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Keywords: Deconvolution kernel, Fourier transform, measurement errors, market microstructure noise, optimization, Value-at-Risk.

JEL Classification Number: G11, G19, C14, C61, C63.

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ABSTRACT

We propose an optimization-based estimation of Value-at-Risk that corrects for the effect of measurement errors in prices. We show that measurement errors might pose serious problems for estimating risk measures like Value-at-Risk. In particular, when the stock prices are contaminated, the existing estimators of Value-at-Risk are inconsistent and might lead to an underestimation of risk, which might result in extreme leverage ratios within the held portfolios. Using Fourier transform and a deconvolution kernel estimator of the probability distribution function of true latent prices, we derive a robust estimator of Value-at-Risk in the presence of measurement errors. Monte Carlo simulations and a real data analysis illustrate satisfactory performance of the proposed method.

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1 Introduction

Since its first appearance in the 1980s, Value-at-Risk (VaR) has become the most widely used risk management tool in financial services industry. Indeed, VaR has gained ground because it is a relatively simple measure to estimate and it was established by the Basel II regulatory framework as the benchmark method for market risk capital requirements calculation; see Basel Committee on Banking Supervision (1996, 2006). In addition to its use as a risk measure, VaR can be used as the basis for portfolio optimization; see e.g. Lwin et al. (2017), Yiu (2004), and Alexander and Baptista (2008). Due to its popularity as a tool for controlling risk, financial managers are rightfully concerned with the precision of VaR estimation. One problem that might affect this estimation is effectively the presence of measurement errors in assets' prices, which can be caused by non-synchronous trading, rounding errors, infrequent trading, market microstructure noise, manipulations (smoothing, extra revenues, fraudulent exchanges, informationless trading), etc. Finance literature has investigated the impact of contaminated prices on the estimation of volatility and developed robust methods to identify the variance of true stock prices; see e.g. Zhou (1996); Andersen et al. (2001); Zhang et al. (2005); Bandi et al. (2006); Barndorff-Nielsen et al. (2011); Hansen and Lunde (2006); and Mancino and Sanfelici (2008). However, no attention is paid to the effect of this contamination on the VaR estimation, and thus no robust estimation technique is available under measurement error.

The presence of measurement error (known as market microstructure noise) in high-frequency prices has been well established in the literature; see Madhavan (2000) for a survey on market microstructure noise. In addition, several papers have investigated the presence of measurement errors in low-frequency prices/returns. For instance, by collecting specific information on risk management from the annual reports of the largest 200 US and international commercial banks for the period 2005-2008, Frésard et al. (2011) find that only a very small fraction of financial institutions (less than 6 %) uses uncontaminated returns to estimate their VaR models. The large fraction uses contaminated data which include intraday revenues, fees, or commissions. They also show that all of the available backtesting procedures are highly sensitive to data contamination. For example, using the "traffic light" approach developed by the Basel Committee, 23.5% of the VaR models are rejected when tested with uncontaminated data, whereas only 10.8% are rejected when tested with returns that include both fees and intraday trading revenues. Therefore, data contamination has undesirable implications for model validation and can lead to the acceptance of misspecified

VaR models, and therefore significantly reduced regulatory capital. Furthermore, Pérignon et al. (2008) assess the accuracy of banks' risk management systems based on daily VaR and profit-and-loss data. They find evidence supporting the idea that banks exhibit a systematic bias in their VaR estimates. They attribute this bias to several factors, including extreme cautiousness, underestimation of diversification effects, and measurement errors.

None of the above papers, however, is interested in developing robust estimation techniques to correct for the effect of measurement errors. In this paper, we propose a semiparametric approach for the estimation of VaR in the presence of measurement errors in stock prices. It is worth mentioning that, both theoretically and computationally, it is more straightforward to deal with the measurement errors in the context of variance than in the context of VaR, as it is generally simpler to work with the moments than with the quantiles. Furthermore, working with an additive measurement error model makes the derivation of robust estimation techniques much easier for variance than for quantiles (VaR). We remind the reader that an additive measurement error model defines the measurement error as the difference between the observed stock price and the true (latent) stock price. For a general discussion on additive and non-additive error models, the reader is referred to Schennach (2016).

To develop our robust estimator of VaR, we consider a different approach than the one used in the literature to derive robust measures of variance. To deal with the contamination, we first use a deconvolution kernel estimator for the density function of the true latent portfolio returns.[1] There is a rich literature on using density deconvolution for estimating probability density functions. Recently, Adusumilli et al. (2020) have studied inference on the cumulative distribution function (CDF) in the context of classical measurement error problem using density deconvolution. They also applied their results to various contexts; such as building confidence bands for CDF and its quantiles, and for performing various goodness-of-fit tests for parametric models of densities and tests for stochastic dominance. For a review on deconvolution methods, the readers can consult Meister (2009), Fan (1991), Hall and Lahiri (2008), Dattner et al. (2011) among others. Thereafter, we use Fourier inversion to calculate the probability distribution function of the latent portfolio

[1] We remind the reader that the concept of deconvolution corresponds to computing the inverse of the convolution operation of two functions. Convolution operation on two functions, say f and g, produces a third function $(f * g)$ that expresses how the shape of one is modified by the other. It is defined as the integral of the product of the two functions after one is reversed and shifted. In the context of a measurement error problem, Kernel deconvolution density estimation consists in estimating the density of a variable of interest (here the density of the true stock price) that is observable only with some measurement error.

returns. We then use power series representations of sine and exponential functions to approximate the integral in the inversion formula and define an optimization problem that make the calculation of VaR computationally feasible. Roughly speaking, power series representations are about representing common functions as polynomials with infinitely many terms, thus integrating a power series is as easy as integrating a polynomial.

The derivation of robust estimator of VaR is first made under the assumption that the density of measurement error is known, but the distribution of the observed portfolio returns is always treated as unknown and estimated nonparametrically. Thereafter, we relax this assumption and suggest a feasible way to deal with the measurement error's distribution. In particular, we follow the literature and assume that the measurement error is normally distributed but with unknown variance that we estimate nonparametrically using high-frequency data and a consistent estimator of variance of measurement error from Zhang et al. (2005). It is worth noting that assuming that the measurement error is normally distributed does not contradict the fact that returns can be non-normally distributed.

We conduct a set of Monte Carlo simulations to examine the performance of our approach under the presence of measurement error. We provide a comparison with a model-free estimator of VaR that does not adjust for the measurement error. We investigate the performance of our method for known and unknown density of the measurement error and the simulation results are very encouraging. Furthermore, we apply our approach to high-frequency data to estimate the VaR of five international market indices. We compare our results with the unadjusted VaR that we obtain using a model-free estimator that simply computes the sample quantiles based on the five indices' returns. The empirical results indicate that ignoring the measurement error leads to an underestimation of risk.

The rest of the paper is organized as follows. In Section 2, we introduce the additive measurement error model for the asset prices and show how to estimate the characteristic and distribution functions of latent portfolio returns. In Section 3, we derive an optimization-based estimator of VaR of latent portfolio returns. Section 4 analyzes the performance [bias, standard deviation, and root mean square error] of our method through a Monte Carlo experiment. In Section 5, we use our approach and high-frequency data to estimate the adjusted VaR of five stock indices. Section 6 concludes, and all mathematical proofs and tables of Monte Carlo simulation results are presented in the Appendix.

2 Framework

The methodology that we develop here works for both individual assets and portfolios. The exposition is made for a portfolio of assets, but an individual asset is a special case by setting all portfolio weights equal to zero, except the weight of the asset in question. Formally, we assume that there are n risky assets in the economy. We denote by $P_t = (p_{1,t}, ..., p_{n,t})'$ and $P_t^* = (p_{1,t}^*, ..., p_{n,t}^*)'$ the vectors of observed and true latent log prices of the n assets at time t, respectively. We suppose that the observed price of each asset j can be contaminated by a measurement error:

$$p_{j,t} = p_{j,t}^* + \epsilon_{j,t}, \text{ for } j = 1, \ldots, n \text{ and } t = 1, \ldots, T, \tag{1}$$

where $p_{j,t}$ ($p_{j,t}^*$) is the observed (true latent) log price of asset j, and $\epsilon_{j,t}$ is the measurement error which we assume to be independent and identically distributed (i.i.d.) with mean zero and variance σ_ϵ^2. The assumption of i.i.d measurement error might be avoided, but this will be at the cost of complicating the calculation of our estimator of VaR. This might require the estimation of the distribution of measurement error (which is very difficult in practice), or in the best case scenario (if we make an assumption on the distribution of measurement error as we do in this paper), we will need to estimate the moments of the measurement error (e.g. its variance as we do in this paper), but also its dependence/correlation structure. The i.i.d. assumption, however, is compelling and widely used in the literature; see for example Zhang et al. (2005), Griffin and Oomen (2011), Schennach (2004, 2016) among others. In addition, the vector errors $\epsilon_t = (\epsilon_{1,t}, ..., \epsilon_{n,t})'$ is assumed to be independent of the vector of true latent prices P_t^*. The elements of ϵ_t are also assumed to be cross-sectionally independent and normally distributed; i.e., $\epsilon_t \sim N(0, \sigma_\epsilon^2 I_n)$, with I_n is an $n \times n$ identity matrix. In this section, we assume that the variance of the measurement error is known, but later on we will relax this assumption and discuss how to estimate σ_ϵ^2.

Using Equation (1), the observed and true (latent) returns of a portfolio of n assets are linked as follows:

$$r_{p,t} = r_{p,t}^* + e_t, \tag{2}$$

where $r_{p,t} = \sum_{j=1}^n \omega_j r_{j,t}$ is the observed portfolio return, $r_{p,t}^* = \sum_{j=1}^n \omega_j r_{j,t}^*$ is the true latent portfolio return, $e_t = \sum_{j=1}^n \omega_j u_{j,t}$ denotes the measurement error in the portfolio returns, with $\omega = (\omega_1, ..., \omega_n)'$ is a known vector of weights that are attributed to each risky asset in the portfolio and $u_{j,t} = \epsilon_{j,t} - \epsilon_{j,t-1}$, for $j = 1, ..., n$, is an error term that is identically distributed (normally distributed) with mean zero and variance $\sigma_u^2 = 2\sigma_\epsilon^2$. Here, the assets' returns correspond to continuously compounded returns (first differences of log assets' prices). It is also worth noting that the distributions of observed and true latent returns are unknown. This implies that even when the measurement error $u_{j,t}$ is normally distributed, the observed and true latent returns can be non-normally distributed.

Since P_t^* and ϵ_t are independent processes, using Equation (2) and Fourier transform, we obtain:

$$\phi_{r_p}(s) = \phi_{r_p^*}(s)\,\phi_e(s), \tag{3}$$

where $\phi_{r_p}(s)$, $\phi_{r_p^*}(s)$, and $\phi_e(s)$ represent the characteristic functions of $r_{p,t}$, $r_{p,t}^*$ and e_t, respectively.

On the one hand, under the assumption $\epsilon_{j,t} \sim N(0, \sigma_\epsilon^2)$, the characteristic function of measurement error e_t is given by:

$$\phi_e(s) = E\left[\exp\left(is\sum_{j=1}^{n}\omega_j u_{j,t}\right)\right] = \exp\left(-s^2\sigma_\epsilon^2\sum_{j=1}^{n}\omega_j^2\right). \tag{4}$$

On the other hand, the characteristic function of the observed portfolio returns $\phi_{r_p}(s)$ can be estimated using its empirical analogue:

$$\hat{\phi}_{r_p}(s) = \frac{1}{T}\sum_{t=1}^{T}\exp\left(isr_{p,t}\right), \tag{5}$$

where T is the number of observations and i is a complex number with $i^2 = -1$. Combining equations (3), (4) and (5), an estimator of the characteristic function of the true latent portfolio returns $\phi_{r_p^*}(s)$ is given by:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} = \frac{1}{T}\sum_{t=1}^{T}\exp\left(isr_{p,t} + s^2\sigma_\epsilon^2\sum_{j=1}^{n}\omega_j^2\right).$$

Nevertheless, later we will be interested in the estimation of the probability distribution function of $r_{p,t}^*$, which is defined (by Fourier transform) as the integral of $exp(-isr_p^*)\hat{\phi}_{r_p^*}(s)$. However, this integral is not well-defined as $\hat{\phi}_{r_p^*}(s)$ is neither integrable nor square integrable over $\mathbb{R}$. In this case, $\hat{\phi}_{r_p^*}(s)$ will not be a good estimator of $\phi_{r_p^*}(s)$ for large values of s. One way to overcome this issue is by regularizing $\hat{\phi}_{r_p^*}(s)$ as follows:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)}K^{ft}(sb), \tag{6}$$

where $K^{ft}(sb)$ is a Fourier transform of a kernel function with a bandwidth b; see Adusumilli, Otsu, and Whang (2020) and Otsu and Taylor (2020). Hereafter, we assume $\phi_e(s) \neq 0$ for all $s \in \mathbb{R}$, and $K^{ft}(s) = 1(-1 \leq s \leq 1)$ with $1(\cdot)$ designating an indicator function. Using the latter kernel, the function $\hat{\phi}_{r_p^*}(s)$ in Equation (6) is supported on $[-1/b, 1/b]$ and bounded whenever $\phi_e(s) \neq 0$, $\forall s \in \mathbb{R}$. Therefore, the regularized estimator $\hat{\phi}_{r_p^*}(s)$ is well-defined. Note that $K^{ft}(s)$ is the Fourier transform of the sinc kernel $K(x) = \sin(x)/(\pi x)$ popularly adopted in the deconvolution method. These assumptions are common in the literature; see for example Otsu and Taylor (2020). Other forms of $K^{ft}(s)$ can also be considered without affecting our main results. Thus, we obtain the following semiparametric estimator of the characteristic function of the latent portfolio returns:

$$\hat{\phi}_{r_p^*}(s) = \frac{\hat{\phi}_{r_p}(s)}{\phi_e(s)} K^{ft}(sb) = \frac{1}{T}\sum_{t=1}^{T} \exp\left(isr_{p,t} + s^2\sigma_\epsilon^2 \sum_{j=1}^{n} \omega_j^2 \right) K^{ft}(sb). \tag{7}$$

Notice that other semiparametric estimators of $\phi_{r_p^*}(s)$ can be obtained by considering other nonparamateric estimators of $\phi_{r_p}(s)$. To avoid some undesirable properties of the empirical characteristic function in (5) for large s, we can use a kernel-based estimator of $\phi_{r_p}(s)$; see e.g., Abdushukurov and Norboev (2017) and Su and White (2007). Formally, we define a kernel function $k(u) : \mathbb{R} \to \mathbb{R}$ that satisfies $\int_{-\infty}^{+\infty} k(u)du = 1$, and we recall the kernel estimator of the cumulative distribution function of $r_{p,t}$, say $F(r_p)$,

$$F_T(r_p) = \frac{1}{T}\sum_{t=1}^{T} \bar{K}_T(r_p - r_{p,t}), \tag{8}$$

where $\bar{K}_T(r_p) = \bar{K}\left(\frac{r_p}{h_T}\right)$ and $\bar{K}(\cdot)$ is a distribution function with the density $k(r_p) = \frac{\partial \bar{K}(r_p)}{\partial r_p}$, and h_T represents a bandwidth parameter that controls the degree of smoothing the distribution function of r_p. Using Equation (8), a kernel estimator of the characteristic function $\phi_{r_p}(s)$ is given by:

$$\hat{\phi}_{r_p}(s) = \int_{-\infty}^{+\infty} \exp(isr_{p,t})\, dF_T(r_{p,t}), \tag{9}$$

where $F_T(r_{p,t}) = \frac{1}{T}\sum_{i=1}^{T} K_T(r_{p,t} - r_{p,i})$. The asymptotic properties of the estimator in (9) can be found in Abdushukurov and Norboev (2017). We next plug the above estimator $\hat{\phi}_{r_p}(s)$ in Equation (6) to obtain an alternative semiparametric estimator of $\phi_{r_p^*}(s)$.

We can now use $\hat{\phi}_{r_p^*}(s)$ to derive an estimator of the probability distribution function of the true latent portfolio returns $r_{p,t}^*$. A standard Fourier-inversion formula [see Gil-Pelaez (1951)] implies

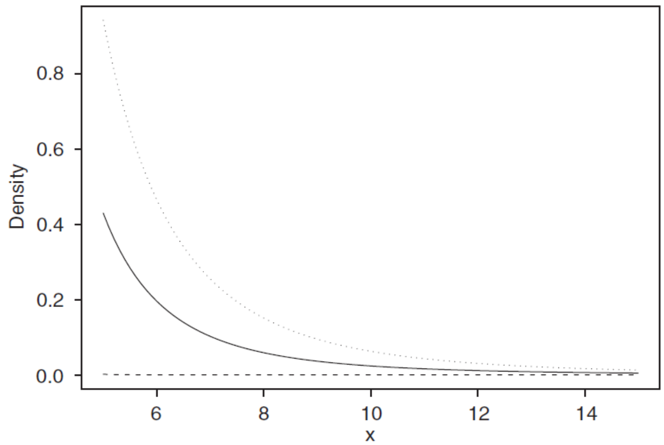
$$\Pr\left(r_{p,t}^* < r_p^*\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\phi_{r_p^*}(s)\exp(-isr_p^*)\right]}{s} ds, \text{ for } r_p^* \in \mathbb{R}, \tag{10}$$

which we can estimate as follows:

$$\widehat{\Pr}\left(r_{p,t}^* < r_p^*\right) = \frac{1}{2} - \frac{1}{T}\sum_{t=1}^{T} L\left(\frac{r_{p,t} - r_p^*}{b}\right), \tag{11}$$

where Im[·] stands for the imaginary part of a complex number, $L(u) = \frac{1}{\pi}\int_0^1 \frac{\sin(su)}{s}\frac{1}{\phi_e(\frac{s}{b})}ds$ is the so-called deconvolution kernel, b is the bandwidth parameter converging to zero at a suitable rate as sample size increases, and ϕ_e is the Fourier transform of the density of the measurement error e.

Figure 1: Tail of convoluting Normal and Student distributions



Note [Source Nason (2006)]: Tail part of p.d.f. of $r_t = r_t^* + e_t$ (solid line); standard normal density for e_t (dashed line); sphered Student's t distribution of 3 degrees of freedom for r_t^* (dotted line).

In the next section, we use the result in Equation (11) to derive an optimization-based estimator of VaR which is robust to the measurement errors that affect stock prices. We shall first consider Figure 1 [see Nason (2006)] to illustrate the undesirable impact that measurement errors can have on risk estimation. The figure shows the right tail parts of standard normal distribution (dashed line), here e_t, Student's t distribution with 3 degrees of freedom (dotted line), here r_t^*, and their sum $r_t = r_t^* + e_t$ (solid line). The latter can be viewed as an additive measurement error model, where r_t represents the observed stock or portfolio return, r_t^* is the true latent stock or portfolio return, and e_t is a measurement error. The density of r_t can be represented as the convolution of the density functions of e_t and r_t^*. The figure shows that the right tail of the distribution of r_t^* (true latent return) dominates the right tail of the distribution of r_t (observed return), and similar situation occurs at the left tail since the distributions of r_t, r_t^*, and e_t are all symmetric. Therefore, the existing estimators of VaR that ignore the measurement error are inconsistent and lead to an underestimation of risk. Underestimating risk might have disastrous effects and the financial crisis of 2008 being an example. We all know that this crisis was in part due to underestimating the risk magnitude of portfolios of subprime mortgages, which resulted in extreme leverage ratios within these portfolios and left institutions unable to cover billions of dollars in losses as subprime mortgage values collapsed. Consequently, providing robust estimation methods for VaR under measurement error should be extremely valuable for risk analysis.

3 VaR under measurement error

VaR is a quantile measure that quantifies the worst expected loss over a given horizon (typically a day or a week) at a given statistical confidence level α (typically 5% or 10%). Several parametric and nonparametric approaches have been used to derive estimators of VaR; for a review the reader can consult Abad, Benito, and López (2014). The level of difficulty of these approaches depends on the assumptions made about the underlying process of returns. As we allow for the latter to capture more stylized effects, the estimation approach becomes more complex. In addition, except when the returns follow elliptical conditional distribution, the estimation of VaR generally requires the use of either simulation or optimization methods. All these estimation techniques, however, are sensitive to the presence of measurement errors in assets' prices. In the following, we use the results from the previous section to derive an optimization-based semi-nonparametric estimator of VaR when the assets' prices are contaminated.

We now follow the convention and let the VaR of the true latent portfolio returns, say $VaR^{\alpha}(r^*_{p,t})$, be a positive quantity. Then, replacing r^*_p by $-VaR^{\alpha}(r^*_{p,t})$ in Equation (11) leads to

$$\widehat{\Pr}\left(r^*_{p,t} < -VaR^{\alpha}(r^*_{p,t})\right) = \frac{1}{2} - \frac{1}{\pi}\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sigma^2_{\epsilon}\sum_{j=1}^{n}\omega_j^2\right)\sin\left(s\left(\frac{r_{p,t}+VaR^{\alpha}(r^*_{p,t})}{b}\right)\right)}{s}ds. \quad (12)$$

Following a similar approach to Duffie and Pan (2001) and Taamouti (2009), $VaR^{\alpha}(r^*_{p,t})$ can then be calculated by inverting the distribution function in (12). However, for reasons we explain below, analytically inverting the function (12) is not feasible and a numerical solution is required. We have the following proposition which can be deduced from Equation (12).

Proposition 1 *The VaR of the true latent portfolio return $r^*_{p,t}$ in (2), at a nominal coverage rate α, denoted by $VaR^{\alpha}(r^*_{p,t})$, is the solution of the following equation:*

$$\frac{1}{T}\sum_{t=1}^{T}\int_0^1 \frac{\exp\left(\frac{s^2}{b^2}\sigma^2_{\epsilon}\sum_{j=1}^{n}\omega_j^2\right)\sin\left(s\left(\frac{r_{p,t}+VaR^{\alpha}(r^*_{p,t})}{b}\right)\right)}{s}ds - \left(\frac{1}{2}-\alpha\right)\pi = 0,$$

where $r_{p,t}$ is the observed portfolio return in (2), σ^2_{ϵ} is the variance of the measurement error, ω_j for $j = 1, ..., n$ are the portfolio weights, and b is the bandwidth parameter.

From Proposition 1, $VaR^{\alpha}(r_{p,t}^{*})$ can be obtained by numerically solving the equation:

$$f\left(VaR^{\alpha}(r_{p,t}^{*})\right)=\frac{1}{T}\sum_{t=1}^{T}\int_{0}^{1}\frac{\exp\left(\frac{s^{2}}{b^{2}}\sigma_{\epsilon}^{2}\sum_{j=1}^{n}\omega_{j}^{2}\right)\sin\left(s\left(\frac{r_{p,t}+VaR^{\alpha}(r_{p,t}^{*})}{b}\right)\right)}{s}ds-\left(\frac{1}{2}-\alpha\right)\pi=0. \tag{13}$$

The function $f\left(VaR^{\alpha}(r_{p,t}^{*})\right)$ can be rewritten as follows:

$$f\left(VaR^{\alpha}(r_{p,t}^{*})\right)=-\pi\left[\widehat{\Pr}\left(r_{p,t}^{*}<-VaR^{\alpha}(r_{p,t}^{*})\right)-\alpha\right]. \tag{14}$$

From Equation (14) and the properties of the probability distribution function [monotonically increasing, $\lim_{x\to-\infty}\widehat{\Pr}\left(r_{p,t}^{*}<x\right)=0$, and $\lim_{x\to+\infty}\widehat{\Pr}\left(r_{p,t}^{*}<x\right)=1$], we can show that Equation (13) admits a unique solution. Another way to calculate $VaR^{\alpha}(r_{p,t}^{*})$ is considering the following optimization problem:

$$\widehat{VaR}^{\alpha}(r_{p,t}^{*})=\underset{VaR^{\alpha}(r_{p,t}^{*})}{Argmin}\left[\frac{1}{T}\sum_{t=1}^{T}\int_{0}^{1}\frac{\exp\left(\frac{s^{2}}{b^{2}}\sigma_{\epsilon}^{2}\sum_{j=1}^{n}\omega_{j}^{2}\right)\sin\left(s\left(\frac{r_{p,t}+VaR^{\alpha}(r_{p,t}^{*})}{b}\right)\right)}{s}ds-\left(\frac{1}{2}-\alpha\right)\pi\right]^{2}. \tag{15}$$

In practice, an *exact* solution for the above minimization problem is not feasible, since the integral term in (15) is difficult to assess. This issue can be solved using a numerical integration based on equally spaced abscissas as in Davies (1973) and Davies (1980); see Duffie and Pan (2001) and Taamouti (2009). However, this approach introduces two types of errors: the discretization error and the truncation error. In this paper, we instead propose a closed-form expression for the Fourier inversion in (10) by regularizing the estimated characteristic function $\hat{\phi}_{r_{p}^{*}}(s)$ using the characteristic function $K^{ft}(sb)$, which is defined on $s\in[-1/b,1/b]$, and by using power series representation of the function $\sin(\cdot)$. Specifically, from power series representations of $\sin(\cdot)$ and $\exp(\cdot)$ functions, we obtain the following corollary [see the proof of Corollary 1 in the Appendix].

Corollary 1 *The VaR of the true latent portfolio return $r_{p,t}^{*}$ in (2), at a nominal coverage rate α, denoted by $VaR^{\alpha}(r_{p,t}^{*})$, is the solution of the following optimization problem:*

$$\widehat{VaR}^{\alpha}(r_{p,t}^{*})=\underset{VaR^{\alpha}(r_{p,t}^{*})}{Argmin}\left[\frac{1}{T}\sum_{t=1}^{T}\sum_{i,j=0}^{\infty}\frac{a^{i}}{i!}\frac{(-1)^{j}\left(\frac{r_{p,t}+VaR^{\alpha}(r_{p,t}^{*})}{b}\right)^{1+2j}}{(1+2j)!(2i+2j+1)}-\left(\frac{1}{2}-\alpha\right)\pi\right]^{2}, \tag{16}$$

where $a=\frac{\sigma_{\epsilon}^{2}\sum_{j=1}^{n}\omega_{j}^{2}}{b^{2}}>0$, $r_{p,t}$ is the observed portfolio return in (2), σ_{ϵ}^{2} is the variance of the measurement error, ω_{j} for $j=1,...,n$ are the portfolio weights, and b is the bandwidth parameter.

The optimization problem in (16) depends on the infinite sum $\sum_{i,j=0}^{\infty}$, which has to be truncated. Thus, the estimation of $VaR^{\alpha}(r_{p,t}^{*})$ will involve a truncation error that we need to control. For $\kappa = \frac{r_{p,t}+VaR^{\alpha}(r_{p,t}^{*})}{b}$, the infinite sum can be decomposed as follows:

$$S = \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = S_{l,k} + R_{l,k} = S_{l,k} + R_{l,k}^{(1)} + R_{l,k}^{(2)} + R_{l,k}^{(3)}, \tag{17}$$

where the truncated and remaining terms $S_{l,k}$ and $R_{l,k}$ ($R_{l,k}^{(1)}$, $R_{l,k}^{(2)}$, and $R_{l,k}^{(3)}$) are defined in the Appendix [see Equation (22)]. Furthermore, we show that the remaining terms $R_{l,k}^{(1)}$, $R_{l,k}^{(2)}$, and $R_{l,k}^{(3)}$ are bounded [see equations (23), (24), and (25) of the Appendix], and consequently $\lim_{l,k \longrightarrow \infty} \left\{ R_{l,k}^{(s)} \right\} = 0$, for $s = 1, 2, 3$ [see the proof in the Appendix]. However, how large should l and k be in practice to make the remaining term negligible is a question that we investigated extensively by simulation. For the data generating processes that we consider in Section 4 and a bandwidth b, which we select according to the rule of thumb $b = c \times (2/\log T)^{1/2}$ [see Otsu and Taylor (2020)], we find that taking $l = k = 10$ in Formula (17) yields satisfactory results. Specifically, we find that there is no improvement in terms of bias and Mean Square Error when we increase l and k beyond 10.

The above calculation of VaR, however, depends on the unknown variance of the measurement error σ_{ϵ}^2. We next discuss how one can estimate this variance using high-frequency data. A more general approach for the estimation of unknown density of the measurement error, say f_e, is also available but requires the use of repeated measurements of the true latent portfolio returns. In other words, if we further assume that the Fourier transform of the measurement error $\phi_e(s)$ is real-valued, that is, the density f_e is symmetric around zero, then - if repeated measurements of the true latent portfolio returns are available - we can estimate $\phi_e(s)$ using the estimator proposed by Delaigle et al. (2008). In finance, however, repeated measurements for asset prices are scarce.

In the context of high-frequency data, a consistent nonparametric estimator of the variance of the measurement error (market microstructure noise) can be obtained as a by-product of the results in Zhang et al. (2005). Formally, assuming that the full grid containing all of the observation points is given by $G = \{t_0, ..., t_m\}$ and using the consistent estimator of the variance of the market microstructure noise provided in Zhang et al. (2005, page 1402), a consistent estimator of the variance of the market microstructure noise e over a time period $[0, t]$ can be obtained as follows:

$$\hat{\sigma}_e^2 = 2\hat{\sigma}_{\epsilon}^2 \left(\sum_{j=1}^{n} \omega_j^2 \right) = \hat{\sigma}_u^2 \left(\sum_{j=1}^{n} \omega_j^2 \right) = \left(\sum_{j=1}^{n} \omega_j^2 \right) \left[1/m \sum_{t_{i-1}, t_i \in \mathcal{G}, t_i \leq t} (p_{t_i} - p_{t_i - 1})^2 \right], \tag{18}$$

where $p_{t_{i-1}}$ (resp. p_{t_i}) is the price of the index at the intraday time t_{i-1} (resp. t_i) and m is the number of sampling intervals over $[0, t]$.

4 Monte Carlo simulations

We conduct Monte Carlo simulations to examine the performance of our VaR estimation technique that adjusts for the effect of measurement error in the prices [hereafter adjusted VaR]. We provide a comparison with a model-free estimator of VaR that does not adjust for the measurement error [hereafter unadjusted VaR]. We assess the performance of our approach under two cases: when the measurement error density f_e is specified and when it is misspecified.

4.1 Case of specified density of measurement error

We suppose that the observed and true portfolio returns are related according to Equation (2). We first consider that the density f_e of the measurement error in Equation (2) is correctly specified and given by a standard normal distribution, i.e., $e_t \sim i.i.d.\, N(0,1)$, and we simulate the true portfolio returns from the following data generating processes (DGPs) that represent different contexts encountered in practice:

Model 1: The true portfolio return $r^*_{p,t}$ follows an AR(1) process:

$$r^*_{p,t} = 0.5 r^*_{p,t-1} + \eta_t, \text{ with } \eta_t \sim i.i.d.\, N(0,1). \tag{19}$$

Model 2: The true portfolio return $r^*_{p,t}$ follows an MA(2) process:

$$r^*_{p,t} = \eta_t + 0.65\eta_{t-1} + 0.24\eta_{t-2}, \text{ with } \eta_t \sim i.i.d.\, N(0,1). \tag{20}$$

Model 3: The true portfolio return $r^*_{p,t}$ is generated from a GARCH (1,1) model:

$$r^*_{p,t} = \sigma_t \eta_t, \text{ with } \eta_t \sim i.i.d.\, N(0,1) \text{ and } \sigma_t^2 = 0.05 + 0.85\sigma_{t-1}^2 + 0.1 r^{*2}_{p,t-1}. \tag{21}$$

We then use the simulated true returns $r^*_{p,t}$ and the standard normal measurement error to simulate the observed returns $r_{p,t}$ using Equation (2). Note that the above DGPs and the corresponding parameters are only selected to reflect the commonly used financial time series models in the simulation design, which offers a wide range of dependent structures. For example, similar choices are also considered in Chen and Tang (2015).

For each of the above models, we analytically calculate the 1%, 5% and 10% VaR of true portfolio return $r^*_{p,t}$. The values are reported in Table 1 and will be used to assess the bias (Bias), standard deviation (Std.), and root mean square error (RMSE) of the adjusted and unadjusted estimates of VaR calculated using our estimation technique and a model-free estimator (see below) that does not take into account the measurement error, respectively. The sample sizes range from $T = 125$

Table 1: True VaR for AR(1), MA(2) and GARCH(1,1) models

VaR of true portfolio returns $r^*_{p,t}$			
	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
Model 1: AR(1)	-2.6861	-1.8992	-1.4797
Model 2: MA(2)	-2.8301	-2.0012	-1.5592
Model 3: GARCH(1,1)	-2.4320	-1.6235	-1.2399

Note: This table reports the true VaR using Model 1 [Equation (19)], Model 2 [Equation (20)], and Model 3 [Equation (21)]. The results are obtained using 10000 replications.

to $T = 500$, which corresponds to data ranging from 6 months to 2 years. From each model, we generate $T + 1000$ observations and then discard the first 1000 observations to minimize the effect of the initial values. All the results are based on 1000 replications, except for the calculation of the true VaRs we use 10000 replications.

To calculate the unadjusted VaR, we simply use the sample quantile estimator based on observed portfolio returns $\{r_{p,t}\}_{t=1}^{T}$ that are contaminated by the measurement error, i.e.

$$\widehat{VaR}^{\alpha}(r_{pt}) = \inf\left\{u : F_{r_p,T}(u) \geq \alpha\right\}, \text{ for } \alpha = 1\%,\ 5\%,\ 10\%,$$

where $F_{r_p,T}(u) = T^{-1}\sum_{t=1}^{T} 1(r_{p,t} \leq u)$ is the empirical cumulative distribution function of contaminated portfolio return $r_{p,t}$, and $1(\cdot)$ is an indicator function.

We next use our approach to compute the adjusted VaR. To control for the truncation errors, in Formula (17) we take $l = k = 10$. In addition, our approach requires the use of bandwidth parameter b. In this simulation study, we follow Otsu and Taylor (2020) to select the bandwidth b according to the rule of thumb $b = c \times (2/\log T)^{1/2}$. We do not provide a theory to guide the choice of a data-driven bandwidth for our semiparametric estimation approach. However, in order to help select this important parameter, we consider a battery of simulations for different values of c, which varies in the grid $\{1, 1.5, 2, 2.5, 3\}$. This allows us to assess the sensitivity of our adjusted VaR estimates to different values of the bandwidth parameter b. As we see in the tables below, we can identify the optimal values of c that work for almost all DGPs under consideration. More precisely, if we choose $c = 2$ or $c = 2.5$, the gains in terms of RMSE over unadjusted approach are generally very significant.

Tables 2 to 4 report the simulation results for the 1% VaR using Model 1 to Model 3. The results for the 5% and 10% VaRs using the same DGPs can be found in Tables A.1 to A.6 of the

Table 2: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.1989	0.7527	0.7786			
$c = 1.5$	0.7052	0.3191	0.7740			
$c = 2.0$	0.1743	0.2335	0.2913	-0.8366	0.4954	0.9723
$c = 2.5$	-0.3834	0.2140	0.4391			
$c = 3.0$	-0.9840	0.2162	1.0075			
			$T = 250$			
$c = 1.0$	0.8214	0.6374	1.0397			
$c = 1.5$	0.8095	0.3763	0.8927			
$c = 2.0$	0.3264	0.1643	0.3654	-0.8356	0.3785	0.9173
$c = 2.5$	-0.2062	0.1619	0.2622			
$c = 3.0$	-0.7629	0.1544	0.7784			
			$T = 500$			
$c = 1.0$	1.3505	0.3548	1.3963			
$c = 1.5$	0.8411	0.4680	0.9625			
$c = 2.0$	0.4481	0.1213	0.4642	-0.8585	0.2681	0.8994
$c = 2.5$	-0.0528	0.1107	0.1226			
$c = 3.0$	-0.5735	0.1059	0.5832			

Table 3: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.4626	0.7203	0.8561			
$c = 1.5$	0.8192	0.2760	0.8644			
$c = 2.0$	0.2806	0.2254	0.3599	-0.8084	0.5085	0.9550
$c = 2.5$	-0.2988	0.2150	0.3681			
$c = 3.0$	-0.8600	0.2031	0.8837			
			$T = 250$			
$c = 1.0$	1.0920	0.5570	1.2258			
$c = 1.5$	0.9087	0.2380	0.9393			
$c = 2.0$	0.4129	0.1660	0.4450	-0.8038	0.3831	0.8905
$c = 2.5$	-0.1044	0.1532	0.1854			
$c = 3.0$	-0.6400	0.1358	0.6543			
			$T = 500$			
$c = 1.0$	1.4877	0.3176	1.5213			
$c = 1.5$	0.9428	0.3809	1.0168			
$c = 2.0$	0.5354	0.1209	0.5489	-0.8179	0.2702	0.8613
$c = 2.5$	0.0495	0.1123	0.1228			
$c = 3.0$	-0.4565	0.1046	0.4683			

Appendix A.2. Considering first the bias of the VaR estimates, the tables show that our estimation approach dominates the unadjusted VaR estimation. These results are encouraging and seem to be consistent with the theory. The performance of our method shows some variation across the different bandwidth choices. However, it is not surprising to see that our semiparametric approach depends on the bandwidth parameter when faced with the measurement error. Considering a battery of simulations for different values of c, we find that the performance of our technique is generally much better for $c = 2.0$ or 2.5. Concerning the standard deviation (Std.) of the VaR estimates, we again find that the adjusted VaR dominates the unadjusted VaR in most of the cases. We also note that there is a clear trade-off between bias and variance, which is intuitive and known in the classical nonparametric estimation literature. However, we see that the balance is well achieved with $c = 2.0$ or 2.5, as is evident from the tables. Regarding the root mean square error (RMSE), a common point to all the results is that the estimated adjusted VaR has in general a smaller RMSE than the unadjusted VaR. Thus, from these simulations we conclude that accounting for measurement error is indeed very important to draw the correct conclusions and must not simply be ignored.

Finally, additional results (not reported, but available upon request) were obtained by considering alternative values (e.g. highly persistent AR process) for the coefficients of Models 1–3 and for the variance of the measurement error ($\sigma_e^2 = 2$). For the coefficients of models, we find that higher persistence makes the variance of both adjusted and unadjusted VaRs significantly higher than in the less persistent case. Although these results show that the adjusted VaR does not perform well compared to the unadjusted VaR, we emphasize that this happens under the very high persistent level of the underlying time series process, and in this case any estimator should be used with care since high persistence is known to have a negative impact on the estimators developed under the stationarity assumption. Furthermore, it is well established that stock returns are weakly persistent, see for example Ding et al. (1993). For the variance of the measurement error, the additional results show that a larger variance of the measurement error affects notoriously the performance of the unadjusted VaR, while our proposed adjusted VaR still works reasonably well and delivers much smaller bias and RMSEs. The latter finding is not unexpected since a larger variance will further reduce the information in the contaminated returns, and as such, an (unadjusted) estimator that does not address the measurement error will fail to uncover useful information from the contaminated returns.

Table 4: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.3953	0.6979	0.8021			
$c = 1.5$	0.6222	0.3182	0.6989			
$c = 2.0$	0.0384	0.1892	0.1930	-0.8670	0.5477	1.0255
$c = 2.5$	-0.5667	0.1683	0.5911			
$c = 3.0$	-1.1629	0.1540	1.1730			
			$T = 250$			
$c = 1.0$	0.1343	0.7663	0.7780			
$c = 1.5$	0.6920	0.4332	0.8164			
$c = 2.0$	0.1967	0.1411	0.2421	-0.8647	0.4362	0.9685
$c = 2.5$	-0.3781	0.1274	0.3989			
$c = 3.0$	-0.9269	0.1141	0.9339			
			$T = 500$			
$c = 1.0$	0.8847	0.7202	1.1407			
$c = 1.5$	0.3774	1.0178	1.0855			
$c = 2.0$	0.3233	0.1034	0.3394	-0.8816	0.3085	0.9340
$c = 2.5$	-0.2052	0.0906	0.2243			
$c = 3.0$	-0.7398	0.0823	0.7444			

4.2 Case of misspecified density of measurement error

We run additional simulations to investigate the performance of our approach when the distribution of the measurement error is misspecified. Specifically, we still estimate our adjusted VaR using Formula (16) in Corollary 1, which we construct from the characteristic function of standard normal, but the measurement errors are in fact generated from non-normal distributions. We consider the same simulation setup as in Section 4.1, but we now generate the measurement error according to: **(i)** a Student's t distribution with 5 degrees of freedom, say $t(5)$; and **(ii)** a mixture of normal distributions, say $0.5N(0,1) + 0.5N(5,1)$.

The bias, standard deviation, and root mean square error of the adjusted and unadjusted 1% VaR estimates under $t(5)$ and mixture distributions are reported in Tables 5-7 and Tables A.7-A.9 of the Appendix A.2, respectively. To save space, we do not report the results for the other coverage rates [5% and 10%], but they are available upon request. For each of the misspecification cases, the above tables show similar patterns to those found for the correctly specified case [Section 4.1]. This indicates a good degree of robustness of our proposed methodology to various deviations to the misspecified measurement errors. In particular, we find that the estimates of the adjusted VaR perform better than the estimates of the unadjusted VaR. As in the specified measurement error density case, we find that the performance of our approach depends on the bandwidth parameter. But again, after providing a battery of simulations for different values of c, we see that the performance of our method is generally much better in terms of bias, standard deviation, root mean square error when $c = 2.0$ or 2.5.

Finally, we consider another simulation exercise where we compare the performance of the adjusted and unadjusted approaches when the measurement error is not present in the data. We use the same simulation setup as in Section 4.1. The simulation results are reported in Tables A.10-A.12. One should expect that the unadjusted approach will perform better than our approach, however when we examine the above tables we see that the adjusted approach is doing well compared to the unadjusted one. It is true that the unadjusted estimator generally leads to a smaller bias, but when the "optimal" bandwidth parameter [$c = 2.5$ or $c = 2.0$] is used, the adjusted estimator has much smaller variance and consequently a smaller mean square error for all the DGPs under consideration.

Table 5: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with Student's $t(5)$ measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	0.3671	0.9794	1.0460			
$c = 1.5$	0.7557	0.4717	0.8909			
$c = 2.0$	0.2247	0.3198	0.3909	-0.9909	0.6771	1.2001
$c = 2.5$	-0.3913	0.2715	0.4762			
$c = 3.0$	-0.9725	0.2211	0.9973			
$T = 250$						
$c = 1.0$	1.0682	1.0054	1.4669			
$c = 1.5$	0.8367	0.5841	1.0204			
$c = 2.0$	0.3573	0.2788	0.4532	-0.9583	0.4680	1.0665
$c = 2.5$	-0.1845	0.1539	0.2403			
$c = 3.0$	-0.7459	0.1485	0.7606			
$T = 500$						
$c = 1.0$	1.7290	0.9493	1.9725			
$c = 1.5$	0.8563	0.7922	1.1665			
$c = 2.0$	0.4818	0.4557	0.6631	-0.9557	0.3196	1.0077
$c = 2.5$	-0.0342	0.1077	0.1129			
$c = 3.0$	-0.5512	0.1059	0.5613			

Table 6: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with Student's $t(5)$ measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.6103	0.9902	1.1632			
$c = 1.5$	0.8831	0.5677	1.0499			
$c = 2.0$	0.2994	0.2492	0.3895	-0.9571	0.6736	1.1704
$c = 2.5$	-0.2848	0.2297	0.3659			
$c = 3.0$	-0.8506	0.2092	0.8759			
			$T = 250$			
$c = 1.0$	1.3156	0.9433	1.6188			
$c = 1.5$	0.9656	0.2983	1.0106			
$c = 2.0$	0.4392	0.2754	0.5184	-0.9018	0.4732	1.0184
$c = 2.5$	-0.0778	0.1736	0.1902			
$c = 3.0$	-0.6322	0.1476	0.6492			
			$T = 500$			
$c = 1.0$	1.8755	0.9434	2.0994			
$c = 1.5$	1.0148	0.6881	1.2261			
$c = 2.0$	0.5783	0.2252	0.6206	-0.9166	0.3273	0.9733
$c = 2.5$	0.0620	0.1121	0.1281			
$c = 3.0$	-0.4517	0.1301	0.4701			

Table 7: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with Student's $t(5)$ measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.0854	0.9749	0.9786			
$c = 1.5$	0.6637	0.3147	0.7345			
$c = 2.0$	0.0459	0.1913	0.1968	-1.0642	0.7198	1.2847
$c = 2.5$	-0.5472	0.1747	0.5744			
$c = 3.0$	-1.1578	0.1582	1.1685			
$T = 250$						
$c = 1.0$	0.5376	1.1656	1.2836			
$c = 1.5$	0.6982	0.6209	0.9343			
$c = 2.0$	0.2183	0.3468	0.4098	-1.0061	0.5119	1.1288
$c = 2.5$	-0.3538	0.1218	0.3742			
$c = 3.0$	-0.9169	0.1164	0.9243			
$T = 500$						
$c = 1.0$	1.3423	1.1259	1.7519			
$c = 1.5$	-0.0641	1.3668	1.3683			
$c = 2.0$	0.3640	0.1790	0.4056	-1.0216	0.3715	1.0871
$c = 2.5$	-0.1873	0.0930	0.2091			
$c = 3.0$	-0.7293	0.0818	0.7339			

Table 8: Time-span of the high-frequency data

Index	**From**	**To**	**No. of Observations**
CAC 40	02/01/2000	29/12/2017	458,353
DAX 30	31/05/2009	29/12/2017	227,508
FTSE 100	04/01/2000	29/12/2017	501,185
FTSE MIB	03/01/2000	29/12/2017	508,175
S&P 500	03/01/2000	29/12/2017	394,898

Note: This table reports the time-span and the number of observations on the sparse trade price series collected at a five minute sampling frequency for the indixes CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500.

5 Empirical application

In this section, we apply the semiparametric approach we proposed in the previous sections to high-frequency data to estimate the VaR of five international stock market indices over one day horizon. We compare our results to the unadjusted approach that estimates VaR by simply computing the sample quantiles of the five indices' returns.

Our data consist of high-frequency tick-by-tick trade prices on the stock market indices CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500, which we obtained from the Thomson Reuter's Tick History (TRTH) database, over the period January 2000 to December 2017. Our interest specifically lies in the sparse trade prices with a five minutes sampling frequency. Table 8 reports the time-span and the number of observations corresponding to the sparse trade price series collected at a five minutes sampling frequency. Evidently, the price series for each index expands to the desired time-span, with the exception of the DAX 30 index, which has trade prices only available from the 31$^{\text{st}}$ May 2009. We then use the above data to calculate the continuously compounded returns over each five minutes interval by taking the difference between the logarithm of the two tick prices immediately preceding each five minutes mark.

As we have seen before, the implementation of the semiparametric approach to estimate VaR requires the knowledge of the variance of the measurement error. This variance, however, is unknown but it can be estimated using high-frequency data as shown at the end of Section 3. The semiparametric approach introduced in Section 3 is applied to estimate the 5% and 10% VaRs of each standardized stock index return [hereafter adjusted VaR]. We standardize the returns (returns divided by their standard deviations) to fairly compare the VaRs of the five stock indices. In ad-

Table 9: The adjusted and unadjusted estimates of VaR for high-frequency financial returns

	Adjusted VaR		**Unadjusted VaR**	
	5%	10%	5%	10%
CAC 40	-2.2623	-1.9142	-1.6491	-1.1448
DAX 30	-2.3325	-1.9741	-1.6446	-1.1624
FTSE 100	-2.2281	-1.8855	-1.5763	-1.0888
FTSE MIB	-2.2415	-1.8969	-1.6105	-1.1255
S&P 500	-2.2373	-1.8936	-1.6038	-1.0849

Note: This table reports the estimated 5% and 10% VaR of standardized returns of CAC 40, DAX 30, FTSE 100, FTSE MIB, and S&P 500, using the approach introduced in Section 3 [adjusted VaR] and the unadjusted approach that does not adjust for the measurement error [unadjusted VaR]. Here we use $c = 2.5$.

dition, for comparison, we estimate the unadjusted 5% and 10% VaRs of these stock indices by simply calculating the sample quantiles using order statistics [hereafter unadjusted VaR].

The results are reported in Table 9. As expected, for both adjusted and unadjusted estimates of VaR, we see that the loss is higher at 5% than 10% statistical confidence levels. Interestingly, for all stock indices and confidence levels, we see that the adjusted estimates of VaRs are much bigger - in absolute value - than the unadjusted estimates. This suggests that ignoring the measurement error might lead to an underestimation of risk. If we take the example of S&P 500 index, the adjusted estimate of 5% VaR is 0.6335 (2.2373-1.6038) higher than the unadjusted one. Thus, an investor who invests, for example, $100 million in the S&P 500 index and uses unadjusted VaR will think that the magnitude of risk at 5% confidence level is equal to $208,494 a day [$100 million $\times$ 1.6038$\times$0.0013 (standard deviation of S&P 500 index)], whereas the true magnitude of risk (according to the adjusted estimate of VaR) is $290,849 a day [$100 million $\times$ 2.2373$\times$0.0013]. Hence, this investor will face an unexpected additional loss of $82,355 a day [$100 million $\times$ 0.6335$\times$0.0013]. Finally, Table 9 shows that the values of adjusted and unadjusted VaRs are similar across the five stock indices, which might indicate that international stock markets are driven by some common factors.

6 Conclusions

We have proposed a semiparametric approach for estimating the VaR of a portfolio of contaminated stock returns. We have shown that measurement errors cause serious problems for estimating risk, and unfortunately the existing methods are inconsistent in the presence of measurement error. Using Fourier transform, we derived a robust estimator of VaR that takes into account the measurement error. We first used a deconvolution kernel estimator for the density function of the true latent portfolio returns to deal with the measurement error. Second, we used Fourier inversion to calculate the probability distribution function of the latent portfolio returns. Thereafter, we used power series representations of sine and exponential functions to approximate the integral in the inversion formula and made the calculation of VaR feasible.

The derivation of robust estimator of VaR was first made under the assumption that the density of measurement error is known, but the distribution of the observed portfolio returns was always treated as unknown and estimated nonparametrically. Thereafter, we relaxed this assumption and suggested a feasible way to deal with the measurement error's distribution. In particular, we followed the literature and assumed that the measurement error is normally distributed but with unknown variance that we estimated nonparametrically using high-frequency data and a consistent estimator of variance of measurement error from Zhang et al. (2005).

Furthermore, we conducted a set of Monte Carlo simulations to examine the performance of our approach. We also provided a comparison with a model-free estimator of VaR that does not take into account the measurement error. We investigated the performance of our approach for known and unknown density of the measurement error and the simulation results were encouraging. Finally, we used our approach and high-frequency data to estimate the adjusted VaR of five US and European stock indices. We compared our results to the unadjusted VaR, which we estimated using a model-free approach that simply computes the sample quantiles based on the five indices' historical returns. The empirical results showed that ignoring measurement error generally leads to an underestimation of risk.

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A Appendix

A.1 Proofs of theoretical results

Proof of Corollary 1. We first calculate the integral:

$$\int_0^1 \frac{\exp\left(as^2\right)\sin(\kappa s)}{s}ds,$$

where $a = \frac{\sigma_\epsilon^2 \sum_{j=1}^n \omega_j^2}{b^2} > 0$ and $\kappa = \frac{r_{p,t}+VaR^\alpha(r_{p,t}^*)}{b}$. To do that, we use power series representation of $\sin(\kappa s)$ function:

$$\sin(\kappa s) = \sum_{i=0}^{\infty} \frac{(-1)^i \kappa^{1+2i} s^{1+2i}}{(1+2i)!}.$$

Furthermore, the power series representation of $\exp\left(as^2\right)$ is given by: $\exp(as^2) = \sum_{l=0}^{\infty} \frac{a^l s^{2l}}{l!}$. Thus,

$$\frac{\exp(as^2)\sin(\kappa s)}{s} = \sum_{i,j=0}^{\infty} \frac{a^i (-1)^j \kappa^{1+2j} s^{2i+2j}}{i!(1+2j)!}.$$

Consequently,

$$\int_0^1 \frac{\exp(as^2)\sin(\kappa s)}{s}ds = \sum_{i,j=0}^{\infty} \frac{a^i(-1)^j\kappa^{1+2j}}{i!(1+2j)!}\int_0^1 s^{2i+2j}ds = \sum_{i,j=0}^{\infty} \frac{a^i}{i!}\frac{(-1)^j\kappa^{1+2j}}{(1+2j)!}\frac{1}{(2i+2j+1)}.$$

Hence, the VaR of the latent portfolio's return $r_{p,t}^*$ with coverage probability α, denoted by $VaR^\alpha(r_{p,t}^*)$, is the solution of the following optimization problem:

$$\begin{aligned}
\widehat{VaR}^\alpha(r_{p,t}^*) &= \underset{VaR^\alpha(r_{p,t}^*)}{Argmin} \left[\frac{1}{T}\sum_{t=1}^{T}\sum_{i,j=0}^{\infty}\frac{a^i}{i!}\frac{(-1)^j\kappa^{1+2j}}{(1+2j)!}\frac{1}{2i+2j+1} - \left(\frac{1}{2}-\alpha\right)\pi\right]^2 \\
&= \underset{VaR^\alpha(r_{p,t}^*)}{Argmin} \left[\sum_{i,j=0}^{\infty}\frac{a^i}{i!}\frac{(-1)^j\frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_{p,t}+VaR^\alpha(r_{p,t}^*)}{b}\right)^{1+2j}}{(1+2j)!\,(2i+2j+1)} - \left(\frac{1}{2}-\alpha\right)\pi\right]^2,
\end{aligned}$$

with $a = \frac{\sigma_\epsilon^2 \sum_{j=1}^n \omega_j^2}{b^2} > 0$. ■

Lemma 1: *A series of the form $S = \sum_n^\infty (-1)^n \varepsilon_n$ where either all ε_n are positive or all ε_n are negative is called an alternating series. Then says: if $|\varepsilon_n|$ decreases monotonically and $\lim_{n\to\infty}\varepsilon_n = 0$ then the partial sum $S_q = \sum_n^q (-1)^n a_n$ approximates S with error bounded by the next omitted term:*

$$R_q = \sum_{n=q}^{\infty}(-1)^n a_n \leq |\varepsilon_{q+1}|.$$

Proof of Lemma 1. First of all, we know that when an alternating series converges to its limit S, this means the partial sum of this alternating series also "alternates" above and below the final limit, i.e., $S_{2q} < S < S_{2q+1}$.

We now show $|S_k - S| \leq \varepsilon_{k+1}$ by considering two cases:

1. When $k = 2q+1$, i.e., k is odd, then we have:

$$|S_{2q+1} - S| = S_{2q+1} - S \leq S_{2q+1} - S_{2q+2} = \varepsilon_{(2q+1)+1}.$$

2. When $k = 2q$, i.e., k is even, then we have:

$$|S_{2q} - S| = S - S_{2q} \leq S_{2q+1} - S_{2q} = \varepsilon_{2q+1}.$$

Both cases rely essentially on the inequality $S_{2q} < S < S_{2q+1}$. ∎

Proof: Bounds of the remaining terms for the truncation of the infinite sum in Equation (16).. First of all, note that the infinite sum $\sum_{i,j=0}^{\infty}$ in the optimization problem in Equation (16) can be decomposed as follows:

$$S = \sum_{i,j=0}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = S_{l,k} + R_{l,k},$$

where the truncated and remaining terms $S_{l,k}$ and $R_{l,k}$ are given by:

$$S_{l,k} = \textstyle\sum_{i=0}^{l} \sum_{j=0}^{k} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)},$$

$$\begin{aligned} R_{l,k} = {} & \underbrace{\sum_{i=l+1}^{\infty} \sum_{j=0}^{k} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(1)}} + \underbrace{\sum_{i=0}^{l} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(2)}} \\ & + \underbrace{\sum_{i=l+1}^{\infty} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)}}_{R_{l,k}^{(3)}}, \end{aligned} \tag{22}$$

with $a = \frac{\sigma_\epsilon^2 \sum_{j=1}^{n} \omega_j^2}{b^2}$ and $\kappa = \frac{r_{p,t} + VaR^\alpha(r_{p,t}^*)}{b}$. The bandwidth b is selected according to the rule of thumb $b = c \times (2/\log T)^{1/2}$; see Otsu and Taylor (2020). Furthermore, based on the high-frequency data [see Section 5] σ_ϵ^2 is of order of magnitude of 10^{-7}. We next derive a bound for each remaining term $R_{l,k}^{(1)}$, $R_{l,k}^{(2)}$ and $R_{l,k}^{(3)}$.

1). The bound for $R_{l,k}^{(1)}$: First, observe that

$$R_{l,k}^{(1)} = \sum_{i=l+1}^{\infty} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = \sum_{j=k+1}^{\infty} (-1)^j \varepsilon_j,$$

is an alternating series, where the term $\varepsilon_j = \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{\kappa^{2j+1}}{(1+2j)!} \frac{1}{(2i+2j+1)}$ is decreasing to zero since $\lim_{j\to\infty} \frac{\kappa^{2j+1}}{(1+2j)!} = 0$. From Lemma 1 of this Appendix, we obtain:

$$
\begin{aligned}
|R^{(1)}_{l,k}| &\leq |\varepsilon_{k+1}| = \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{1}{(2i+2k+3)} \\
&\leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{1}{(2i+2k+3)} \leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{\exp(a)}{(2l+2k+5)},
\end{aligned} \tag{23}
$$

since $\sum_{i=l+1}^{\infty} \frac{a^i}{i!} \leq \sum_{i=0}^{\infty} \frac{a^i}{i!} = e^a$ and a is a positive number defined in Corollary 1. Thus, $\lim\limits_{l,k\longrightarrow\infty} \left\{ R^{(1)}_{l,k} \right\} = 0$.

2). The bound for $R^{(2)}_{l,k}$: Following the same argument as the one for the bound of $R^{(1)}_{l,k}$, we have

$$
R^{(2)}_{l,k} = \sum_{i=0}^{l} \sum_{j=k+1}^{\infty} \frac{a^i}{i!} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} = \sum_{j=k+1}^{\infty} (-1)^j \varepsilon_j,
$$

is an alternating series, where the term $\varepsilon_j = \sum_{i=0}^{l} \frac{a^i}{i!} \frac{\kappa^{2j+1}}{(1+2j)!} \frac{1}{(2i+2j+1)}$ is decreasing to zero since $\lim_{j\to\infty} \frac{\kappa^{2j+1}}{(1+2j)!} = 0$. From Lemma 1 of this Appendix, we obtain:

$$
\begin{aligned}
|R^{(2)}_{l,k}| &\leq |\varepsilon_{k+1}| = \sum_{i=0}^{l} \frac{a^i}{i!} \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{1}{(2i+2k+3)} \\
&\leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \sum_{i=0}^{l} \frac{a^i}{i!} \frac{1}{(2i+2k+3)} \leq \frac{|\kappa|^{2k+3}}{(2k+3)!} \frac{\exp(a)}{(2k+3)}.
\end{aligned} \tag{24}
$$

Thus, as for bound of $R^{(1)}_{l,k}$, we have $\lim\limits_{l,k\longrightarrow\infty} \left\{ R^{(2)}_{l,k} \right\} = 0$.

3). The bound for $R^{(3)}_{l,k}$**:** Observe that:

$$
\begin{aligned}
|R^{(3)}_{l,k}| &= \left| \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \sum_{j=0}^{k} \frac{(-1)^j \kappa^{1+2j}}{(1+2j)!} \frac{1}{(2i+2j+1)} \right| \leq \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \frac{1}{(2i+1)} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!} \\
&\leq \frac{1}{(2l+3)} \sum_{i=l+1}^{\infty} \frac{a^i}{i!} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!} \leq \frac{\exp(a)}{(2l+3)} \sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!}.
\end{aligned} \tag{25}
$$

Now, if we define $\nu_j = \frac{|\kappa|^{1+2j}}{(1+2j)!}$, we obtain:

$$
\frac{\nu_{j+1}}{\nu_j} = \left(\frac{|\kappa|^{3+2j}}{(3+2j)!}\right) / \left(\frac{|\kappa|^{1+2j}}{(1+2j)!}\right) = \frac{\kappa^2}{(2j+3)(2j+2)}.
$$

From the above equation and based on D'Alembert criterion, the series $\sum_{j=0}^{k} \frac{|\kappa|^{1+2j}}{(1+2j)!}$ converges for $k \longrightarrow \infty$, and consequently $\lim\limits_{l,k\longrightarrow\infty} \left\{ R^{(3)}_{l,k} \right\} = 0$. ■

A.2 Monte Carlo simulation results

Table A.1: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the AR(1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.5183	0.8712	1.0137			
$c = 1.5$	0.1771	0.4417	0.4759			
$c = 2.0$	-0.2947	0.2129	0.3636	-0.5917	0.3217	0.6735
$c = 2.5$	-0.8063	0.1977	0.8301			
$c = 3.0$	-1.3169	0.2108	1.3337			
			$T = 250$			
$c = 1.0$	0.1543	0.7206	0.7369			
$c = 1.5$	0.2503	0.3918	0.4649			
$c = 2.0$	-0.1672	0.1570	0.2293	-0.6069	0.2318	0.6497
$c = 2.5$	-0.6449	0.1500	0.6621			
$c = 3.0$	-1.1200	0.1472	1.1296			
			$T = 500$			
$c = 1.0$	0.7285	0.3469	0.8069			
$c = 1.5$	0.2865	0.5020	0.5781			
$c = 2.0$	-0.0668	0.1176	0.1353	-0.6102	0.1680	0.6329
$c = 2.5$	-0.5034	0.1073	0.5147			
$c = 3.0$	-0.9610	0.1038	0.9666			

Table A.2: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the MA(2) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.2425	0.8210	0.8561			
$c = 1.5$	0.2376	0.3270	0.4042			
$c = 2.0$	-0.2231	0.2139	0.3091	-0.5709	0.3280	0.6585
$c = 2.5$	-0.7270	0.2029	0.7548			
$c = 3.0$	-1.2461	0.2011	1.2622			
			$T = 250$			
$c = 1.0$	0.4182	0.5876	0.7212			
$c = 1.5$	0.3095	0.3115	0.4391			
$c = 2.0$	-0.1084	0.1546	0.1888	-0.5815	0.2363	0.6276
$c = 2.5$	-0.5666	0.1447	0.5848			
$c = 3.0$	-1.0555	0.1397	1.0648			
			$T = 500$			
$c = 1.0$	0.8183	0.2813	0.8653			
$c = 1.5$	0.3712	0.3382	0.5022			
$c = 2.0$	-0.0060	0.1119	0.1121	-0.5844	0.1665	0.6077
$c = 2.5$	-0.4353	0.1081	0.4485			
$c = 3.0$	-0.8856	0.1019	0.8914			

Table A.3: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 95% VaR for the GARCH(1,1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-1.3376	0.7844	1.5507			
$c = 1.5$	0.0393	0.1983	0.2021			
$c = 2.0$	-0.4766	0.1742	0.5074	-0.6867	0.3290	0.7615
$c = 2.5$	-1.0081	0.1662	1.0217			
$c = 3.0$	-1.5350	0.1512	1.5424			
			$T = 250$			
$c = 1.0$	-0.6805	0.8463	1.0860			
$c = 1.5$	0.0782	0.5265	0.5323			
$c = 2.0$	-0.3426	0.1213	0.3635	-0.6912	0.2329	0.7294
$c = 2.5$	-0.8387	0.1095	0.8458			
$c = 3.0$	-1.3359	0.1101	1.3404			
			$T = 500$			
$c = 1.0$	0.1386	0.7616	0.7741			
$c = 1.5$	-0.4823	1.4075	1.4878			
$c = 2.0$	-0.2274	0.0931	0.2457	-0.6961	0.1716	0.7170
$c = 2.5$	-0.7022	0.0846	0.7073			
$c = 3.0$	-1.1662	0.0791	1.1689			

Table A.4: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the AR(1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.8049	0.9783	1.2668			
$c = 1.5$	0.0087	0.3893	0.3894			
$c = 2.0$	-0.3910	0.2157	0.4466	-0.4674	0.2782	0.5439
$c = 2.5$	-0.8328	0.2104	0.8589			
$c = 3.0$	-1.2632	0.2023	1.2793			
$T = 250$						
$c = 1.0$	-0.0657	0.7661	0.7689			
$c = 1.5$	0.0727	0.4323	0.4384			
$c = 2.0$	-0.2895	0.1511	0.3265	-0.4697	0.1988	0.5101
$c = 2.5$	-0.6933	0.1486	0.7090			
$c = 3.0$	-1.0974	0.1466	1.1072			
$T = 500$						
$c = 1.0$	0.4559	0.3727	0.5888			
$c = 1.5$	0.0738	0.5994	0.6039			
$c = 2.0$	-0.1918	0.1104	0.2213	-0.4723	0.1378	0.4920
$c = 2.5$	-0.5699	0.1074	0.5800			
$c = 3.0$	-0.9528	0.1064	0.9587			

Table A.5: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the MA(2) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.5198	0.9414	1.0754			
$c = 1.5$	0.0610	0.3142	0.3201			
$c = 2.0$	-0.3433	0.2112	0.4030	-0.4449	0.2777	0.5245
$c = 2.5$	-0.7749	0.2008	0.8005			
$c = 3.0$	-1.2025	0.2012	1.2192			
$T = 250$						
$c = 1.0$	0.1582	0.6488	0.6678			
$c = 1.5$	0.1302	0.3182	0.3438			
$c = 2.0$	-0.2335	0.1390	0.2718	-0.4552	0.1971	0.4961
$c = 2.5$	-0.6397	0.1434	0.6555			
$c = 3.0$	-1.0391	0.1401	1.0485			
$T = 500$						
$c = 1.0$	0.5451	0.2662	0.6066			
$c = 1.5$	0.1761	0.3547	0.3960			
$c = 2.0$	-0.1444	0.1074	0.1800	-0.4589	0.1397	0.4797
$c = 2.5$	-0.5182	0.1004	0.5279			
$c = 3.0$	-0.9014	0.1006	0.9070			

Table A.6: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 90% VaR for the GARCH(1,1) model

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-1.7152	0.8601	1.9188			
$c = 1.5$	-0.1212	0.2618	0.2885			
$c = 2.0$	-0.5634	0.1592	0.5854	-0.5541	0.2652	0.6143
$c = 2.5$	-1.0119	0.1514	1.0231			
$c = 3.0$	-1.4607	0.1405	1.4675			
			$T = 250$			
$c = 1.0$	-0.9770	0.9381	1.3545			
$c = 1.5$	-0.0774	0.4375	0.4443			
$c = 2.0$	-0.4398	0.1168	0.4551	-0.5554	0.1867	0.5860
$c = 2.5$	-0.8608	0.1095	0.8677			
$c = 3.0$	-1.2842	0.0972	1.2879			
			$T = 500$			
$c = 1.0$	-0.1552	0.8065	0.8213			
$c = 1.5$	-0.6550	1.4626	1.6025			
$c = 2.0$	-0.3411	0.0821	0.3508	-0.5584	0.1346	0.5744
$c = 2.5$	-0.7417	0.0763	0.7456			
$c = 3.0$	-1.1405	0.0735	1.1429			

Table A.7: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with normal mixture measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	0.2242	0.7193	0.7534			
$c = 1.5$	0.6918	0.3010	0.7544			
$c = 2.0$	0.1664	0.2214	0.2770	-0.7016	0.4428	0.8297
$c = 2.5$	-0.3986	0.2046	0.4481			
$c = 3.0$	-0.9766	0.2054	0.9980			
			$T = 250$			
$c = 1.0$	0.7986	0.5333	0.9603			
$c = 1.5$	0.7782	0.2178	0.8081			
$c = 2.0$	0.3127	0.1629	0.3525	-0.7086	0.3347	0.7836
$c = 2.5$	-0.2172	0.1512	0.2646			
$c = 3.0$	-0.7580	0.1445	0.7716			
			$T = 500$			
$c = 1.0$	1.1445	0.2733	1.1767			
$c = 1.5$	0.8383	0.2956	0.8889			
$c = 2.0$	0.4314	0.1174	0.4470	-0.7259	0.2371	0.7636
$c = 2.5$	-0.0642	0.1052	0.1233			
$c = 3.0$	-0.5691	0.1018	0.5781			

Table A.8: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with normal mixture measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	0.4628	0.6784	0.8213			
$c = 1.5$	0.7629	0.3033	0.8210			
$c = 2.0$	0.2613	0.2240	0.3442	-0.6833	0.4551	0.8210
$c = 2.5$	-0.2938	0.2038	0.3576			
$c = 3.0$	-0.8780	0.2010	0.9007			
$T = 250$						
$c = 1.0$	0.9930	0.4417	1.0868			
$c = 1.5$	0.8637	0.2246	0.8925			
$c = 2.0$	0.3937	0.1594	0.4247	-0.6957	0.3492	0.7785
$c = 2.5$	-0.1060	0.1481	0.1822			
$c = 3.0$	-0.6497	0.1418	0.6649			
$T = 500$						
$c = 1.0$	1.2706	0.2701	1.2990			
$c = 1.5$	0.9337	0.1779	0.9505			
$c = 2.0$	0.5111	0.1101	0.5228	-0.7028	0.2426	0.7435
$c = 2.5$	0.0339	0.1051	0.1104			
$c = 3.0$	-0.4660	0.1048	0.4776			

Table A.9: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with normal mixture measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.4889	0.7829	0.9230			
$c = 1.5$	0.5925	0.2473	0.6420			
$c = 2.0$	0.0293	0.1824	0.1848	-0.6861	0.5253	0.8641
$c = 2.5$	-0.5666	0.1688	0.5912			
$c = 3.0$	-1.1610	0.1520	1.1709			
			$T = 250$			
$c = 1.0$	0.0647	0.7885	0.7911			
$c = 1.5$	0.6880	0.2817	0.7434			
$c = 2.0$	0.1748	0.1350	0.2209	-0.7050	0.4365	0.8292
$c = 2.5$	-0.3801	0.1190	0.3983			
$c = 3.0$	-0.9383	0.1197	0.9459			
			$T = 500$			
$c = 1.0$	0.8506	0.5781	1.0285			
$c = 1.5$	0.6384	0.6114	0.8839			
$c = 2.0$	0.3025	0.1013	0.3190	-0.7220	0.3082	0.7851
$c = 2.5$	-0.2127	0.0823	0.2281			
$c = 3.0$	-0.7445	0.0834	0.7491			

Table A.10: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the AR(1) model with no measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.3574	0.3240	0.4824			
$c = 1.5$	0.8470	0.8722	1.2158			
$c = 2.0$	0.5068	0.1921	0.5420	0.0277	0.3983	0.3993
$c = 2.5$	-0.1303	0.1852	0.2264			
$c = 3.0$	-0.7713	0.1883	0.7939			
			$T = 250$			
$c = 1.0$	-0.1230	0.2436	0.2729			
$c = 1.5$	0.3654	1.2783	1.3295			
$c = 2.0$	0.6653	0.1323	0.6784	0.0466	0.2971	0.3008
$c = 2.5$	0.0637	0.1347	0.1490			
$c = 3.0$	-0.5226	0.1254	0.5374			
			$T = 500$			
$c = 1.0$	0.0718	0.2694	0.2788			
$c = 1.5$	-1.1150	0.8139	1.3804			
$c = 2.0$	0.8120	0.0985	0.8179	0.0055	0.2223	0.2224
$c = 2.5$	0.2365	0.0923	0.2539			
$c = 3.0$	-0.3254	0.0936	0.3386			

Table A.11: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the MA(2) model with no measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
$T = 125$						
$c = 1.0$	-0.2503	0.3240	0.4258			
$c = 1.5$	1.0800	0.8722	1.2594			
$c = 2.0$	0.6093	0.1921	0.6355	0.0473	0.4139	0.4166
$c = 2.5$	-0.0138	0.1852	0.1801			
$c = 3.0$	-0.6562	0.1883	0.6789			
$T = 250$						
$c = 1.0$	-0.0079	0.2436	0.3222			
$c = 1.5$	0.6609	1.2783	1.3469			
$c = 2.0$	0.7626	0.1323	0.7751	0.0301	0.3212	0.3226
$c = 2.5$	0.1716	0.1347	0.2138			
$c = 3.0$	-0.4069	0.1254	0.4260			
$T = 500$						
$c = 1.0$	0.2630	0.2694	0.4857			
$c = 1.5$	-0.6736	0.8139	1.3169			
$c = 2.0$	0.9082	0.0985	0.9131	0.0007	0.2360	0.2360
$c = 2.5$	0.3309	0.0923	0.3434			
$c = 3.0$	-0.2166	0.0936	0.2340			

Table A.12: Finite-sample biases, standard deviations (Std.'s) and root mean square errors (RMSEs) of the adjusted and unadjusted estimates of 99% VaR for the GARCH(1,1) model with no measurement errors

	Adjusted VaR			Unadjusted VaR		
Bandwidth	Bias	Std.	RMSE	Bias	Std.	RMSE
			$T = 125$			
$c = 1.0$	-0.4236	0.2925	0.5148			
$c = 1.5$	0.8471	0.7002	1.0991			
$c = 2.0$	0.3449	0.1411	0.3726	0.0664	0.5345	0.5386
$c = 2.5$	-0.3159	0.1542	0.3515			
$c = 3.0$	-0.9509	0.1129	0.9576			
			$T = 250$			
$c = 1.0$	-0.1976	0.2575	0.3246			
$c = 1.5$	-1.0542	1.2495	1.6348			
$c = 2.0$	0.5271	0.1018	0.5368	0.0455	0.4636	0.4658
$c = 2.5$	-0.1000	0.0999	0.1413			
$c = 3.0$	-0.7034	0.0846	0.7085			
			$T = 500$			
$c = 1.0$	0.0330	0.3749	0.3764			
$c = 1.5$	-1.5633	0.1288	1.5686			
$c = 2.0$	0.6685	0.0762	0.6728	0.0106	0.3416	0.3417
$c = 2.5$	0.0773	0.0692	0.1038			
$c = 3.0$	-0.4977	0.0644	0.5019			