Foundations of Physical Law
2 Mathematical ideas and methods

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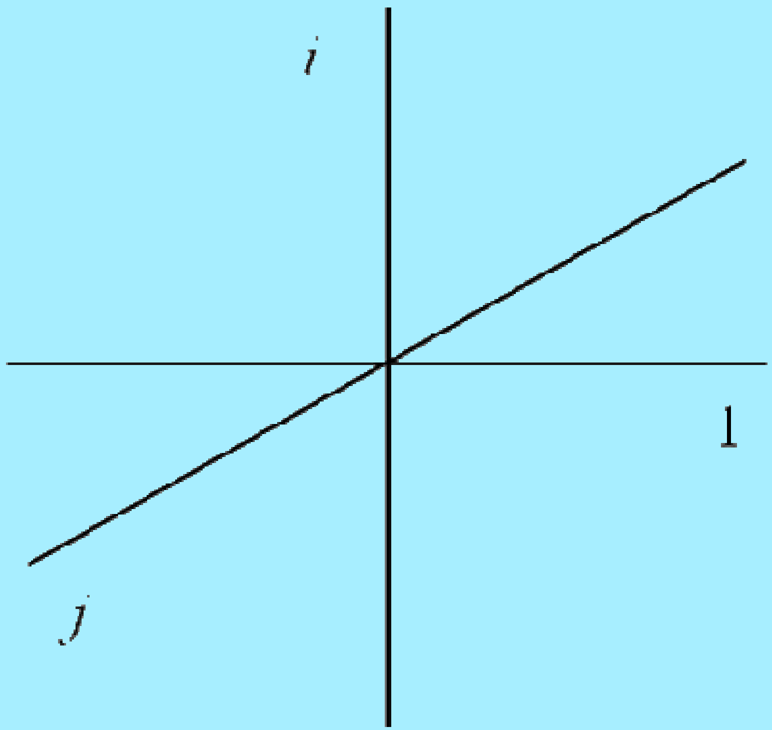
Quaternions

In 1843 Sir William Rowan Hamilton realised, for the first time, that he had discovered the meaning of 3-dimensionality. He was trying to extend the idea of complex numbers to the next level.

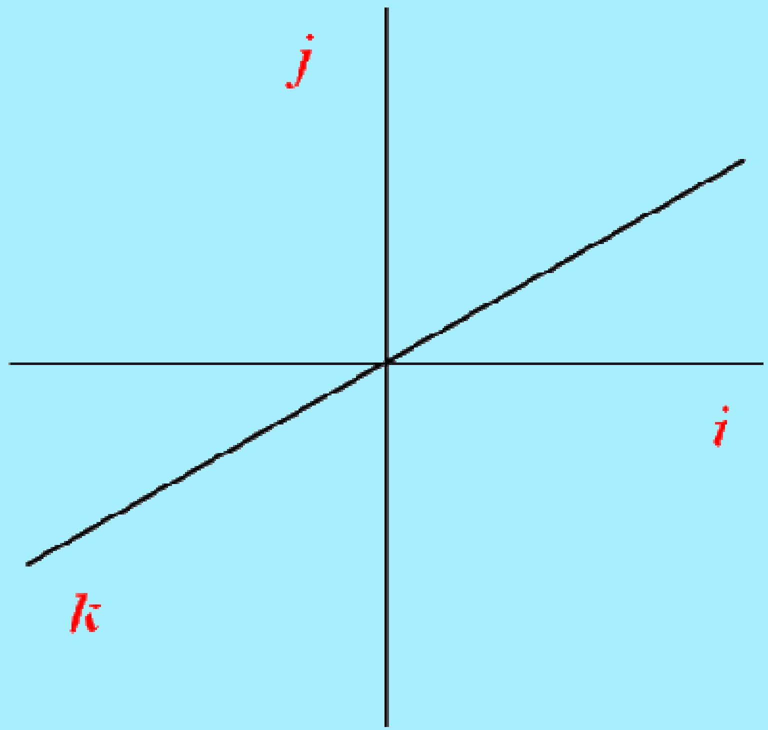
He proposed drawing a third, z-axis on the Argand diagram, perpendicular to the other two, which couldn't contain real numbers, as these are all on the x-axis,

but could conceivably contain another set of imaginary numbers based on a unit $j = \sqrt{(-1)}$ which is different from i. The problem with Hamilton's idea is that it doesn't work algebraically. There is no meaningful product of numbers with units i and j.

Quaternions



extended Argand diagram



quaternions

Quaternions

The answer meant violating one of the cardinal principles of algebra as then known: the principle of commutativity. i.e., we assume that the product *ba* is exactly the same answer as taking the product *ab*.

Hamilton remove the real axis entirely and had 3 imaginary axes, with units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, all different from each other but all equating to √(–1), and following a rotation cycle, so that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

$$\boldsymbol{ij} = \boldsymbol{k}$$

$$\boldsymbol{ki} = \boldsymbol{j}$$

$$\boldsymbol{jk} = \boldsymbol{i}$$

Quaternions

This works well but it requires a reversal of sign when we reverse the order of multiplication. That is:

$$\boldsymbol{ji} = -\boldsymbol{k}$$
$$\boldsymbol{ik} = -\boldsymbol{j}$$
$$\boldsymbol{kj} = -\boldsymbol{i}$$

The units are *anticommutative*. The reason can be seen immediately. Take the product $\boldsymbol{ij}\,\boldsymbol{ji}$. We multiply $\boldsymbol{j}\,\boldsymbol{j}$ first, and their product is –1. The remaining term is – $\boldsymbol{ii}$, which clearly equals 1. So

$$\boldsymbol{ij}\,\boldsymbol{ji} = -\,\boldsymbol{ii} = 1$$

But this can only be true if $\boldsymbol{ji} = -\boldsymbol{k}$.

Quaternions

Accepting this had to be done, Hamilton now had a closed algebra, with four basic units 1, ***i***, ***j***, ***k***, which he called *quaternions*, and which was double the size of ordinary complex algebra with units 1, *i*, and four times the size of real algebra, based on unit 1.

Can we extend again and still maintain the same rules, say with one real and four imaginary units: 1, ***i***, ***j***, ***k***, ***l***?

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Can we extend again and still maintain the same rules, say with one real and four imaginary units: 1, ***i***, ***j***, ***k***, ***l***?

No! You can't get consistency if you extend the number of imaginary units beyond three. Frobenius proved this in 1878.

Octonions

There is just one exception.

You can create a consistent system with one real and *seven* imaginary units, 1, ***i***, ***j***, ***k***, ***e***, ***f***, ***g***, ***h***, called the *octonions*.

However, to do this you have to break another algebraic rule, the law of associativity in which, say, $(ab)c$ is always equal to $a(bc)$.

So, octonions, unlike quaternions, are antiassociative as well as anticommutative, leading to products equivalent to $(ab)c = -a(bc)$.

Division algebras

The trick can't be repeated at any other level, so we are left with just four so-called division algebras:

Real	norm 1	commutative	associative
Complex	norm –1	commutative	associative
Quaternions	norm –1	anticommutative	associative
Octonions	norm –1	anticommutative	antiassociative

Anticommutativity and 3-dimensionality

The ‘3-ness’ isn’t the primary cause of the 3-dimensionality of space. It is simply a result of anticommutativity.

If we have two axes, ***i*** and ***j***, that are anticommutative with each other, then we cannot draw any other axis that is anticommutative with them, unless it is ***ij***, which we also call ***k***. Anticommutativity forces 3-dimensionality. The strange arbitrariness of the number 3 is explained.

Commutative things, of course, can be defined to infinity. If ***i*** and ***j*** were commutative, we could have ***i***, ***j***, ***k***, ***l***, ***m***, etc. without limit. Anticommutative things ‘know’ about each other’s presence and have to act accordingly; commutative things do not.

Multiplication of quaternions

A quaternion number will have a real and three imaginary parts, say,

$$\boldsymbol{a} = w + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z,$$

where w, x, y, z are just positive or negative scalars or real numbers.

Let us now suppose we take the product with another quaternion, say

$$\boldsymbol{a}' = w' + \boldsymbol{i}x' + \boldsymbol{j}y' + \boldsymbol{k}z'.$$

$$\boldsymbol{aa}' = ww' - (xx' + yy' + zz')$$
$$+ \boldsymbol{i}\,(w + w' + yz' - zy') + \boldsymbol{j}\,(w + w' + zx' - xz') + \boldsymbol{k}\,(w + w' + xy' - yx')$$

Multiplication of quaternions

Hamilton called w the *scalar* part of the quaternion $w + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$, and $+ \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$ the *vector* part.

He also described $ww' - (xx' + yy' + zz')$ as the *scalar product* and $\boldsymbol{i}\,(w + w' + yz' - zy') + \boldsymbol{j}\,(w + w' + zx' - xz') + \boldsymbol{k}\,(w + w' + xy' - yx')$ as the *vector product*.

He also introduced a quaternion differential operator,

$$\nabla = \boldsymbol{i}\,\partial / \partial x + \boldsymbol{j}\,\partial / \partial y + \boldsymbol{k}\,\partial / \partial z.$$

The significance of quaternions

He speculated that the real or scalar part of the quaternion represented time and the imaginary or vector part space, and that the quaternion structure showed the long-sought link between them.

Hamilton also realised that unit quaternions could be used to represent rotations in 3-dimensional space.

He was convinced that he had discovered the reason why space had to be 3-dimensional and that quaternions would be the key to unlocking the secrets of the universe.

The significance of quaternions

Maxwell provided alternative quaternion treatments of mathematical operations in his famous *Treatise on Electricity and Magnetism* of 1873.

But then it all went horribly wrong.

People started complaining about the fact that quaternions, when squared in Pythagoras' theorem, produced the wrong sign of product, negative instead of positive.

They also disliked the connection between the real and imaginary parts, preferring a structure that had just three real parts and no imaginary part.

Vector algebra

Ultimately, Gibbs and Heaviside formulated a new *vector* theory, which made the 3-dimensional or vector part real, discarded the fourth component, and recreated the scalar and vector products as two separate operations.

Their vector theory was simply a rule book.

Vector algebra

Vector 'algebra' is not an algebra at all as it has no multiplication and its operations do not exhibit closure.

Vectors can have two 'products', neither of which results in another vector.

The scalar product produces a scalar and the vector product a new type of quantity, a *pseudovector* (or axial vector), which transforms differently to vectors,

while the scalar product of a vector and a pseudovector produces yet another type of quantity, a *pseudoscalar*, which is quite different from a scalar.

The annihilation of quaternions

The supporters of the new theory decided there wasn't room for both vector theory and quaternions and an intensive vilification campaign led to the annihilation of the quaternionists.

Their mathematics was utterly worthless and totally without application.

Hamilton's view that quaternions were the key to the universe was one of the most self-deluding ideas ever attained by a great mathematician. His career, so promising at first, had ended in total tragedy.

Even when relativity made the connection between space and time, no one was interested.

Complexified quaternions

Yet Hamilton was not only right all along about the space-time connection, so removing the first objection, he had already produced the mathematics that would have removed the second objection as well.

This was in his very first development of the original idea.

Since quaternions were quite distinct from ordinary complex numbers, why not combine the two and produce complexified quaternions?

Our base set is now 1, *i*, ***i***, ***j***, ***k***, and, multiplying everything out, we will also generate terms like *i****i***, *i****j***, *i****k***.

Complexified quaternions

For reasons that will soon become clear, I also write $i\boldsymbol{i} = \mathbf{i}$, $i\boldsymbol{j} = \mathbf{j}$, $i\boldsymbol{k} = \mathbf{k}$. If we take the products of these terms, we can write:

$$(i\boldsymbol{i})^2 = (i\boldsymbol{j})^2 = (i\boldsymbol{k})^2 = -i(i\boldsymbol{i})\,(i\boldsymbol{j})\,(i\boldsymbol{k}) = 1$$

$$(i\boldsymbol{i})\,(i\boldsymbol{j}) = i(i\boldsymbol{k})$$

$$(i\boldsymbol{k})\,(i\boldsymbol{i}) = i(i\boldsymbol{j})$$

$$(i\boldsymbol{j})\,(i\boldsymbol{k}) = i(i\boldsymbol{i})$$

The complexified quaternion units $i\boldsymbol{i} = \mathbf{i}$, $i\boldsymbol{j} = \mathbf{j}$, $i\boldsymbol{k} = \mathbf{k}$ are, of course, anticommutative in exactly the same way as ordinary quaternion units, but we now notice an extra feature, the i term outside the bracket that has appeared on the right-hand side of the equations.

Complexified quaternions

This becomes clearer if we write the equations in our alternative, more compactified, notation:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -i\mathbf{ijk} = 1$$
$$\mathbf{ij} = i\mathbf{k}$$
$$\mathbf{ki} = i\mathbf{j}$$
$$\mathbf{jk} = i\mathbf{i}$$

These objects have the properties that we require of vectors. In particular, they square to positive values. But they have something else in addition, an extra property whose meaning didn't emerge until well into the twentieth century. They also incorporate *spin*, that is, the mysterious property introduced by quantum mechanics.

Multivariate vectors

They are isomorphic to the Pauli matrices, originally introduced into nonrelativistic quantum mechanics to incorporate the experimentally-discovered concept of spin.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hestenes termed **i**, **j** and **k** as the units of a *multivariate vector* algebra, with vectors **a** and **b** following a full multiplication rule, incorporating both scalar and vector products in the same way as quaternions:

$$\mathbf{ab} = \mathbf{a.b} + i\,\mathbf{a} \times \mathbf{b}$$

Multivariate vectors

Hestenes showed that if we used the full product $\nabla\nabla\psi$ for a multivariate vector ∇ (basically, Hamilton's own definition of the symbol!) instead of the scalar product $\nabla.\nabla\psi$ for an ordinary vector ∇,

we could obtain spin ½ for an electron in a magnetic field from the nonrelativistic *Schrödinger equation*.

Though the first explanation of spin ½ came from the relativistic Dirac equation, the effect is nothing to do with relativity. It comes from properties deep within 3-dimensionality. The reason why Dirac first obtained it is because he effectively included these properties in the extra algebra he needed to make his equation linear.

Multivariate vectors

In fact, all physical vectors are really multivariate vectors and not ordinary vectors at all. When I use the word ‘vector’ I will use the multivariate definition unless there is a specific reason to do otherwise.

I will also always use **bold** symbols for vectors to distinguish them from the ***bold italics*** used to represent quaternions and the *italics* used for ordinary complex numbers.

Unlike ordinary vector algebra, multivariate algebra is a real algebra. It has closure and a genuine product. It also makes sense of such things as pseudovectors and pseudoscalars, which appear arbitrarily in ordinary vector algebra.

Multivariate vectors

Let's take a simple example. Imagine we have a rectangle with sides **a** and **b**. To find the area, we take the product **ab** = **a.b** + *i* **a** × **b**. Since **a** and **b** are orthogonal, the first term on the right-hand side disappears, leaving us with an imaginary vector in a direction perpendicular to **a** and **b**. The area is an imaginary or *pseudovector*, say *i***A**.

If we then suppose that the rectangle is the base of a solid body with height **c** in the direction of this pseudovector, then the volume will be the product *i***Ac** = *i***A.c** + *i i***A** × **c**. This time, since the vector and pseudovector are parallel, it is the second term which disappears, leaving the product as an imaginary scalar. So volume (the 'triple product' in fact, as well as in name) is a pseudoscalar.

Multivariate vectors

With these extra terms, vector algebra is a larger algebra than quaternion algebra. In fact, quaternions can be seen as a subalgebra of vectors, composed of the pseudovectors and scalars.

Pseudovectors, which include such concepts as torque and angular momentum, are identical in principle to quaternions, and we can (with appropriate sign adjustments) switch from quaternion to vector representations and vice versa simply by multiplying the units by *i*.

Vector algebra in this form also incorporates the 4-vector algebra of relativity, which requires the units (**i**, **j**, **k**, *i*) and is equivalent to the complexified version of a quaternion (with units ***i***, ***j***, ***k***, 1).

Clifford algebra

The most remarkable extension of quaternions comes with the algebra invented by Clifford in the 1870s. (Clifford first wrote down the expression $\mathbf{ab} = \mathbf{a.b} + i\, \mathbf{a} \times \mathbf{b}$.)

Clifford algebra (also called geometrical algebra) is one of the most powerful tools ever offered to the physicist. It seems to be the mathematical code built deep in the structure of physics.

It is still massively under-used, though Dirac recognised early on that the algebra he had devised for his equation for relativistic quantum mechanics was, in fact, a Clifford algebra.

Clifford algebra

Clifford algebra unites real, complex numbers, quaternions and vectors into a single system of infinite potential complexity.

It defines a system with m units which are square roots of 1 (norm 1) and n which are square roots of –1 (norm –1), where m and n are integers of any size. We write this as $Cl(m, n)$ or $G(m, n)$. However, this is not necessarily a unique specification for it is often possible to produce the same algebra with a quite different specification of m and n.

One way of building up Clifford algebras is to use commuting sets of quaternions, which may also be complexified.

Clifford algebra of 3-D space

In its full specification it has 8 basic units, each of which can be + or –.

i **j** **k**	vector		
*i***i** *i***j** *i***k**	bivector	pseudovector	quaternion
i	trivector	pseudoscalar	complex
1	scalar		

Clifford algebra of 3-D space

It has 3 subalgebras: bivector / pseudovector / quaternion, composed of:

i**i** i**j** i**k**	bivector	pseudovector	quaternion
1	scalar		

trivector / pseudoscalar / complex, composed of:

i	trivector	pseudoscalar	complex
1	scalar		

and scalar, with just a single unit:

1	scalar

Clifford algebra of 3-D space

We use the term 'bivector' for the product of two vectors and 'trivector' for the product of three. It is important for us to recognise that we could specify the entire algebra, either by using either **i j k** or a combination of its 3 subalgebras. This will become very significant in our work.

A particularly interesting algebra emerges if we combine this algebra with an identical algebra of 3-dimensional space to which this is commutative.

i **j** **k**	vector		
*i***i** *i***j** *i***k**	bivector	pseudovector	quaternion
i	trivector	pseudoscalar	complex
1	scalar		

The combined algebra

When we multiply these two algebras by each other, term by term, we produce an algebra that has 64 basic units, which are + and – versions of:

i	**j**	**k**	*i***i**	*i***j**	*i***k**	*i*	1
i	**j**	**k**	*i***i**	*i***i**	*i***k**		
ii	**ij**	**ik**	*i***ii**	*i***ij**	*i***ik**		
ji	**jj**	**jk**	*i***ji**	*i***jj**	*i***jk**		
ki	**kj**	**kk**	*i***ki**	*i***kj**	*i***kk**		

The combined algebra

Since vectors are complexified quaternions and quaternions are complexified vectors, we obtain an identical algebra if we use complexified double quaternions:

i	***j****	***k***	*i**i***	*i**j***	*i**k****	*i*	1
i	***j***	***k***	*i**i***	*i**i***	*i**k***		
ii*	***ij***	***ik***	*i**ii***	*i**ij***	*i**ik***		
ji*	***jj***	***jk***	*i**ji***	*i**jj***	*i**jk***		
ki*	***kj***	***kk***	*i**ki***	*i**kj***	*i**kk***		

The combined algebra

Yet another variation can be found using a combination of vectors (blue) and quaternions (red):

$\boldsymbol{i}$	$\boldsymbol{j}^*$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}^*$	i	1
$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{i}$	$i\mathbf{k}$		
$\mathbf{i}\boldsymbol{i}^*$	$\mathbf{i}\boldsymbol{j}$	$\mathbf{i}\boldsymbol{k}$	$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{k}$		
$\mathbf{j}\boldsymbol{i}^*$	$\mathbf{j}\boldsymbol{j}$	$\mathbf{j}\boldsymbol{k}$	$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{k}$		
$\mathbf{k}\boldsymbol{i}^*$	$\mathbf{k}\boldsymbol{j}$	$\mathbf{k}\boldsymbol{k}$	$i\mathbf{k}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{k}$		

The special nature of 3-D

In Clifford algebra the same algebra can have different dimensionalities simultaneously. It depends on how you distribute between m and n.

Two square roots of –1 can become a single square root of 1 and vice versa. And this can be done multiples of times.

You can have a 10-D in one perspective simultaneously with 3-D in another.

Ultimately, 3-D is special and is due to a single, simple property. Any higher D can always be resolved into structures based on 3, and even complex numbers can be represented as incomplete quaternion sets.

Groups

Groups are an important aspect of the mathematics of symmetry. A finite or infinite number of elements form a group if they contain:

(1) a binary operation (e.g. multiplication, permutation) between any two elements to produce another
(2) an identity element, so that the binary operation between the identity element and another element produces that element
(3) an inverse to each element, so that the binary combination of any element and its inverse produces the identity element
(4) closure, so that the binary operation between any two elements always produces an element within the group

Groups: C_2

The simplest group C_2 is of order 2 and so has only 2 elements, which, for example, are 1 and –1.

The binary operation is multiplication.
The identity element is 1.
Each element is its own inverse,
and we have closure because each binary operation between 1 and –1 only ever produces either 1 or –1.

Complex numbers give us a group of order 4 (C_4) made out of the base units 1, –1, i and $-i$.

Groups: quaternions and vectors

Quaternions (*Q*) give us a group of order 8, from the base units 1, –1, ***i***, –***i***, ***j***, –***j***, ***k***, –***k***.

Multivariate vectors, which complexify the quaternion algebra, are a group of order 16. In all these cases negative as well as positive units are required for closure.

Octonions (*O*), however, are not a group because their multiplication is not associative. For any group, finite or infinite, it is possible to produce all the elements of the group using a finite number of elements as *generators*. For a finite group, this may be less than the number of elements in the group. Often, this can be done in many different ways.

Groups: cyclic

C_2 and C_4 are examples of *cyclic* groups. There is a cyclic group at every order, and how they operate can be illustrated by writing down the multiplication table for C_3.

*	I	a	a^2
I	I	a	a^2
a	a	a^2	I
a^2	a^2	I	a

A cyclic group of order n, has elements, I, a, a^2, ... a^{n-1}, with a^n reverting to the identity. So, if we wanted to express the complex number units as a cyclic group of order 4, then I, a, a^2, a^3 would become 1, i, –1, –i, in that order.

Groups: $\mathcal{D}_2$

There is a second group of order 4, which is called D_2 or the Klein-4 group. It is called D_2 or dihedral 2 because it is the group of rotations of the rectangle (identity, and rotations along 3 different axes), and it is the only other group than C_4 of order 4.

It can be generated by a 'double' algebra known as H_4, which is made up of 4 units, constructed from two commutative sets of quaternions, 1, ***i***, ***j***, ***k***, and 1, ***i***, ***j***, ***k***.

Here, the red and blue units multiply commutatively with each other in the ordinary way, but the red units anticommute with *each other*, as do the blue units.

Groups: $\mathcal{D}_2$

The H_4 algebra units can be constructed as 1, ***ii***, ***jj***, ***kk***, and you can see that the units ***ii***, ***jj***, ***kk*** have now become commutative with other, unlike the units of their parent systems. So ***iijj*** = ***jjii***, etc. The H_4 algebra units become a group with the multiplication table:

*	1	***ii***	***jj***	***kk***
1	1	***ii***	***jj***	***kk***
ii	***ii***	1	***kk***	***jj***
jj	***jj***	***kk***	1	***ii***
kk	***kk***	***jj***	***ii***	1

Like quaternions but ignoring the negative signs.

Groups: $\mathcal{D}_2$

Another way of generating the same table is by creating elements made up of variations of three 'components', $\pm x$, $\pm y$ and $\pm z$:

A	x	y	z
B	$-x$	$-y$	z
C	x	$-y$	$-z$
D	$-x$	y	$-z$

The binary operation (*) doesn't have to be anything to do with multiplication. So we can make up our own rules as long as they are rigidly followed and lead to closure.

Groups: $\mathcal{D}_2$

Let's say we have:

$$x * x = -x * -x = x$$
$$x * -x = -x * x = -x$$
$$x * y = y * -x = 0$$

and similarly for y and z. Then we will have a group table:

Groups: $\mathcal{D}_2$

Let's say we have:

$$x * x = -x * -x = x$$
$$x * -x = -x * x = -x$$
$$x * y = y * -x = 0$$

and similarly for *y* and *z*. Then we will have a group table:

*	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

Groups: $\mathcal{D}_2$

I hope to show later that, overall, there is no more important group in the whole of physics.

We can also arrange for a dual version of this group, with assignments such as:

A*	$-x$	y	z
B*	x	$-y$	z
C*	$-x$	$-y$	$-z$
D*	x	y	$-z$

Groups: $\mathcal{D}_2$

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D*	x	y	$-z$

Vector-quaternion algebra: group of order 64

The commutative product of two vector algebras or of a vector algebra with a quaternion algebra is a group of order 64.

1	*i*				–1	–*i*			
i **i**	***i*** **j**	***i*** **k**	*i* ***k***	***j***	–***i*** **i**	–***i*** **j**	–***i*** **k**	–*i* ***k***	–***j***
j **i**	***j*** **j**	***j*** **k**	*i* ***i***	***k***	–***j*** **i**	–***j*** **j**	–***j*** **k**	–*i* ***j***	–***i***
k **i**	***k*** **j**	***k*** **k**	*i* ***j***	***i***	–***k*** **i**	–***k*** **j**	–***k*** **k**	–*i* ***j***	–***i***
i **i** ***i***	*i* **i** ***j***	*i* **i** ***k***	*i* **k**	**j**	–*i* **i** ***i***	–*i* **i** ***j***	–*i* **i** ***k***	–*i* **k**	–**j**
i **j** ***i***	*i* **j** ***j***	*i* **j** ***k***	*i* **i**	**k**	–*i* **j** ***i***	–*i* **j** ***j***	–*i* **j** ***k***	–*i* **i**	–**k**
i **k** ***i***	*i* **k** ***j***	*i* **k** ***k***	*i* **j**	**i**	–*i* **k** ***i***	–*i* **k** ***j***	–*i* **k** ***k***	–*i* **j**	–**i**

Vector-quaternion algebra: group of order 64

Apart from the 4 units of ordinary complex algebra, 1, *i*, –1, –*i*, the other sixty units arrange themselves in 12 groups of 5.

Remarkably, *any* of these groups of 5 will generate the *entire group*, and this is in fact the minimum number of generators.

Significantly, they all have the same overall structure. We take the 8 base units provided by a 4-vector (**i**, **j**, **k**, *i*) combined with a quaternion (***i***, ***j***, ***k***, 1), and break the symmetry of one of the two 3-dimensional structures to create the minimum structure for the generators.

Vector-quaternion algebra: group of order 64

From the perfect symmetry of

i **i** **j** **k** 1 ***i*** ***j*** ***k***

we rearrange to produce:

i	**i**	**j**	**k**	1
k		***i***		***j***

and finally:

*i****k*** ***i*****i** ***i*****j** ***i*****k** 1***j***

Vector-quaternion algebra: group of order 64

Here, the symmetry of the ***red quaternion operators*** is broken in that they are attached to different types of object: pseudoscalar *i*, vector **i j k**, and scalar 1. The symmetry of the vector operators, however, is retained because each of **i j k** is associated with the same object. We can imagine that, if we wanted to make the minimum package containing two 3-dimensional structures, the symmetry of one would be broken in this way.

Exactly the same result would have followed if we had used double vectors (or the units of two 'spaces'), where the 5 generators would have been

*i***k** **ii** **ij** **ik** **j**

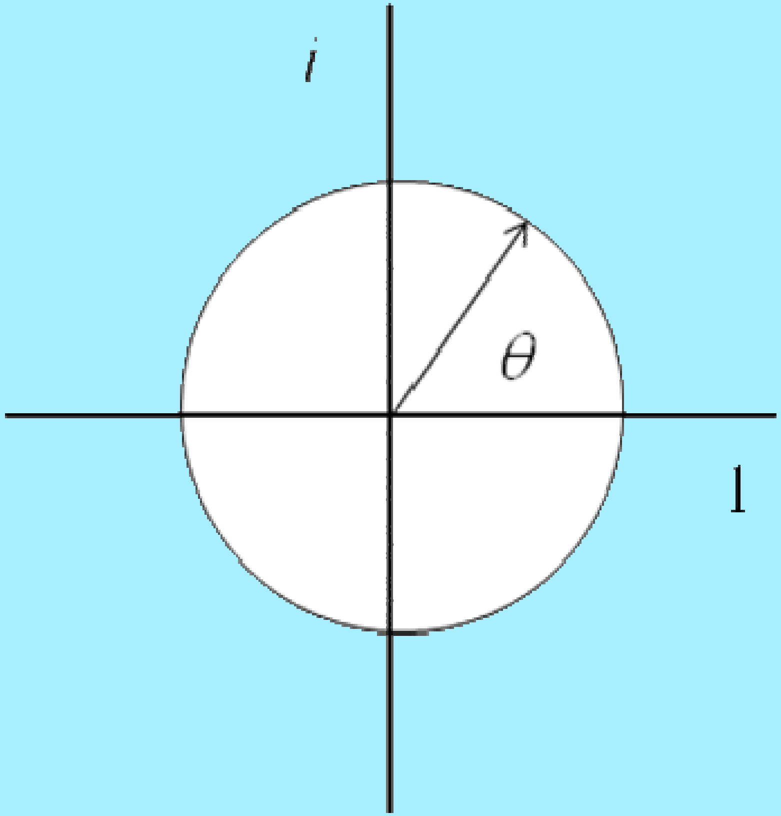
Infinite or Lie groups: U(1)

These have an infinite number of elements and a finite number of generators. Often generated from the rotational aspects of finite groups.

Unitary groups, $U(n)$ defined as the groups of $n \times n$ unitary matrices, and so have a complex determinant with norm 1. A complex square matrix, such as the ones which define $U(n)$ groups, is unitary if the product of the matrix U and its conjugate transpose U^* is the identity matrix I. $U(n)$ groups have n^2 generators.

Simplest, $U(1)$, illustrated by a circle of radius 1 on the Argand diagram centred round the origin. The group then consists of all the complex numbers, $x = iy$, on the circumference of this circle.

Infinite or Lie groups: U(1)



$U(1)$ *symmetry*

Infinite or Lie groups: U(1)

If we imagine a radius vector drawn from the origin to any point on the circumference, the length of the vector will remain at unity regardless of the angle that the vector makes with the x or y axis, which is equivalent to an arbitrary phase term θ between 0 and 2π.

$U(1)$ group is significant as the one connected with the inverse-square law of force, which is derived from the spherical symmetry of space round a point source, and is particularly associated with the electric interaction, which has no other component.

In the quantum version of this interaction, the single generator of the group becomes the photon, the boson which mediates the interaction.

Infinite or Lie groups: SU(2)

The special unitary groups of degree n, or $SU(n)$ are groups of $n \times n$ unitary matrices, like $U(n)$ groups, but this time with the extra specification that the matrices have determinant 1.

$SU(n)$ groups have $n^2 - 1$ generators, the extra constraint of determinant 1 reducing the degrees of freedom, and hence the number of generators, by 1.

$SU(2)$ (which is isomorphic to the group of unit quaternions and can be mapped smoothly onto the 3-sphere) is significant as the symmetry group of the weak interaction in the Standard Model, the 3 generators being equivalent to the 3 weakly interacting bosons, W^+, W^- and W^0.

Infinite or Lie groups: SU(3)

In the Standard Model the W^0 mixes with the $U(1)$ field to produce the Z^0 and the photon. It is also the symmetry group for fermion spin.

$SU(3)$ is the symmetry group for the strong interaction, with 8 generators, which can be identified with the gluons or bosons mediating the interaction.

A more advanced discussion of Lie groups would show that those groups of most interest in physics, including $SU(2)$, $SU(3)$, and the exceptional groups F_4, E_6, E_7 and E_8, can be derived from the symmetries associated with the real numbers, complex numbers, quaternions and octonions, and are ultimately an expression of the special nature of 3-D space, glimpsed in Hamilton's discovery of quaternions.

Nilpotents

Nilpotents are defined as objects for which *any* finitely repeated operation will yield zero, but here we are concerned only with the case where the operation is squaring.

Trivially, of course, 0 is nilpotent, but we can also create nonzero nilpotents when anticommutation removes the cross terms in the product by mutual cancellation.

Pythagoras' theorem can be structured entirely in nilpotent form, e.g.

$$(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)^2 = 5^2 - 4^2 - 3^2 = 0.$$

$$(i\mathbf{k}5 + \mathbf{i}\mathbf{4} + \mathbf{j}3)^2 = 0.$$

Idempotents

Idempotents are objects which square to *themselves*. Let us premultiply ($i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3$) by $\mathbf{k}$ and scaling a factor a to be determined later.

$$a\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)\;a\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)$$

Now $$a\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)\,\mathbf{k} = a\,(-i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)$$

and $$(-i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3) = (-2i\mathbf{k}5 + i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)$$

So

$$a\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)\;a\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3) = a\mathbf{k}\,(-2i\mathbf{k}5)\;a\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)$$
$$= 10ia^2\,\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3)$$

So we make $a = 1 / 10i$ and $\mathbf{k}\,(i\mathbf{k}5 + \mathbf{i}4 + \mathbf{j}3) / 10i$ becomes an idempotent, or square root of itself.

Real numbers

Real numbers begin with the natural numbers, 1, 2, 3, etc., which are quickly generalised to the integers, which may be negative as well as positive.

The next stage is to take fractions of integers and define these as rational numbers. Then come algebraic numbers, which are solutions of algebraic equations, that is equations in which a polynomial expression in one variable is equated to 0, for example $x^2 - 2 = 0$.

Finally, there are transcendental numbers, such as π and e, which are not generated by algebraic equations, but can usually be expressed only in terms of an infinite series. All of these combined are the real numbers.

Real numbers

In the Cantor argument rational and algebraic numbers can be counted by being put into a one-to-one relation with the integers, but the transcendental numbers (and hence the real numbers as a whole) cannot.

Between any two numbers that can be counted in this way there are an infinite number of real numbers that cannot. The real numbers, therefore, are uncountable or non-denumerable.

This argument is valid and most people think it is uniquely so, in that the opposite construction of countable real numbers is false. This, however, is not the case.

Real numbers

Even though the Cantor argument is valid in that real numbers can be defined in this way, it is *not uniquely true*.

Real numbers can be constructed in such a way that they can be put into a one-to-one correspondence with the integers and so can be made denumerable in the same way as all other numbers.

This is because to construct a real number requires an algorithmic process, and algorithmic processes can be counted. So, if we think of real numbers as simply ‘there’ in nature, then they cannot be counted. If we think of them as always the result of a construction, then they can.

Standard and nonstandard analysis

The procedure for counting real numbers in this way was first generated by Skolem in 1934 and is called ‘non-standard arithmetic’ (as opposed to ‘standard arithmetic’, based on the Cantor argument).

Löwenheim-Skolem theorem, any consistent finite, formal theory has a denumerable model, in which the elements of its domain are in a one-to-one correspondence with the positive integers.

Subsequently applied to the real numbers in the construction of space in non-Archimedean geometry. A parallel development also applies to calculus in which non-standard analysis aligns itself with non-standard arithmetic in the same way as standard analysis aligns itself with the arithmetic based on the Cantor continuum.

Standard and nonstandard analysis

We are taught differentiation using infinitesimals. You draw a graph of a function and draw a line between two points on the graph with horizontal and vertical separations δx and δy, then use the function of the graph to find an expression for $\delta y / \delta x$.

You then make the two points approach each and say that at an infinitesimal distance apart you cancel any term with δx in it, leaving you with an expression for the differential dy / dx.

This is a very effective procedure, used by seventeenth century mathematicians and cast into the form familiar to us by Leibniz, but it was always considered nonrigorous.

Standard and nonstandard analysis

To do differentiation ‘properly’ meant ‘taking the limit’, a process introduced by Newton (who had grown dissatisfied with infinitesimals) and perfected by Cauchy, in which there were no such cancellations.

But, in the twentieth century, Abraham Robinson developed a new procedure in which infinitesimals could be made as rigorous as limits (incidentally, using a nilpotent-type structure).

Standard and nonstandard analysis

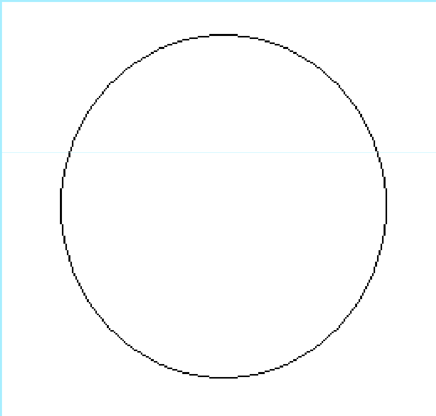
This ‘non-standard’ procedure and the standard analysis based on limits produce exactly the same results, and neither method is always superior to the other in proving mathematical theorems.

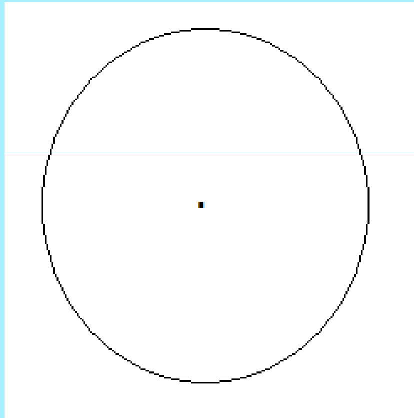
Sometimes one method is superior, sometimes the other.

The methods of standard and non-standard analysis and arithmetic, and Archimedean and non-Archimedean geometry, are completely dual and cannot be distinguished by any known mathematical theorem or methodology.

Topology

A simply-connected space is one without singularities. A multiply-connected space is one with a topological singularity.





simply-connected space   *multiply-connected space*

The topology of $SU(n)$ groups, notably, is simply-connected, while that of $U(n)$ groups is not.

Topology

If we imagine ‘parallel-transporting’ a vector round a closed circuit in the simply-connected space, then, on returning to the starting position it will be pointing in the original direction.

If the space is multiply-connected, however, it will, on its return, be pointing in the *opposite direction*.

There will be a phase change of 180^{o} or π radian. To return to the start pointing in its original direction, it will have to do a *double circuit*.

An illustration of a general phenomenon, the Berry phase or geometric phase, which can always be mapped onto a topological representation.

Key numbers: duality, commutativity and symmetry-breaking

A few key integers turn up regularly in physics and sometimes in other areas of science (particularly, in biology). Some have a very primitive origin in the deepest levels of physics.

The most important are 2, 3 and 5. Generally, where 2 occurs it can be tracked down to duality, where 3 occurs it comes from anticommutativity, and where 5 occurs it represents symmetry-breaking.

5 always comes from complexity, but 2 and 3 come from a genuinely primitive level. Nearly all the group structures that are important in physics, including such seemingly complicated ones as E_8, can be interpreted in terms of these numbers, and the individual introduction of the numbers tracked down to their roots.

The End