Foundations of Physical Law
4 A fundamental symmetry

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A potential primitive structure

With only one conjecture – itself a reasonable extension from experimental evidence – we have a potential structure for the most primitive level in physics.

If the structure is valid, it suggests resolutions of many of the issues connected with fundamental ideas such as space, time and 3-dimensionality, and this without a single new equation, exactly as we would expect at this level.

Our next target, before we begin to explore the first level of complexity, is to find the mathematical basis, if any, behind this structure, and so tighten even further the conditions required by a fundamental theory.

Dualities

We have examined three dualities between four parameters that seem to be fundamental. The pairings are between quantities that have opposite characteristics, but each is determined by the other.

Duality of this kind is found right through mathematics, and finding it also in physics seems to suggest that the subjects are even closer in their origins than we had previously thought, and that each plays a significant role in extending the other.

Remarkably, the abstract dualities seem to be responsible for characteristics that we may have once considered solidly *physical*. In the structure we have been investigating, we notice that each parameter is paired with a different partner for each duality.

Conserved	*Nonconserved*
Mass	Space
Charge	Time
Real	*Imaginary*
Mass	Charge
Space	Time
Anticommutative	*Commutative*
Space	Mass
Charge	Time

The factor 2

Such dualities are everywhere in physics, their signature is the factor 2.

Duality, as such, often more important in creating this factor than the particular duality, allowing us to switch between them. This suggests that the real explanation lies buried at a very deep level.

For a seemingly fundamental explanation of a physical phenomenon, the explanation, although correct, is often not unique.

½ spin of the electron, and doubling of the observed magnetic moment when the electron is aligned in a magnetic field, compared to the value expected from standard classical theory. Is it quantum, or relativistic, or relativistic quantum, or something else?

The factor 2

The answer is both none of them and all of them!

The standard derivation from the Dirac equation using commutators. The anticommutativity of the momentum operator generates a factor 2, which becomes ½ when transferred to the other side of the equation.

However, the first successful explanation of the effect made the doubling derive from a relativistic correction of the frame of rotation, known as the Thomas precession. So, this suggests, at least, that we need either quantum theory or relativity to make the correction.

The factor 2

Yet there is another, astonishingly simple, explanation of the magnetic effect using only *classical* theory. Here, we have two energy equations.

One describes the kinetic energy which an object acquires during changing conditions (classically $\frac{1}{2}mv^2$), while the other describes the potential energy (typically, mv^2, using the virial theorem, $V = 2T$) which would maintain the system in a steady state.

Using the kinetic energy equation *at the moment when the magnetic field is switched on*, we obtain the correct factor without involving either quantum theory or relativity!

The factor 2

None of the explanations is false. All are completely true, but none uniquely so, and the real explanation is somewhere else. The fact of duality is more significant than any of the particular applications. There are a large number of phenomena in physics where this factor occurs (see lecture 10), and all of them have multiple explanations!

classical kinetic energy equation	conserved / nonconserved duality
Thomas precession (relativity)	real / imaginary duality
Dirac equation (quantum mechanics)	commutative / anticommutative duality

A group of order 4

Our lengthy analysis in the last lecture suggested that some symmetry was at work between the fundamental parameters. What symmetry this is can be seen by arranging the dual properties in a table:

mass	conserved	real	commutative
time	nonconserved	imaginary	commutative
charge	conserved	imaginary	anticommutative
space	nonconserved	real	anticommutative

A group of order 4

We can make the symmetry more obvious by using the symbols, x, y and z, to represent the properties, with $-x$, $-y$ and $-z$ representing the exactly opposite properties, or, as we can conveniently describe them, the 'antiproperties':

mass	x	y	z
time	$-x$	$-y$	z
charge	x	$-y$	$-z$
space	$-x$	y	$-z$

A key group

This is one of the group structures that we examined in the second lecture, the Klein-4 group or noncyclic group of order 4. We can produce this group by devising a binary operation of the form:

$$x * x = -x * -x = x$$
$$x * -x = -x * x = -x$$
$$x * y = y * -x = 0$$

and similarly for y and z. Effectively, any combination of a single property or antiproperty with itself gives the *property*; but a combination of a property with its antiproperty gives the *antiproperty*; while the combination of any property with any other property or antiproperty vanishes.

Groups: $\mathcal{D}_2$

This gives us the group table:

*	mass	charge	time	space
mass	mass	charge	time	space
charge	charge	mass	space	time
time	time	space	mass	charge
space	space	time	charge	mass

Groups: $\mathcal{D}_2$

which, as we have said previously, has the same structure as the H_4 double algebra:

*	1	***ii***	***jj***	***kk***
1	1	***ii***	***jj***	***kk***
ii	***ii***	1	***kk***	***jj***
jj	***jj***	***kk***	1	***ii***
kk	***kk***	***jj***	***ii***	1

In the current representation, mass is the identity element, while each element is its own inverse.

A group of order 4

But this representation is not unique. We can easily rearrange the algebraic symbols to make any of space, time or charge the identity element.

For example, we could have assigned the symbols in the form:

mass	$-x$	y	$-z$
time	x	$-y$	$-z$
charge	$-x$	$-y$	z
space	x	y	z

Groups: $\mathcal{D}_2$

In this representation, space becomes the identity element, and the group table is now:

*	space	time	mass	charge
space	space	time	mass	charge
time	time	space	charge	mass
mass	mass	charge	space	time
charge	charge	mass	time	space

Groups: $\mathcal{D}_2$

The significance of the group structure should strike the viewer immediately. If the representation is true, then we can no longer continue to do physics as though it doesn't exist.

A symmetry of this kind will become an astonishingly powerful tool for generating further physical information, so justifying the methodology used in deriving it. It seems to be an exact symmetry, not merely an approximation to some more fundamental truth, suggesting that we really are operating at a fundamental level.

We will soon show that the only partial conjecture that we made – concerning charge – can be fully justified by the results.

Groups: $\mathcal{D}_2$

The method by which it was derived, and the principles on which it was based, suggest that the group might be *exclusive*.

In that case, we have an extra constraint, that there is no physical information that is not contained within it, which we can use to derive laws of physics and states of matter. Such a constraint would be even more powerful and general than those generated at present in physics by the conservation laws.

Of course, with information at such a basic level, it will take a great deal of ingenuity to develop the more complex models which apply to physics as we mostly know it, but we hope to show that many will arise as purely natural developments.

Groups: $\mathcal{D}_2$

The binary operation for the group need not be restricted to the one we have defined. The generality of the symmetry suggests that *any* binary operation between two elements must be extended to the whole group.

Already have a binary operation between the units of space and time, and those of mass and charge, through the 4-vector and quaternion representations.

The same numerical relation must therefore apply to all the parameters, and, because any element can be the identity, and each is its own inverse, there must be a relation between the inverse units of each parameter and the units of each other, and this must apply to the units of the inverse of each parameter and itself (cf *S*, *T*, *U* dualities of string theory).

Groups: $\mathcal{D}_2$

Ultimately, these conditions are satisfied if we can define a fundamental unit for each parameter, and, of course, these exist in the Planck length, $(\hbar G / c^3)^{1/2}$, the Planck time, $(\hbar G / c^5)^{1/2}$, the Planck mass, $(\hbar c / G)^{1/2}$, and the Planck charge, $(\hbar c)^{1/2}$.

The units $\hbar$, c and G, from which these are constructed, are, of course, of no fundamental significance, and merely represent the inherited choices of Babylonian astronomers and French revolutionaries.

Groups: $\mathcal{D}_2$

The relations of the elements with their own inverses in creating an identity is natural given the importance of *squaring* in the algebra that creates them.

The squared values of all the parameters have physical significance, those of space-time in creating Pythagorean or vector addition, and then combining with those of mass-charge to define physical interactions.

Typically nonconserved parameters, space and time, square undefined units. Conserved parameters, mass and charge, square specifically identifiable ones.

Groups: $\mathcal{D}_2$

Also, since the fundamental relations between the units have a numerical aspect, it is possible to construct new composite quantities (for example, momentum, force, energy, angular momentum),

which additionally express various aspects that are characteristic of them, such as invariance, variability, dimensionality, and a real or imaginary nature.

Groups: $\mathcal{D}_2$

This is necessary because the parameters only come as a package. They have no independent existence.

Also, the conservation of mass and charge would have no meaning unless this came *directly linked* with variation in space and time, as the conserved quantities would be otherwise completely inaccessible.

This is why we invent concepts like momentum, $\mathbf{p} = m\, d\mathbf{r} / dt$, and extend it then to force, $\mathbf{F} = d\mathbf{p} / dt$, because we need the second order for time. The quantity is a kind of minimal level of packaging, and it becomes universal when we make it totality zero. The zeroing of fundamental composite quantities is a very general process in foundational physics.

Groups: $\mathcal{D}_2$

The same happens when force is defined by the ratio of the vector addition of squared conserved quantities to that of nonconserved ones.

Again, the result is totality zero and we can relate it mathematically to the other one.

We can in this relatively simple way create relationships which express many of the fundamental laws of physics, in particular those of mechanics and electromagnetic theory, in a directly mathematical form, and show that they demonstrate different aspects of the properties which the group constrains upon its elements.

Groups: $\mathcal{D}_2$

This important and extensive development will not be covered here in detail, since we will derive most of our results directly through quantum mechanics, but it features in *Zero to Infinity*, chapter 8.

We will, however, assume results from classical mechanics in any of its versions and from electromagnetic theory wherever needed.

Groups: $\mathcal{D}_2$

The tables of properties and antiproperties and their algebraic representations also lead to some interesting reflections. First of all, we note that the parameters are completely interchangeable as abstract objects, something which is entirely within the spirit of modern algebra.

This is very remarkable when we consider how they make such different impressions on our consciousness. Mass and charge appear to be tangible things, whereas we imagine space and time as more abstract. This, in fact, is an illusion, as experimental evidence seems to show that 'material objects' are ultimately composed only of abstract points. However, the illusion has had such a strong hold on the human psyche that the complete realisation that there is only abstraction is still yet to take place.

Groups: $\mathcal{D}_2$

Another reflection is the strong indication here that physics at this level has a total conceptual content of exactly zero, just as we originally proposed.

We don’t have to use algebraic terms such as x and $–x$ to represent properties and antiproperties, and we don’t have to use the expressions ‘property’ and ‘antiproperty’ for the dual concepts.

But there is a clear indication that each fundamental conceptual property in nature is negated in some sense by a property that is exactly opposite. In general terms, there is every indication that the symmetry is absolutely exact, and that it is the exclusive source of information about the physical world.

Visual representations

If space, time, mass and charge form a group structure with an exact and unbroken symmetry, then we need to know the properties of only one of these terms to find those of all the others.

These will emerge automatically from the group, like kaleidoscopic images.

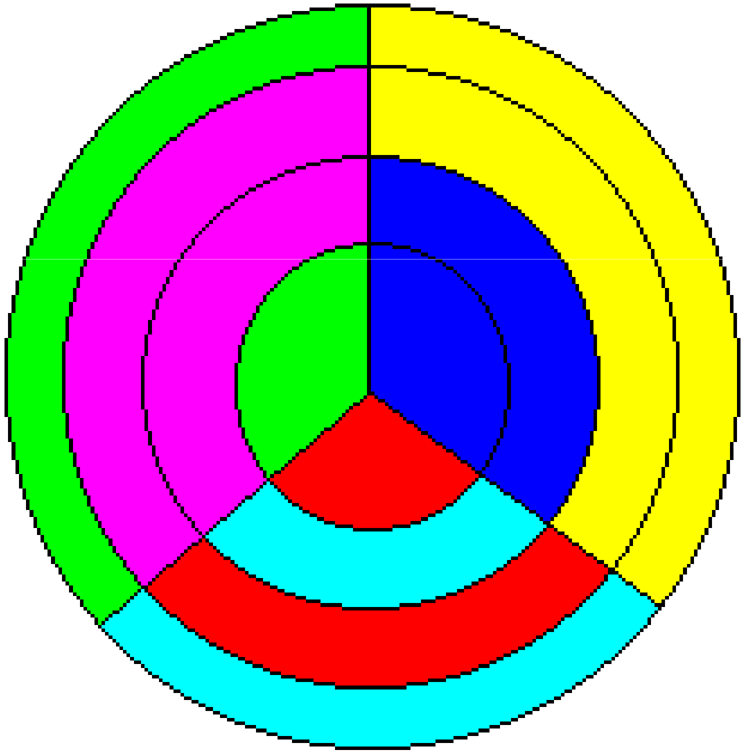
The arbitrary choice of which parameter becomes the ‘identity’ element can be seen in a number of visual representations, which, incidentally show the deep connections of the group with the fundamental nature of 3-dimensionality.

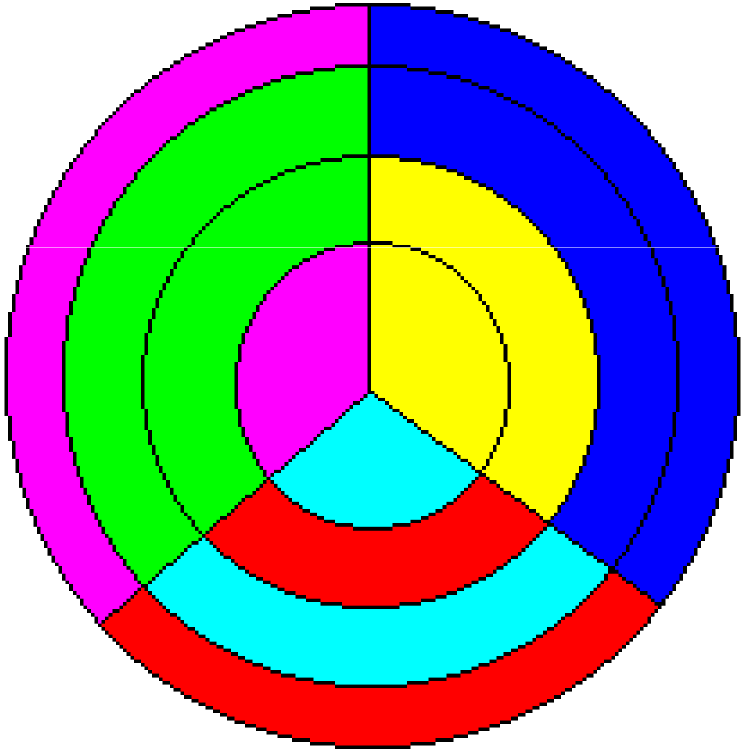
Visual representations

3 fundamental properties and corresponding antiproperties match perfectly with the existence of 3 primary colours, red, blue and green, and 3 complementary secondary colours, cyan, yellow and magenta. Again, there is no absolutely unique representation. Just as we can use any parameter as the identity element, so we can choose colours arbitrarily to represent properties and or antiproperties (and even this designation is arbitrary).

In the colour representation, space, time, mass and charge become concentric circles, each of which is divided into three sectors. Now, for illustration, we take any primary colour to represent a property, with the corresponding secondary colour as the antiproperty. Here, we show two examples, each of which has many interpretations.

Colour representation



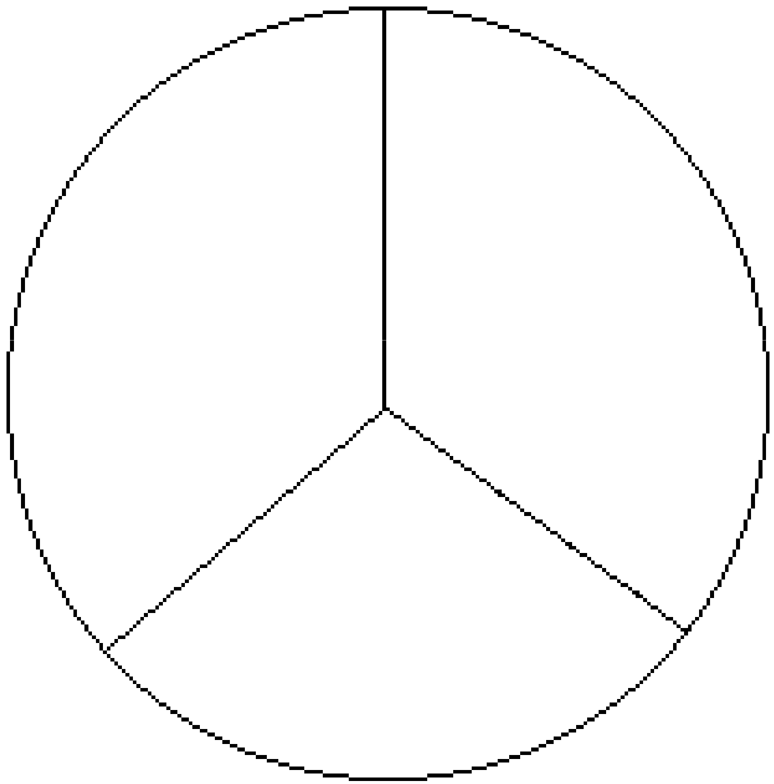


Colour representation

Take the left-hand diagram, and make the central triplet mass. Then we assume, say, that green represents real, and magenta imaginary; that blue is commutative and yellow anticommutative; and that red is conserved and cyan nonconserved.

Then the next circle is imaginary, commutative and nonconserved (that is, time); the next one is imaginary, anticommutative and conserved (that is, charge); and the outer circle is real, anticommutative and nonconserved (that is, space). Each sector, in any version, always adds to zero, represented by white. The representation is striking for showing that, in many ways, it is the pattern that is important, rather than the specifics. The structure only makes sense as a complete package.

Summation of the colour sectors



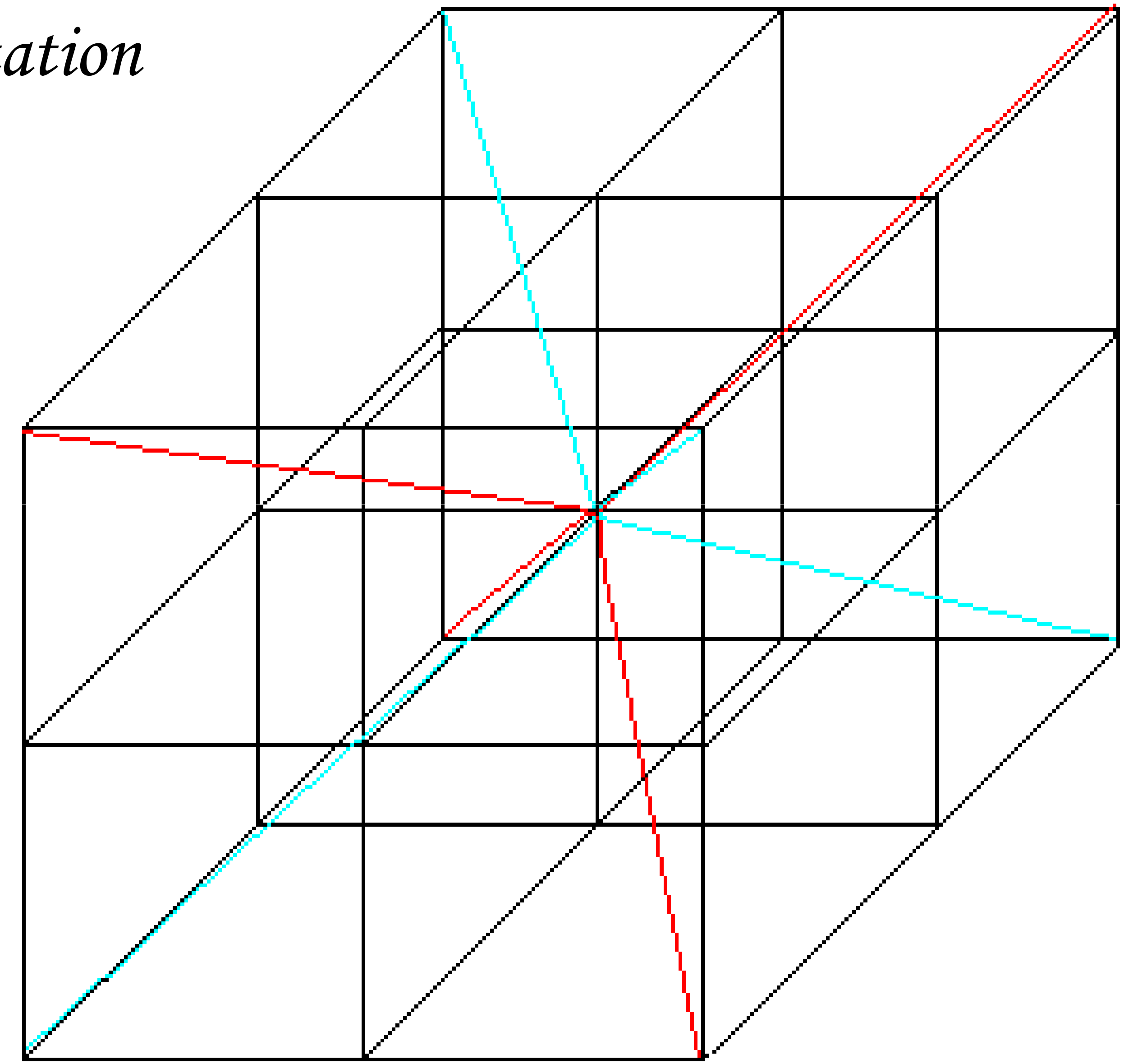
3-D representation

As an alternative to using colour, we could make direct use of the labels *x*, *y*, *z* to represent axes in 3-dimensional space.

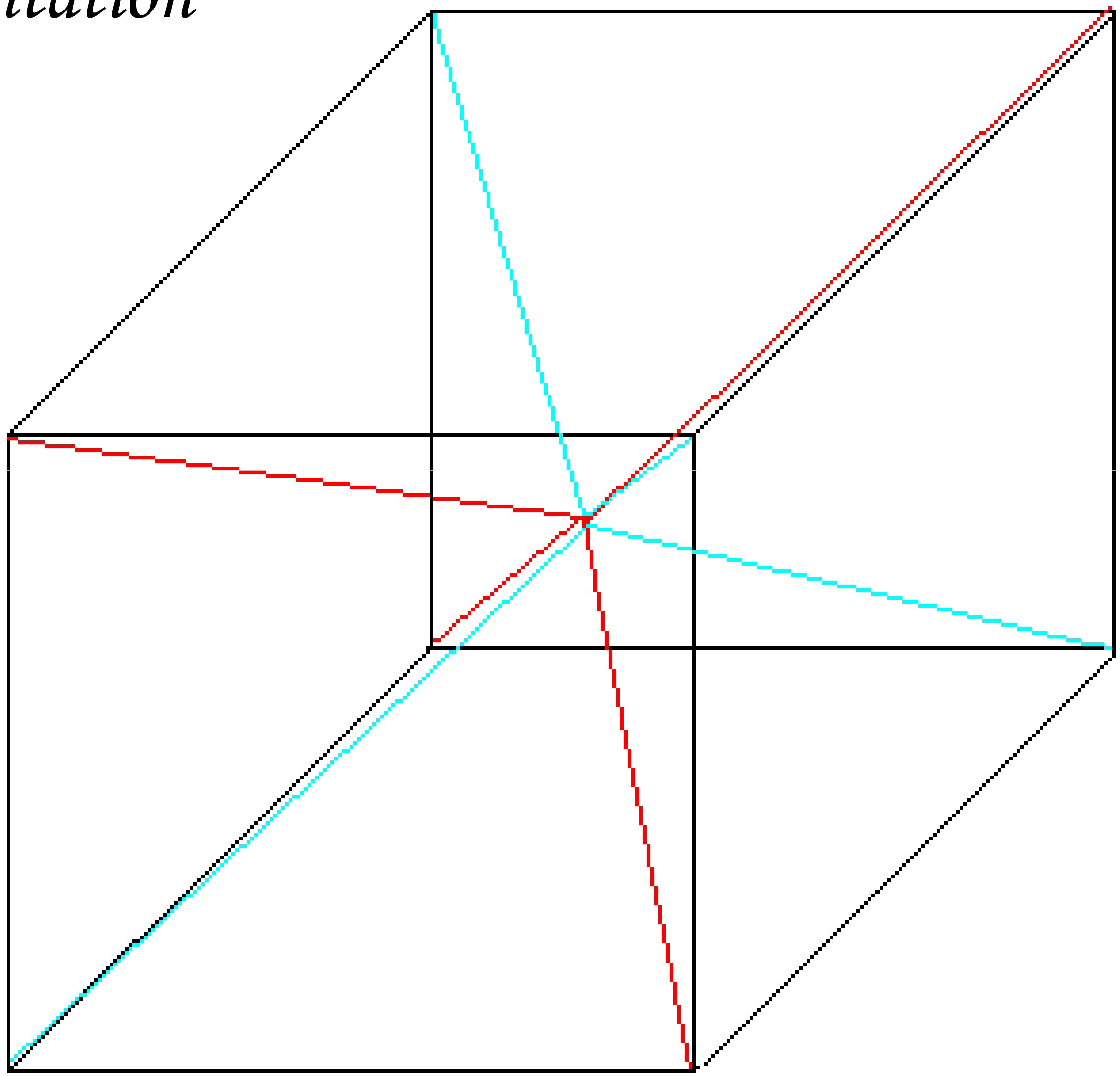
This time, it is the + and – directions that represent property and antiproperty.

The four parameters then become equivalent to lines drawn from the centre of a cube to four of its corners.

3-D Representation



3-D Representation



3-D representation

The cyan lines then represent the alternative arrangement of the signs of x, y and z (say, by switching the signs of any or all of x, y or z) which we showed in the second lecture would lead to the same group:

$-x$	$-y$	$-z$
x	y	$-z$
$-x$	y	z
x	$-y$	z

This dual arrangement (also seen in the second colour example) might also be physical, though not primary, say in the relativistic quantum mechanics of the Dirac equation, where there is an apparent reversal in some of the characteristics of the fundamental parameters.

The dual group

The easiest to change is the real / imaginary distinction. In this case, we have:

mass*	x	$-y$	z
time*	$-x$	y	z
charge*	x	y	$-z$
space*	$-x$	$-y$	$-z$

or

mass*	conserved	imaginary	commutative
time*	nonconserved	real	commutative
charge*	conserved	real	anticommutative
space*	nonconserved	imaginary	anticommutative

The reason for this will become clear when we discuss the packaging of the parameter group.

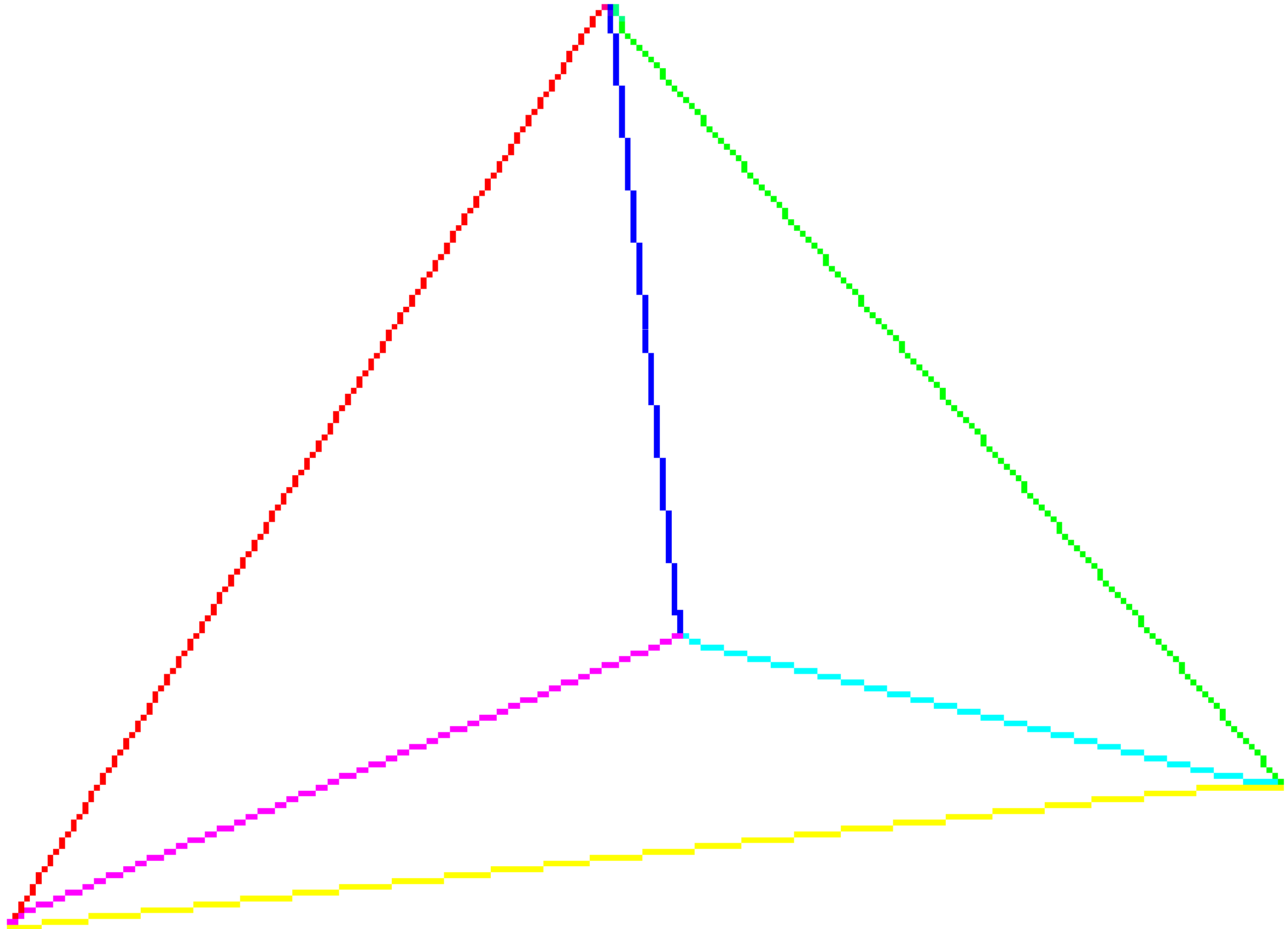
Tetrahedral Representation

In yet another representation, the parameters can be placed at the vertices of a regular tetrahedron.

The six edges in primary and secondary colours now represent the respective properties and antiproperties, or vice versa.

Alternatively, since the tetrahedron is a dual structure in itself, we can represent the parameters by the *faces*. Once again, an extra duality appears, as with the colour and cubic representations.

Tetrahedral Representation



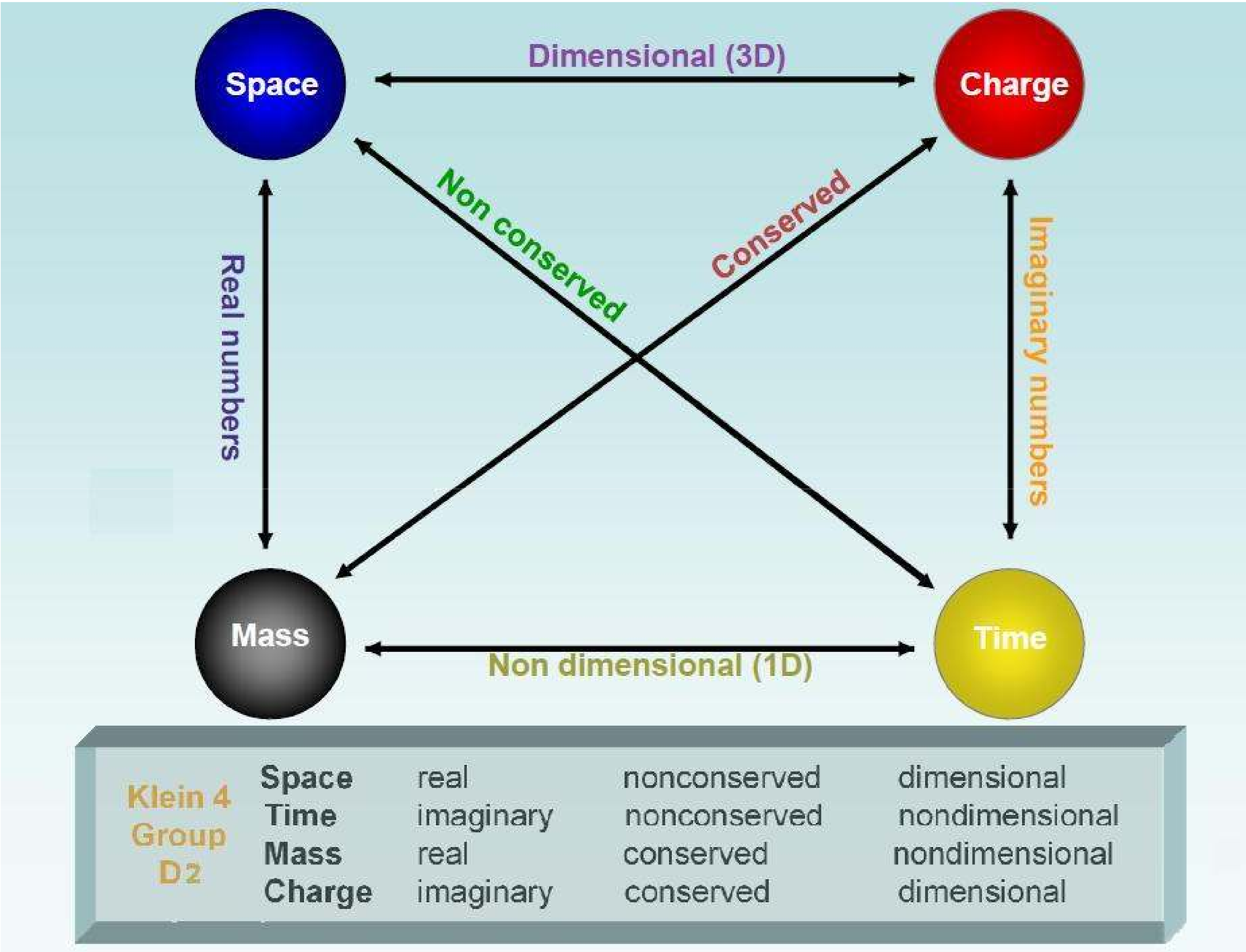


diagram © Vanessa Hill 2013

Combining the group and dual group

There is something like a C_2 symmetry between the dual D_2 structures, and the $C_2 \times D_2$ of order creates a larger structure of the form:

*	*M*	*C*	*S*	*T*	*M**	*C**	*S**	*T**
M	*M*	*C*	*S*	*T*	*M**	*C**	*S**	*T**
C	*C*	*M**	*T*	*S**	*C**	*M*	*T**	*S*
S	*S*	*T**	*M**	*C*	*S**	*T*	*M*	*C**
T	*T*	*S*	*C**	*M**	*T**	*S**	*C*	*M*
*M**	*M**	*C**	*S**	*T**	*M*	*C*	*S*	*T*
*C**	*C**	*M*	*T**	*S*	*C*	*M**	*T*	*S**
*S**	*S**	*T*	*M*	*C**	*S*	*T**	*M**	*C*
*T**	*T**	*S**	*C*	*M*	*T*	*S*	*C**	*M**

Combining the group and dual group

Remarkably, this structure is identical to that of the quaternion group (Q):

*	1	***i***	***j***	***k***	–1	–***i***	–***j***	–***k***
1	1	***i***	***j***	***k***	–1	–***i***	–***j***	–***k***
i	***i***	–1	***k***	–***j***	–***i***	1	–***k***	***j***
j	***j***	–***k***	–1	***i***	–***j***	***k***	1	–***i***
k	***k***	***j***	–***i***	–1	–***k***	–***j***	***i***	1
–1	–1	–***i***	–***j***	–***k***	1	***i***	***j***	***k***
–***i***	–***i***	1	–***k***	***j***	***i***	–1	***k***	–***j***
–***j***	–***j***	***k***	1	–***i***	***j***	–***k***	–1	***i***
–***k***	–***k***	–***j***	***i***	1	***k***	***j***	–***i***	–1

Two spaces?

In relation to the 'unreasonable effectiveness' of mathematics in physics and of physics in mathematics, what we think of as *physical* properties – and the only ones that actually exist at the fundamental level – can be expressed almost entirely as pure algebra.

Two of the dualities – real and imaginary, and commutative and noncommutative – are obviously so. The third – conserved and nonconserved – is almost certainly in the same category.

Probably, the algebraic reason is something like the fact that the nonconserved parameters – time and space – are constructed partly from an incomplete quaternion group, i.e. complex numbers, based on the imaginary 'pseudoscalar' *i*.

Two spaces?

This aspect of complex numbers is more apparent when the algebra is constructed from a universal rewrite system algebra, as was done in work carried out with my colleague Bernard Diaz (*Zero to Infinity*, chapter 1, see lecture 10).

A similar kind of reasoning concerning incompletion was the idea which initially led to the discovery of quaternions.

Two spaces?

We can, in fact, restructure the algebras and subalgebras using the vector / bivector / trivector terminology of Clifford algebra:

Space	**i**, **j**, **k**	vector		
	*i***i**, *i***j**, *ik* ≡ ***i***, ***j***, ***k***	bivector	pseudovector	quaternion
	i	trivector	pseudoscalar	
	1	scalar		
Charge	***i***, ***j***, ***k*** ≡ *i***i**, *i***j**, *i***k**	bivector	pseudovector	quaternion
	1	scalar		
Time	*i*	trivector	pseudoscalar	
	1	scalar		
Mass	1	scalar		

Two spaces?

What is immediately noticeable is that the algebras of mass, time and charge are, mathematically, subalgabras of the algebra of space. Now, if we put together the three subalgebras, we obtain an equivalent of the algebra itself. The combination of *i* and *i***i**, *i***j**, *i***k** (with or without 1) will create the missing **i**, **j**, **k**.

In effect, putting charge, time, and mass together gives us a mathematical equivalent to another space **i**, **j**, **k** alongside real space, **i**, **j**, **k**. Physically, this composite is not a space, because it is not a single quantity, and so it will never be measurable or observable in the same way as space. However, mathematically it is the same.

Two spaces?

We can consider it as an *antispace*.

It incorporates all the things that are *not* space, allowing us to equate the totality of space and this composite object to zero.

For space as the identity element, the respective algebraic sums for the properties would be: space, x, y, z, antispace, $-x$, $-y$, $-z$.

The same could, of course, be done for time, mass or charge, but as these are not measurable quantities, the effect is rather less significant.

Totality zero

Overall, the symmetry between the parameters seems to be telling us that nature or the universe is a conceptual ‘nothing’ or zero, with no defining characteristics.

It isn’t even possible to decide whether we should view it ontologically (God’s eye’ view) or epistemologically (the view of an observer).

And the factor 2 that appears everywhere in physics comes from the fact that any defined aspect of nature also produces a mirror image of itself which negates its existence.

Totality zero

We have already noted the increasing tendency in physics to suggest that the universe has some kind of zero sum of quantities like force and energy, but the various suggestions seem to stop short of the possibility that there is absolutely *nothing at all* in the 'universe' or in 'nature', even *conceptually*.

This may seem startling, because we seem to be surrounded by 'something' everywhere we look, but really, from the *inside*, we can have no real idea about 'nothing' referring to nature as a whole. We can write down equations with zero on the right hand side, but we can't realise any of them physically. Getting 'close to zero', if it isn't actually zero, isn't really close at all.

Totality zero

Clearly, the zeroing is much more subtly arranged than we would find simply by taking something like 1 – 1 = 0, as we can see from the group symmetry that we have uncovered, which suggests zero without immediate cancellation.

But, however we perceive it, totality zero is an especially powerful constraint because it forces us into a holistic view of the universe, such as quantum mechanics seems to require.

A unified algebra

To take physics further we need to put together the 'package' which incorporates all the individual components into a coherent unified system. This creates the first level of complexity. We already have the mathematics. We can, for example, take the Clifford algebra approach, and put together two vector spaces, which are commutative with each other, with fundamental units consisting of + and – versions of

i	**j**	**k**	*i***i**	*i***j**	*i***k**	*i*	1
i	**j**	**k**	*i***i**	*i***j**	*i***k**	*i*	1
	vector			*bivector*		*trivector*	*scalar*

A unified algebra

The product of each term with every other, or tensor product, consists of 64 terms, which are + and – values of the following:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{i}$	$i\mathbf{k}$		
$\mathbf{ii}$	$\mathbf{ij}$	$\mathbf{ik}$	$i\mathbf{ii}$	$i\mathbf{ij}$	$i\mathbf{ik}$		
$\mathbf{ji}$	$\mathbf{jj}$	$\mathbf{jk}$	$i\mathbf{ji}$	$i\mathbf{jj}$	$i\mathbf{jk}$		
$\mathbf{ki}$	$\mathbf{kj}$	$\mathbf{kk}$	$i\mathbf{ki}$	$i\mathbf{kj}$	$i\mathbf{kk}$		

A unified algebra

We could equally well have begun with the four algebras of space, time, mass and charge:

i j k	*i*	1	***i j k***
space	time	mass	charge
vector	*pseudoscalar*	*scalar*	*quaternion*

A unified algebra

This would give us the completely equivalent vector-quaternion algebra, which would emerge from exchanging _i_***i***, i***j***, _i_***k*** for **i**, **j**, **k** and ***i***, ***j***, ***k*** for i**i**, i**j**, i**k**, and which requires + and – values of:

i	___j___*	***k***	_i_***i***	_i_***j***	_i_***k****	_i_	1
i	**j**	**k**	_i_**i**	_i_**i**	_i_**k**		
i___i___*	**i*****j***	**i*****k***	_i_**i*****i***	_i_**i*****j***	_i_**i*****k***		
j___i___*	**j*****j***	**j*****k***	_i_**j*****i***	_i_**j*****j***	_i_**j*****k***		
k___i___*	**k*****j***	**k*****k***	_i_**k*****i***	_i_**k*****j***	_i_**k*****k***		

A unified algebra

We have already obtained these algebras and identified them as a group of order 64. Here, we have 8 generators of the algebra, which, using the two vector spaces, **i**, **j**, **k**, and **i**, **j**, **k**, we could reduce to 6.

But neither of these is the minimum, which we have already shown reduces to 5, all of course, elements of the group. This can be done in many ways, but all those that incorporate all the base elements look something like

	$i\boldsymbol{k}$	$\boldsymbol{i}\mathbf{i}$	$\boldsymbol{i}\mathbf{j}$	$\boldsymbol{i}\mathbf{k}$	$\boldsymbol{j}$
or	$i\mathbf{k}$	$\mathbf{ii}$	$\mathbf{ij}$	$\mathbf{ik}$	$\mathbf{j}$

A unified algebra

All the sets of 5 generators have the same pattern, as we have seen by splitting up the 64 units into 1, –1, *i* and – *i*, and 12 sets of 5 generators, each of which generates the entire group:

1	i				–1	$-i$			
$\boldsymbol{i}\mathbf{i}$	$\boldsymbol{i}\mathbf{j}$	$\boldsymbol{i}\mathbf{k}$	$i\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}\mathbf{i}$	$-\boldsymbol{i}\mathbf{j}$	$-\boldsymbol{i}\mathbf{k}$	$-i\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}\mathbf{i}$	$\boldsymbol{j}\mathbf{j}$	$\boldsymbol{j}\mathbf{k}$	$i\boldsymbol{i}$	$\boldsymbol{k}$	$-\boldsymbol{j}\mathbf{i}$	$-\boldsymbol{j}\mathbf{j}$	$-\boldsymbol{j}\mathbf{k}$	$-i\boldsymbol{j}$	$-\boldsymbol{i}$
$\boldsymbol{k}\mathbf{i}$	$\boldsymbol{k}\mathbf{j}$	$\boldsymbol{k}\mathbf{k}$	$i\boldsymbol{j}$	$\boldsymbol{i}$	$-\boldsymbol{k}\mathbf{i}$	$-\boldsymbol{k}\mathbf{j}$	$-\boldsymbol{k}\mathbf{k}$	$-i\boldsymbol{j}$	$-\boldsymbol{i}$
$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{k}$	$i\mathbf{k}$	$\mathbf{j}$	$-i\mathbf{i}\boldsymbol{i}$	$-i\mathbf{i}\boldsymbol{j}$	$-i\mathbf{i}\boldsymbol{k}$	$-i\mathbf{k}$	$-\mathbf{j}$
$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{k}$	$i\mathbf{i}$	$\mathbf{k}$	$-i\mathbf{j}\boldsymbol{i}$	$-i\mathbf{j}\boldsymbol{j}$	$-i\mathbf{j}\boldsymbol{k}$	$-i\mathbf{i}$	$-\mathbf{k}$
$i\mathbf{k}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{k}$	$i\mathbf{j}$	$\mathbf{i}$	$-i\mathbf{k}\boldsymbol{i}$	$-i\mathbf{k}\boldsymbol{j}$	$-i\mathbf{k}\boldsymbol{k}$	$-i\mathbf{j}$	$-\mathbf{i}$

A unified algebra

Even this arrangement is not unique, but any rearrangement would retain the same pattern in which the symmetry of one of the two 3-dimensional structures (**i**, **j**, **k** or ***i***, ***j***, ***k***; and **i**, **j**, **k**) was broken while the symmetry of the other was preserved. Physics always tends to go for the most minimal representation, and though something like

$i\boldsymbol{k}$ $\quad\quad$ $\boldsymbol{i}\mathbf{i}$ $\quad$ $\boldsymbol{i}\mathbf{j}$ $\quad$ $\boldsymbol{i}\mathbf{k}$ $\quad\quad$ $\boldsymbol{j}$

does not appear to be as symmetrical at first sight as

$\mathbf{i}$ $\quad$ $\mathbf{j}$ $\quad$ $\mathbf{k}$ $\quad$ i $\quad$ 1 $\quad$ $\boldsymbol{i}$ $\quad$ $\boldsymbol{j}$ $\quad$ $\boldsymbol{k}$

it contains the same information, and, ultimately, the same symmetries.

A unified algebra

It is thus in creating the minimum packaging for the information contained in the parameter group that we find the ultimate explanation of why the symmetry of charge is broken, at the first level of complexity (packaging), whereas that of space is not.

The process is completely dual, so it would be quite possible to create a physics in which the process was reversed, and the geometry of space was altered rather than the charge structure, say using Finsler geometry, but, for comparison with physics as we know it, it seems more convenient to retain the symmetry of space rather than that of charge.

A unified algebra

The symmetry between the three components of charge and their interactions can be seen to be broken at the level of observation, that is, when we package it with space.

To preserve the symmetry of the observed quantity, real space (that of **i**, **j**, **k**), we necessarily have to break the symmetry of 'charge' (***i***, ***j***, ***k***) or the unobservable mathematical 'space' (**i**, **j**, **k**) that links charge with mass and time.

A unified algebra

Effectively, starting with the 8 units needed for the 4 parameters:

i	**i**	**j**	**k**	1	***i***	***j***	***k***
time		space		mass		charge	

we 'compactify' to the 5 generators by removing the three 'charge' units and attaching one to each of the other three parameters:

i	**i**	**j**	**k**	1
k		***i***		***j***

and finally:

i***k***	***i*****i**	***i*****j**	***i*****k**	1***j***

A unified algebra

As a result, we create 3 new 'composite' parameters, each of which has aspects of time, space or mass, but also some characteristics of charge.

$i\boldsymbol{k}$	$\boldsymbol{i}\mathbf{i}$	$\boldsymbol{i}\mathbf{j}$	$\boldsymbol{i}\mathbf{k}$	$1\boldsymbol{j}$
E	p_x	p_y	p_z	m

We can attach to these unit structures any *scalar* labels we like, and here we select those that will subsequently identified as those for energy, momentum and rest mass. The significant thing here is that these quantities are defined by their algebraic units not by their scalar values.

A unified algebra

Since we started only with space, time, mass and charge, this becomes the *first appearance* of these *conjugate* quantities in physics, and it would seem that superposition of two sets of parameters with different characters to create generators for the group combining their algebras actually *creates* them.

It also simultaneously fixes quantization and relativity as fundamental components of the package, each of these being effectively the establishment of numerical relations between the units of previously unrelated physical quantities.

A unified algebra

The new structure is essentially what we normally describe as phase space, but it is not independent of either of the 'spaces' that create it.

The conjugate pairings, time and energy, and space and momentum, are not actually independent, for the set involving energy and momentum is in part created from the more primitive set involving time and space.

While fully independent quantities are commutative with each other, dependent quantities are not.

A unified algebra

Energy and time are therefore anticommutative at the level of the most fundamental units, as are momentum and space.

This is exactly what is expressed in Heisenberg's uncertainty principle: 2 × the product of the fundamental units of the two anticommutative terms produces the most fundamental quantum unit of their combination, = h / 2π, the quantum unit of angular momentum.

Nilpotency

Physics operates in such a way that the total package of all information is zero, and the combined structure we have created by packaging the entire source of information available to us, $(i\mathbf{k}E + \mathbf{i}\mathbf{i}px + \mathbf{i}\mathbf{j}py + \mathbf{i}\mathbf{k}pz + \mathbf{j}m)$, becomes a norm 0 object, or a nilpotent. So

$$(i\mathbf{k}E + \mathbf{i}\mathbf{i}px + \mathbf{i}\mathbf{j}py + \mathbf{i}\mathbf{k}pz + \mathbf{j}m)^2 = E^2 - p^2 - m^2 = 0$$

which immediately creates the numerical relations we require between all the parameters.

Nilpotency

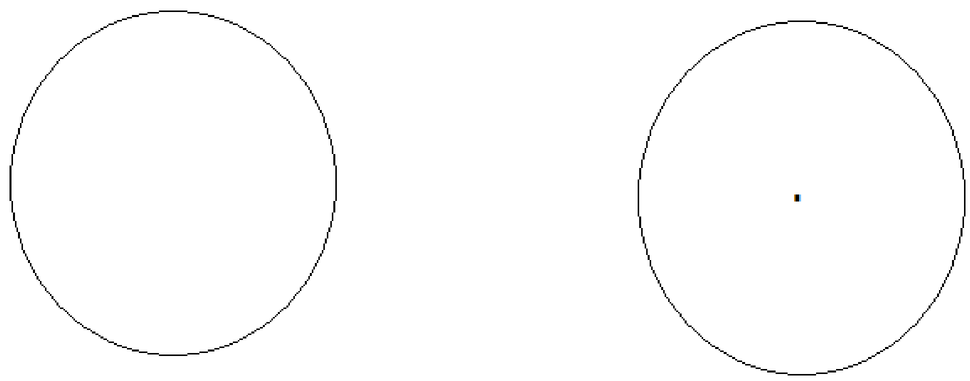
Now, from a fundamental point of view, we can begin to see that the nilpotent structure is equivalent to creating a point source or charge singularity in 3-D space.

In effect, we combine two spaces, or space and antispace, to effectively cancel and create a region in which the ‘spatial’ extent is zero. Through the nilpotent condition, the two spaces share dual information, though it is differently organized in each.

The observed 3-D space becomes multiply-connected because it is acting as *two* spaces, only one of which is observed. The space that remains unobserved is described as ‘vacuum space’ in quantum mechanics.

Nilpotency

A circuit of a closed path in real space will require a double rotation to return to the starting-point because it is only in this space for half the time. The charge singularity will itself be a multiply-connected space, and require a double circuit, which will manifest itself as spin ½.



simply-connected space   *multiply-connected space*

Nilpotency

We can regard the 5 group generators as the most efficient packaging of all the information contained in the group structure of space, time, mass and charge, and codified in their algebraic structures.

We should be able to use it to generate the physics that we know is contained in the interactions between fermions, in particular the Dirac equation and the relativistic quantum mechanics of fermions and bosons.

In fact, this emerges in an extraordinarily transparent form, in which many developments follow immediately from the algebraic structure. This will be covered mostly in the following lecture, but it will be useful to do a preliminary analysis here.

Dirac equation

The apparently classical expression

$$(ikE + \mathbf{i}ip_x + i\mathbf{j}p_y + i\mathbf{k}p_z + jm)\ (ikE + \mathbf{i}ip_x + i\mathbf{j}p_y + i\mathbf{k}p_z + jm) = 0$$

or

$$(ikE + i\mathbf{p} + jm)\ (ikE + i\mathbf{p} + jm) = 0$$

can be immediately restructured as relativistic quantum mechanics using a canonical quantization of the first bracket ($E \rightarrow i\partial / \partial t$, $\mathbf{p} \rightarrow -i\nabla$) and its application to a phase factor, which, for a free particle, would be $e^{-i(Et - \mathbf{p.r})}$. So that

$$(ik\partial / \partial t + i\nabla + jm)\ (ikE + i\mathbf{p} + jm)\ e^{-i(Et - \mathbf{p.r})} = 0$$

which, as we will demonstrate in the next lecture, is the Dirac equation for the free fermion. operator).

Dirac equation

In effect, the equation shows the simultaneous application of the dual 'spaces' involved in the nilpotent structure, the 'amplitude' term $(i\mathbf{k}E + \mathbf{i}\mathbf{p} + \mathbf{j}m)$ representing the localised, real space of the point-particle, and the operator $(i\mathbf{k}\partial / \partial t + \mathbf{i}\nabla + \mathbf{j}m)$ acting on the phase factor $e^{-i(Et - \mathbf{p.r})}$ representing the variation over the delocalised vacuum space.

The phase factor in this form gives us the expression for a space and time that can varied without restriction, and the operator acting on it sets up the conservation conditions that have to be applied simultaneously.

Dirac equation

Ultimately, we will see that the amplitude and phase are not independent information.

The entire information is incorporated within the operator ($i\mathbf{k}\partial / \partial t + \mathbf{i}\nabla + \mathbf{j}m$), which sets up the 'nonconservation' conditions for space and time, which lead to the creation of the energy and momentum as conserved quantities,

in addition to angular momentum (for we will eventually recognise that the term ($i\mathbf{k}E + \mathbf{i}\mathbf{p} + \mathbf{j}m$) is, in principle, an angular momentum operator).

The symmetry breaking between charges

The packaging process affects time, space and mass by creating the energy-momentum-rest mass conjugate. But it must also affect charge, for it simultaneously creates three new ‘charge’ units, which take on the respective characteristics of the parameters with which they are associated.

*i***k**	***i*i *i*j *i*k**	***j***
weak charge	strong charge	electric charge
pseudoscalar	*vector*	*scalar*

The symmetry breaking between charges

In the Standard Model, the symmetry between the weak, strong and electric interactions is broken in such a way that they respond respectively to the symmetry groups $SU(2)$, $SU(3)$ and $U(1)$. These group structures have no fundamental explanation in the Standard Model.

However, it should be possible to see that they are generated through the 2-component pseudoscalar ($SU(2)$), 3-component vector ($SU(3)$) and single component scalar ($U(1)$) nature of the weak, strong and electric charges as they are incorporated within the nilpotent structure. This will become much more explicit after we have established a system of quantum mechanics.

The parameters in the dual group

We can now return to some unresolved issues. One is the nature of the dual group to space, time, mass and charge.

The extra quaternion units in the expression ($i\mathbf{k}E$ + $\mathbf{i}\mathbf{p}$ + $\mathbf{j}m$) clearly change the norm of the timelike term ($i\mathbf{k}E$) from –1 to 1, and those of the spacelike and masslike terms ($\mathbf{i}\mathbf{p}$ and $\mathbf{j}m$) from 1 to –1,

so making the quantized energy and momentum and rest mass terms equivalent to time*, space* and mass*.

The parameters in the dual group

The same would be true if we used the nilpotent structure ($i\mathbf{k}t + \mathbf{i}\mathbf{r} + \mathbf{j}\tau$) for the relativistic space-time invariance, where τ is the proper time.

The quantized angular momentum would then be equivalent to the charge* term, in line with the already established link between charge and angular momentum.

The group of order 8 incorporating the D_2 parameter group and its mathematical dual, which is isomorphic to the quaternions, would then be the quantized phase space for the fermion.

A broken octonion

A different question is the relation between the algebraic *units* of the parameters.

This is a different 8-fold structure to that between the parameters and their duals.

If mass, charge, time and space form a group of order 4, then the combination of their base units (1, ***i***, ***j***, ***k***, *i*, *i****i***, *i****j***, *i****k***), or (*m*, *s*, *e*, *w*, *t*, *x*, *y*, *z*), expressed in the form of complexified double quaternions, could be said to be that of a ‘broken octonion’.

A broken octonion

It is also intriguingly close to Penrose's twistor structure with 4 'real' parts and 4 'imaginary' parts, though the additional structure here is crucial in separating out two sets of 3-D objects and two full vector spaces.

The breaking isn't because a large fundamental structure is exposed to a symmetry-breaking 'mechanism', but because the large structure is made up of units with an independent origin, which have asymmetric aspects from the beginning.

The parameters arranged in algebraic units

*	m	s	e	w	t	x	y	z
m	m	s	e	w	t	x	y	z
s	s	$-m$	w	$-e$	x	t	$-z$	y
e	e	$-w$	$-m$	s	y	z	$-t$	$-x$
w	w	e	$-s$	$-m$	z	$-y$	x	$-t$
t	t	$-x$	$-y$	$-z$	$-m$	s	e	w
x	x	t	$-z$	y	$-s$	$-m$	$-w$	e
y	y	z	t	$-x$	$-e$	w	$-m$	$-s$
z	z	$-y$	x	t	$-w$	$-e$	s	$-m$

The octonion mapping

*	1	i	j	k	e	f	g	h
1	1	i	j	k	e	f	g	h
i	i	–1	k	–j	f	–e	–h	g
j	j	–k	–1	i	g	h	–e	–f
k	k	j	–i	–1	h	–g	f	–e
e	e	–f	–g	–h	–1	i	j	k
f	f	e	–h	g	–i	–1	–k	j
g	g	h	e	–f	–j	k	–1	–i
h	h	–g	f	e	–k	–j	i	–1

Angular momentum and type of charge

The other issue is particularly significant because it illustrates the predictive value of the fundamental methodology.

Earlier, we predicted a quite extraordinary result as a consequence of Noether's theorem.

This equated the conservation of angular momentum or the rotation symmetry of space with the conservation of *type* of charge, i.e. the inability of weak, strong and electric charges to transform into each other.

The result looks impossible to demonstrate or to fit to a mathematical description, but now we can give the explanation.

Angular momentum and type of charge

Essentially, angular momentum conservation is made up of *three separate conservation laws* which are completely independent but all required at the same time.

We have to separately conserve the magnitude, the direction, and the handedness (i.e. whether the rotation is right- or left-handed), and the symmetries we require for these conservation laws are the $U(1)$, $SU(3)$ and $SU(2)$ symmetries involved with the electric, strong and weak charges.

Angular momentum and type of charge

In principle, these symmetries are versions of the spherical symmetry of 3-D space around a point charge.

Spherical symmetry, they say, is preserved by a rotating system

whatever the length of the radius vector	$U(1)$;
whatever system of axes we choose	$SU(3)$;
whether we choose to rotate the system left- or right-handed	$SU(2)$.

Angular momentum and type of charge

Conservation of charge ≡ conservation of spherical symmetry for a point source, and has to preserve all three aspects.

As we have seen from our analysis of symmetry-breaking, the $SU(3)$ and $SU(2)$ aspects are dealt with by the respective strong and weak charges, with their vector and pseudoscalar characteristics.

These are additional to the $U(1)$ symmetry, to which all three charges contribute (just as they do to the Coulomb interaction) because all three charges also have scalar characteristics.

The electric charge is unique, however, in contributing only to this symmetry.

Angular momentum and type of charge

So all three charges have to be conserved independently of each other, in the same way as the direction, handedness and magnitude of the angular momentum.

It must one of the strongest possible tests of a theory to predict such a totally unexpected result and then to find a simple reason why it must be valid.

The End