a full quantum field representation, it is interesting to look at more conventional approaches to field quantization. One of these helps us to demonstrate the relation between charge and energy operators, on which our formalism was constructed. In standard theory, field quantization requires gauge fixing (or removal of gauge invariance) before propagators can be constructed. The canonical quantization of the electromagnetic field uses Coulomb gauge, but this means that Lorentz invariance must be broken. The path integral approach allows us to use any gauge, and so maintain Lorentz invariance, but the problem now is the introduction of nonphysical or 'fictitious' Fadeev-Popov ghost fields. A version used in string theory (BRST) eliminates the ghost fields by packaging all the information into a single operator, applied to the Lagrangian.

Significantly, the BRST operator (δ_{BRST}) is a nilpotent. This operator can be used to construct a Noether current (J_μ), corresponding to a nilpotent BRST conserved fermionic charge (Q_{BRST}). The condition for defining a physical state then becomes

$$Q_{\text{BRST}} \left|\psi\right\rangle = 0.$$

In the Dirac nilpotent formulation, ($\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$), which applies only to physical (mass shell) states, is already second quantized, and a nilpotent operator of the form δ_{BRST}. It is, also, a nilpotent *charge* operator of the form Q_{BRST}, but extended to incorporate weak and strong, as well as electromagnetic, charges. It is, finally, in its eigenvalue form, identical to $\left|\psi\right\rangle$. So the three possible meanings for the expression ($\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$) apply, respectively, to: E and $\mathbf{p}$ interpreted as differential operators in time and space; E, $\mathbf{p}$ and m as coefficients determining the nature of the charges specified by $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$; and E and $\mathbf{p}$ interpreted as eigenvalues of energy and momentum. The nilpotent Dirac operator thus supplies simultaneously all the characteristics which the separate BRST terms δ_{BRST}, Q_{BRST}, and $\left|\psi\right\rangle$ require.

Mass generation

If we consider the boson structures we have outlined as defining the vertices for boson production via the weak interaction, then it appears from the impossibility of creating a massless spin 0 boson and the two sharply defined helicity states for hypothetical massless fermions (lecture 5) that the pure weak interaction requires left-handed fermions and right-handed antifermions. In other words, it requires both a charge-conjugation violation and a simultaneous parity or time-reversal violation.

We can see in principle how this leads to mass generation by some process at least resembling the Higgs mechanism. Let us imagine a fermionic vacuum state with zero mass, say ($i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p}$). An ideal vacuum would maintain exact and absolute C, P and T symmetries. Under C transformation, ($i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p}$) would become ($-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p}$), with which it would be indistinguishable under normalization. No bosonic state is required for such a transformation, where the states are identical. If, however, the vacuum state is degenerate in some way under charge conjugation (as supposed in the weak interaction), then (***ik****E* + ***i*****p**)`will be transformable into a state which can be distinguished from it, and the bosonic state (***ik****E* + ***i*****p**) (–***ik****E* – ***i*****p**) will necessarily exist, the (***ik****E* + ***i*****p**) here being the new state, and the (–***ik****E* – ***i*****p**) being the annihilation of the old state. This can only be true, however, if the state has nonzero mass and becomes the spin 0 'Higgs boson' (***ik****E* + ***i*****p** + ***j****m*) (–***ik****E* – ***i*****p** + ***j****m*). The mechanism, which produces this state, and removes the masslessness of the boson, requires the fixing of a gauge for the weak interaction (a 'filled' weak vacuum), which manifests itself in the massive intermediate bosons, *W* and *Z*.

From the structures of bosons and the consideration of fermion spin, it would seem that mass and helicity are closely related. If the degree of left-handed helicity is determined by the ratio (±) ***i*****p** / (±) ***ik****E*, then the addition of a mass term will change this ratio. Similarly, a change in the helicity ratio will also affect the mass. If the weak interaction is only responsive to left-handed helicity states in fermions, then right-handed states will be intrinsically passive, so having no other function except to generate mass. The presence of two helicity states will be a signature of the presence of mass. The *SU*(2) of weak isospin, which, in effect, expresses the invariability of the weak interaction to the addition of an opposite degree of helicity (due to the presence of, say, mass or electric charge) is thus related indirectly to the *SU*(2) of spin, which is a simple description of the existence of two helicity states. It is significant that the *zitterbewegung* frequency, which is a measure of the switching of helicity states, depends only on the fermion's mass. Mass is in some sense created by it, or is in some sense an expression of it. The restructuring of space and time variation or energy and momentum, via the phase factor, during an interaction, leads to a creation or annihilation of mass, which manifests itself in the restructuring of the *zitterbewegung*.

String theory

String theory was developed to try to unify the four interactions and to remove the divergences assumed to be a consequence of assuming that point-like particles were the sources of physical interactions. Fundamental theorems suggested that 10 dimensions were the number required to remove all the anomalies from physics. The problem was to show why these 10 reduced to 4 dimensions of space-time in the world we actually observe, and the argument was that they were 'compactified'. The famous analogy with a hosepipe suggested that a compactification in this form, rolling the dimension in a circle around an uncompactified dimension, had connections with the Kaluza-Klein theory which had extended Einstein's general relativity to 5 dimensions to include electric charge. The circle compactification also had the *U*(1) symmetry required of the electric interaction. Whether the dimensionalities of string theory are all spatial or temporal dimensions becomes indeterminate if the compactification to the extra dimensions occurs below the Planck length, as is generally assumed. String theory was never intended to be a model-dependent theory, but this aspect was meant to be a stage in leading to a more abstract, general theory to which the models would be approximations. The object was to find the correct vacuum which would generate the required compactification. There are famously many possible string theories, but they can be classified into 5 types, and a theory known as membrane theory or M-theory proposed that the different types could be combined if an extra dimension was added to connect them. The objects of M-theory were 2-dimensional spatial objects in 1-dimensional time, as opposed to the 1-dimensional spatial objects in 1-dimensional time of string theory. The combination of the 5 types of objects in 1-dimensional string theory in M-theory, however, has not yet happened. There is only an indication that it might be possible.

The main interest for us in string theory is not in producing a new one, but in the claim about the fundamental significance of 10 dimensions in removing anomalies, for it is immediately apparent that the nilpotent operator (± ***ik****E* ± ***i*****p** + ***j****m*) can, in fact, be regarded as a 10-dimensional object embedded in Hilbert space or equivalent. The reason why it can be regarded as 10-dimensional originates in the fact that it expresses a fundamental duality between two 'spaces'. Though each of these is intrinsically 3-dimensional, each becomes expanded to five in the process of combination. Thus, there are five dimensions for *iE*, **p**, *m* and 5 for ***k***, ***i***, ***j***; and six of these (all but *iE* and **p**, or the equivalent time and space) are compactified in the sense of being fixed. The double 5-dimensionality also has connections with the Kaluza-Klein theory, which was originally two separate theories, one of which tried to explain invariant mass and the other electric charge, for the fifth term in the nilpotent has both these characteristics and it also represents a $U(1)$ symmetry.

Now, John Baez has claimed that the 10 dimensions indicate the existence of 2-dimensional strings (1 dimension of space, 1 of time) in an 8-dimensional space, which could be octonion, and has suggested that an octonion space is the true basis of physics. Clifford algebra has the property of incorporating different dimensionalities in the same expression, and the nilpotent algebra is also an 8-dimensional object in two related senses. Two of the dimensions are made redundant because of the nilpotent structure, and the whole structure originates in the combination of the 4 dimensions of space and time and the 4 of mass and charge as a *broken* octonion. Yet another way of looking at the fermionic nilpotent is as a 3-'dimensional' structure in ***k***, ***i***, ***j***, though point-like in space, with one of the three 'dimensions' redundant, and this also corresponds to the 2-dimensional objects of string theory and the 3-dimensional ones of membrane theory. In this sense, the 2 or 3 redundant dimensions, which would, in string or M-theory be a string or brane connecting two 3- or 4-dimensional systems, become reduced to a point with zero dimension.

A classic prescription for a perfect string theory is one in which 'self-duality in phase space determines vacuum selection' (Alon Faraggi). The nilpotent certainly fulfils this criterion and it is also a mass-shell system and incorporates the right groups. It incorporates gravity / gauge theory correspondence and the holographic principle. Though we have no need for a model-dependent theory to incorporate the interactions, it is important to be able to satisfy all the conditions that appeared to make string theory, *or a more fundamental abstract theory, of which the model-dependent theories are approximations*, seemingly necessary. In a sense the reduction of the 2-dimensional strings or 3-dimensional membranes to points in real space is the ultimate reduction of the model-dependent objects of string theory or M-theory to abstraction – a 'string theory without strings'. It is significant that the nilpotent formalism achieves this through solving the problem of vacuum.

In string theory, mass is generated by the vibrations of the strings, which replace point particles. However, this mass-generating mechanism is already incorporated in the point particle concept (as the Lamb shift makes clear), and relates to the Berry phase and *zitterbewegung*. It comes from the dual vector spaces needed to define a point particle, because the duality ensures that *zitterbewegung* (and hence vacuum fluctuation, the Lamb shift, etc.) is the origin of fermionic mass, and it *requires* a pole or singularity. In the nilpotent theory, rest mass always comes from defining a singularity through a double vector space. The very act of defining a point particle is also the same as ensuring that it undergoes vacuum fluctuations, or equivalent, and therefore generates mass. Again connecting with string theory, it is the same duality as that between gravity and gauge theory or between the local and nonlocal.

One fermion theory

Of the various attempts to provide a 'physical' picture of quantum mechanics, one is especially interesting with reference to the nilpotent connection between fermion and vacuum. This is Wheeler's 'one-electron theory of the universe', now extended to a 'single fermion theory'. Here, a single fermion in different spatial and temporal states becomes equivalent to many fermions appearing simultaneously. All the other fermions after the first are this fermion in different space and time states, positive and negative. In the nilpotent formalism, of course, each single fermion sees all other fermions as constituting a vacuum which is a mirror image of itself. It is relevant here that the usual objections against the one-fermion theory no longer apply, as the total fermion structure requires equal numbers of fermions and antifermions existing simultaneously in *two different* but completely dual vector spaces – 'real' or observed space and vacuum space. In addition to exact equality between fermions and antifermions, the dual spaces also ensure that there is no mutual annihilation.

A consequence of this representation is that we can take an ensemble of fermions as a single fermion, and so justify applying the nilpotent condition in some form to larger structures, as is also evident from the fact that a version of the nilpotent Dirac equation using a discrete version of calculus (involving commutators rather than differentials) applies to classically discrete, as well as quantum systems. The scale-independence of the single fermion / ensemble duality also fits exactly with the renormalization group procedure.

An ensemble is not localised as narrowly as a single fermion, so its vacuum will not be quite as nonlocal. If we take the whole universe to be a fermion, we can imagine all the possible space and time conditions (and bit flips in the terminology of Seth Lloyd) as constituting this universe – which is what we have called vacuum. This includes all the states to which the single fermion could possibly aspire over time. In other words a real single fermion includes the entire possible history of the universe within its event horizon (the backward causality we referred to), making sense of our thermodynamics and evolutionary theory, as a unique birth-ordering. However, this does not require determinism because we can only define the entire history if we localize the fermion exactly, which of course we cannot do. It is only an ideal. So, we have an exact idea of what we mean by nonlocal, as all the other potential states, in space and time, which would of course be determined by the real states. The symmetry is perfect. Fixing a particular moment in time is localizing in time, in the same way as we localize in space.

Dualities in nilpotent quantum theory

The nilpotent quantum theory is built upon fundamental dualities, and many dualities are intrinsic components of its structure. The operator and wavefunction are dual, but there are also dualities at many other levels. A related duality between fermion and vacuum originates comes from the principle that, by defining a fermion state, we are also defining a fundamental singularity. To define a singularity we are forced to use a dualistic structure by simultaneously defining what is not singular. While we can view the fermion as a singularity with connections leading out to the rest of the universe, the vacuum acts as a kind of 'inverse singularity', with connections from the rest of the universe leading into the singularity constituting the fermion state.

This duality ensures that vacuum is not something separated from the fermion. It is an intrinsic component of its definition, and of the spinor structure needed to define the fermion as a singular state. It is the reason why the fermion has half-integral spin – we can only define it by simultaneously splitting the universe into two halves which are mirror images of each other. The duality manifests itself physically in the phenomenon of *zitterbewegung*. Using either operator or amplitude, we define ($\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$) as a 4-spinor, with 4 terms (each of which is nilpotent) arranged as a column / row vector. In the convention we have used, the 'real' state (the one subject to physical observation) is determined by the signs of E and **p** in the first term. The other three states are like three 'dimensions' of vacuum, the states into which the real term could transform by respective P, T or C transformations. The duality ensures that fermion and vacuum occupy separate 3-dimensional 'spaces', which are combined in the γ algebra defining the singularity state. It can be shown that these 'spaces', though seemingly different, are truly dual, each containing the same information, and that this duality manifests itself directly in many physical forms.

Some examples of the duality can be listed. The following effects have explanations based on real space (coded blue), and alternative explanations based on 'vacuum space' or, as I sometimes describe it for totality zero, 'antispace' (coded red):

Pauli exclusion	antisymmetric wavefunctions	nilpotency
angular momentum	p_1, p_2, p_3	iE, p, m
spin ½	anticommutation of **p**	Thomas precession
SR velocity addition	2 space components	space and time
holographic principle	space × space	space × time

In the case of Pauli exclusion, we can represent both p_1, p_2, p_3 and iE, p, m on orthogonal axes, to give a resultant 'vector' that is unique for each state. There are two mappings, on $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, $\boldsymbol{\sigma}^3$ and on $\boldsymbol{\Sigma}^1$, $\boldsymbol{\Sigma}^2$, $\boldsymbol{\Sigma}^3$, and these are dual. Both sets of coordinates yield information about the same physical quantity: angular momentum. For full specification, angular momentum requires three separate pieces of information – magnitude, direction and handedness – and this is provided when iE, **p** and m are combined. It is also provided when we use all the information incorporated in the **p** vector alone.

We have already discussed how we can derive spin ½ for fermions either by taking the commutator of the spin pseudovector $\boldsymbol{\sigma} = -\mathbf{1}$ and the Hamiltonian, and deriving the half-integral spin from the anticommuting aspects of the components of **p**, or, in terms of the Thomas precession, which is a relativistic correction. That is, we can derive spin ½ using either the (multivariate) vector properties of space (using $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, $\boldsymbol{\sigma}^3$) *or* the relativistic connection between space and time (using $\boldsymbol{\Sigma}^1$, $\boldsymbol{\Sigma}^2$, $\boldsymbol{\Sigma}^3$); the 3-dimensional 'spaces' involved are totally dual. The same applies to the velocity addition law in special relativity, which can be derived using either two dimensions of space (which generates the $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, $\boldsymbol{\sigma}^3$ structure of Euclidean space) or one of space relativistically connected with one of time (which generates the $\boldsymbol{\Sigma}^1$, $\boldsymbol{\Sigma}^2$, $\boldsymbol{\Sigma}^3$ connection between space, time and proper time). The holographic principle provides another example where this occurs (and this is completely defined for the fermionic case by the nilpotent structure); here, the bounding 'area' specifying a system can be defined either by two spatial coordinates or one of space and one of time.

In all these examples, the two vector spaces required to define fermion structure as a singularity are completely dual, though the symmetry of one is preserved while that of the other is broken. As in the parameter group from which it is derived, one condition is necessary to define the opposite in the other, and, in this way, the opposing conditions ultimately provide the same information, in the same way as localised fermion state and nonlocalised vacuum, or operator acting on phase factor and amplitude. *Zitterbewegung* is generally interpreted as a switching between a fermion state and its vacuum, but it is also an expression of the duality between the 'real' space of $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$, $\boldsymbol{\sigma}^3$ and the 'vacuum space' of $\boldsymbol{\Sigma}^1$, $\boldsymbol{\Sigma}^2$, $\boldsymbol{\Sigma}^3$, neither of which is privileged. Both give an equally correct description of the state and must be simultaneously valid, even though we can only observe one at any given moment, and even the choice of broken / unbroken rotation symmetries between the components can be reversed by switching the space of observation from 'real' to 'vacuum' space.

The intrinsically dualistic nature of the fermion is most readily apparent when it is described by the self-dual nilpotent form of quantum mechanics, which is founded on the commutative combination of two vector spaces, each of which is exactly dual to the other. It is remarkable that physics has never been successfully founded on a concept of a single space, however distorted, but it does seem to respond well to being structured on two! From the initial duality, many others emerge, for example, those between fermion and vacuum, fermion and vacuum boson, operator and amplitude, nilpotent and idempotent, broken and unbroken symmetries. These dualities allow the same mathematical structures (or the same structures but for sign changes) to describe apparently dissimilar objects, and so explain how the creation of a fermionic singularity effectively splits the universe into two halves that are mathematically and physically, if not observationally, equivalent.

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