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Implementation in vNM Stable Set

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Implementation in vNM Stable Set*

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Abstract

We fully identify the class of social choice functions that are implementable in von Neumann Morgenstern (vNM) stable set (von Neumann and Morgenstern, 1944) by a rights structure. A rights structure formalizes the idea of power distribution in a society. Following Harsanyi'critique (Harsanyi, 1974), we also study implementation problems in vNM stable set that are robust to farsighted reasoning.

Keywords: Stable Set, Implementation, Rights Structure, Farsightedness

JEL Codes: C71; D02; D71; D82.

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1 Introduction

THE FIRST SOLUTION CONCEPT for a general model of binding agreements is introduced by von Neumann and Morgenstern (1944) in their monumental work on game theory. "The solution," so eloquently named by the authors and now widely referred to as the von Neumann Morgenstern (vNM) stable set, builds on a notion of dominance. An outcome x dominates an outcome y if a coalition of agents has the power to move from y to x and each coalition member strictly prefers x to y. The vNM stable set satisfies two properties. *Internal stability*: No outcome in the set dominates another outcome inside it. *External stability*: For every outcome outside the set, an outcome inside the set dominates it.

The outcomes in the vNM stable set are consistent because they do not dominate one another, and such a consistency is reached by excluding outcomes that are dominated by outcomes inside the set. Conceptually, the vNM stable set expresses the idea of social organization. Indeed, it can be viewed as a variety of alternatives, social norms, or "standards of behavior" (Greenberg, 1990), conveying the orientation of a society.

Despite its applications in several areas, we still do not have a general theory for the vNM stable set. We know that the vNM stable set is usually not unique (Lucas, 1968) and may fail to exist (Lucas, 1992). Also, the problem of its computation is undecidable (Deng and Papadimitriou, 1994). These facts imposed the *core* (Gillies, 1959) as the central solution concept for games where coalitions are the fundamental decision units. The core is the set of outcomes immune to any coalitional deviation. However, the core does not ask if a coalitional deviation is credible or not. Following Ray and Vohra (2019), the notion of credibility of coalitional deviation can be stated recursively. A deviation is credible if no other credible deviation challenges it. The vNM stable set embodies this idea naturally, since it can be equivalently defined as the set of outcomes that are not dominated by any outcome in the vNM stable set (von Neumann and Morgenstern, 1944). Of course, the vNM stable set includes the core, but it may also include other elements. This feature is significant because the core might not be able to fully describe all agents' bargaining possibilities (Ehlers, 2007; Núñez and Rafels, 2013). In contrast, the vNM stable set may offer consistent predictions.

From a normative point of view, the vNM stable set's primacy as a solution concept is undoubted. However, its normative investigation is almost an unexplored territory. Indeed, Serrano (2021) complains that among all leading game-theoretic solution concepts, the vNM stable set is the only solution suffering from this drawback. This paper contributes to the normative theory of the vNM stable set in the realm of implementation theory.

Implementation theory offers a normative framework for the design of institutions, emphasizing the problem of incentives. A common interpretation of an implementation problem is that a hypothetical planner wants to achieve socially desirable outcomes without knowing agents' preferences. The social objectives the planner wants to achieve are summarized in a social choice function (SCF), that is, in a single-valued function mapping agents' preferences into an outcome. To achieve his goals, the planner decentralizes the decision-making by designing a mechanism or game form. Roughly speaking, a mechanism represents the communication and decision aspects of the organization. Formally, it specifies a message space for each agent and an outcome function mapping vectors of messages into social decisions. A mechanism implements an SCF if its equilibrium outcome corresponds to the outcome of the SCF, irrespective of agents' preferences.

Although successful results have been obtained in the last decades in identifying the classes of SCFs that can be implemented with this approach, it is still not clear how to replicate via a game form the recursive character of the vNM stable set. Moreover, a nagging criticism of the theory (Abreu and Matsushima, 1992; Jackson, 1992) is that the devised implementing mechanisms have unnatural features that reduce the relevance of the theory. In particular, many implementing mechanisms employ some sort of "integer game" or "modulo game," which are used to eliminate strategies with unacceptable outcomes from the equilibria.

To overcome these issues, we follow the approach developed by (Sertel, 2001; Koray and Yildiz, 2018), who propose a notion of rights structure as an explicit specification of the "power distribution" underlying social interaction. With this approach, an implementation problem consists of designing a rights structure such that its equilibrium outcome corresponds to the outcome of the SCF, irrespective of agents' preferences. In solving this problem, the planner needs to describe the available alternatives via a set of possible states and specify which coalitions of agents have the power to move from one state to another. The power distribution implements an SCF when the outcome corresponding to its vNM stable states corresponds to the outcome of the SCF, irrespective of agents' preferences. As reflected by recent contributions (Koray and Yildiz, 2018; Korpela, Lombardi and Vartiainen, 2020; Korpela, Lombardi and Vartiainen , 2021), this "blocking" approach to implementation theory suits the normative investigation of cooperative solution concepts well.

We introduce three conditions, vNM EFFICIENCY, vNM MONOTONICITY and TEST CYCLE, which fully characterize the class of functions that are implementable in vNM stable set via a rights structure. We focus on single-payoff vNM stable sets, in line with recent contributions to cooperative single-payoffs solutions. Examples are Béal, Durieu, and Solal (2008); Mauleon, Vannetelbosch, and Vergote (2011); Ray and Vohra (2015); Dutta and Vohra (2017); Ray and Vohra (2019); Bloch and van den Nouweland (2021); Herings, Mauleon and Vannetelbosch (2020). We also identify two simple sufficient conditions for implementation, vNM EFFICIENCY and independence of irrelevant alternatives (IIA). The two conditions are also necessary for implementation in some well-known domains. This finding, besides, sheds some new light on the role of IIA in implementation theory, which is coherent with recent investigations in voting theory (Dasgupta and Maskin, 2020).

Finally, by considering the Harsanyi's critique, we develop a theory robust to agents' rational sophistication. Harsanyi (1974) argued that farsighted agents, who can conjecture about the ultimate consequences that a deviation can lead to, may select outcomes outside the vNM stable set. We achieve robustness to farsighted reasoning by requiring that the planner devises a rights structure that (doubly) implements in vNM stable set and in the largest consistent set (Chwe, 1994).

The largest consistent set successfully describes farsighted behavior and, in our setting, encompasses most of the farsighted equilibrium notions, such as the farsighted stable set (Ray and Vohra, 2015), the (strong) rational expectation farsighted stable set (Dutta and Vohra, 2017) and the absolutely maximal farsighted stable set (Ray and Vohra, 2019). A rights structure that double implements in vNM stable set and in the largest consistent set induces the agents to select the socially optimal outcome irrespective of whether they are myopic or farsighted. We find that vNM EFFICIENCY, when combined with indirect independence of irrelevant alternatives (iIIA), an extension of IIA, is sufficient for the double implementation. These conditions are also necessary for double implementation in some domains.

The rest of the paper is organized as follows. Section 2 provides the rights structure framework and shows how it suits the theory of the vNM stable set. Section 3 introduces the notion of implementation in vNM stable set via a rights structure and presents our full characterization. Applications are presented in Section 4. Section 5 provides an alternative characterization via simpler conditions. Section 6 concludes. All proofs are relegated to the Appendix.

2 Preliminaries

We consider a finite (nonempty) set of *agents*, denoted by N, and a finite (nonempty) set of *alternatives*, denoted by Z. For each agent $i(\in N)$, a *preference relation* over Z is a complete and transitive binary relation $R_i \subseteq Z \times Z$. We denote by P_i the asymmetric part of R_i, i.e., xP_iy if and only if xR_iy and not yR_ix, while the symmetric part of R_i is denoted by I_i, i.e., xI_iy if and only if xR_iy and yR_ix. A *preference profile* $R \equiv (R_i)_{i\in N}$ lists the preferences of all agents in N. Let $\mathcal{R}$ be the collection of all admissible preference profiles. A *coalition* K is any non-empty subset of N. For any preference profile $R(\in \mathcal{R})$ and coalition $K \subseteq N$, we write xR_Ky and xP_Ky to denote respectively that xR_iy holds for all $i \in K$ and xP_iy holds for all $i \in K$. For any $R \in \mathcal{R}$, and any $x, y \in Z$, let $K(R, x, y)$ be a coalition defined by the rule: $i \in K(R, x, y) \iff xP_iy$. That is, $K(R, x, y)$ is the set of agents that strictly prefer x to y at R. As usual, $L_i(x, R)$ denotes the *lower contour set* of x at R for agent i.

The goal of the planner is to implement a *social choice function* (SCF) $f : \mathcal{R} \longrightarrow Z$. The *range* of $f : \mathcal{R} \longrightarrow Z$ is the set

$$f(\mathcal{R}) \equiv \{x \in Z | x \in f(R) \text{ for some } R \in \mathcal{R}\}.$$

For all $x \in Z$, let $f^{-1}(x) \equiv \{R \in \mathcal{R} | f(R) = x\}$ be the inverse image of x.

For all $R \in \mathcal{R}$ and all $x, z \in Z$, we say that z is *equivalent* to x at R if xI_Nz, and that z is *image equivalent* to x at R if $z \in f(\mathcal{R})$ and x is equivalent to z at R. We write $I^f(x, R) = \{z \in f(\mathcal{R}) \mid zI_Nx\}$ for the set of all image equivalent outcomes to x at R.

Finally, the *graph* of $f : \mathcal{R} \longrightarrow Z$ is the set

$$Gr(f) \equiv \{(x, R) | x \in f(R), R \in \mathcal{R}\}.$$

To implement $f : \mathcal{R} \longrightarrow Z$, the planner constructs a *rights structure* $\Gamma = (S, h, \gamma)$, where S is the *state space*, $h : S \longrightarrow Z$ the *outcome function*, and $\gamma : S \times S \rightrightarrows N$ a *code of rights*, which specifies, for each pair of distinct states (s, t), the collection of coalitions $\gamma(s, t) \subseteq 2^N$ that is entitled to move from state s to t. If $\gamma(s, t) = \emptyset$, then no coalition is entitled to move from s to t. From an economic design perspective, the rights structure is the planner's design variable and corresponds to a "mechanism" in the economic theory jargon. To save notation, we denote $S^x = \{s \in S | h(s) = x\}$, with a typical element s^x, the set of states where the outcome is x.

A rights structure and a preference profile return a *social environment* (Chwe, 1994), a general framework to model strategic interaction among agents or groups.

Definition 1 (Social Environment)**.** A *social environment* is a pair $\langle\Gamma, R\rangle$ consisting of a rights structure Γ and a preference profile R.

Definition 2 establishes the *dominance* relation: a state $y \in S$ dominates another state $x \in S$ if there is a coalition such that (i) it can move from x to y and (ii) each of its members strictly prefer to do so.

Definition 2 (Dominance)**.** Given a social environment $\langle\Gamma, R\rangle$ and states $s, s' \in S$, the state $s' \in S$ dominates $s \in S$ under γ at $R \in \mathcal{R}$, if there is a coalition $K \subseteq N$ such that: (i) $K \in \gamma(s, s')$; and (ii) $h(s')P_K h(s)$.

Given $\langle\Gamma, R\rangle$, if s dominates s' under γ at R, then we write $s >_{(\Gamma,R)} s'$.
Definition 3 introduces the notion of vNM stable set, for any social environment $\langle\Gamma, R\rangle$.

Definition 3. Let $\langle\Gamma, R\rangle$ be a social environment. The set $V(\Gamma, R) \subseteq S$ is a vNM stable set at (Γ, R) if it satisfies the following conditions:

Internal Stability: for all $s, s' \in V(\Gamma, R)$, not $s' >_{(\Gamma,R)} s$.

External Stability: for all $s \notin V(\Gamma, R)$, there exists $s' \in V(\Gamma, R)$ such that $s' >_{(\Gamma,R)} s$.

Internal Stability requires that no state in the set is dominated by any other state in the set. External Stability requires that each state outside the set is dominated by a state inside the set. Internal and external stability work together: no two allocations threaten each other, and jointly, the stable allocations dominate all non-stable allocations. As von Neumann and Morgenstern (1944) pointed out, the notion of the vNM stable set can be stated as a single condition. For a given social environment (Γ, R) and any subset $A \subseteq S$ define $Dom_{(\Gamma,R)}(A)$, the *dominion* of A, as the subset of states that are dominated by some element of A, formally:

$$Dom_{(\Gamma,R,>)}(A) \equiv \{s \in S | \exists s' \in A : s' >_{(\Gamma,R)} s\}$$

Then, any vNM stable set at (Γ, R) is

$$V(\Gamma, R) \equiv S - Dom_{(\Gamma,R,>)}(V(\Gamma, R))$$

that is the set of states that are not dominated by any state in the vNM stable set.

A vNM stable set $V(\Gamma, R)$ is named single-payoff if for all $s, s' \in V(\Gamma, R)$ it holds that $h(s) = h(s')$. We denote by $vNM(\Gamma, R)$ the union of all vNM stable sets at $\langle \Gamma, R \rangle$.

3 Implementation in vNM stable set

We are now ready to define our notion of implementation. An SCF $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set via a rights structure if, at each preference profile R, the alternative chosen by $f : \mathcal{R} \longrightarrow Z$ coincides with the outcome induced by the vNM stable set prediction.

Definition 4 (Implementation in vNM stable set)**.** A rights structure Γ implements $f : \mathcal{R} \longrightarrow Z$ in the vNM stable set if $f(R) = h \circ vNM(\Gamma, R)$ for all $R \in \mathcal{R}$. If such a rights structure exists, $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set by a rights structure.

Remark 1. *If Γ implements $f : \mathcal{R} \longrightarrow Z$ in the vNM stable set, then the vNM stable set of Γ at any $R \in \mathcal{R}$ is unique and single-payoff.*

In coalition theory, the single-payoff cooperative solutions are widely studied. Prominent examples are Béal, Durieu, and Solal (2008); Mauleon, Vannetelbosch, and Vergote (2011); Ray and Vohra (2015); Dutta and Vohra (2017); Ray and Vohra (2019); Bloch and van den Nouweland (2021); Herings, Mauleon and Vannetelbosch (2020).

Note that, by Remark 1, if $x = f(R)$, then S^x is the unique vNM stable set at R.

Koray and Yildiz (2018) and Korpela, Lombardi and Vartiainen (2020) show that (Maskin) monotonicity is necessary for implementation in core via a rights structure. Monotonicity requires that if an outcome x is optimal at R,[1] preferences changes from R to R', and the outcome x does not fall in any agent's preference ordering relative to any other alternative, then x remains optimal at R'[2]. The following example shows that monotonicity is not necessary for implementation in the vNM stable set via a rights structure.

Example 1. There are two agents $\{1, 2\}$, three outcomes $\{x, y, z\}$ and two preference profiles $R, R' \in \mathcal{R}$. The table below specifies agents' preferences. The SCF $f : \{R, R'\} \longrightarrow \{x, y, z\}$ is such that $f(R) = x$ and $f(R') = y$. Note that $f : \mathcal{R} \longrightarrow Z$ is not monotonic: x is optimal at R. No agent experiences a preference reversal around x when the state changes from R then R', but x is not optimal at R'. The SCF is implementable in the vNM stable set. The right-hand side of Figure 1 is an example of implementing rights structure. First, we impose that states are outcomes. An oriented graph represents the rights structure. The vertices are the states. The edges represent the code of rights: Agent 1 can move from x to y and from z to x. Agent 2 can move from y to x and from z to y and *vice versa*.

R		R'	
1	2	1	2
y	x	y	x, y
x	z	x, z	z
z	y		

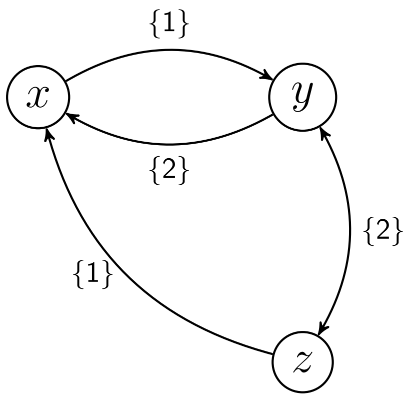

Figure 1: An example of a non-monotonic SCF and an implementing rights structure.

According to this rights structure, the unique vNM stable set at R and R' are, respectively, $vNM(\Gamma, R) = \{x\}$ and $vNM(\Gamma, R') = \{y\}$. To see this, take, as an example, the preference profile R. Then, $\{x\}$ trivially satisfies internal stability.

[1] x is optimal at R means that $f(R) = x$.
[2] Formally, for all $R, R' \in \mathcal{R}$, $L_i(f(R), R) \subseteq L_i(f(R), R')\ \forall i \in N \implies f(R) = f(R')$

External stability is also satisfied since z and y are dominated by x. Note that $\{x\}$ is the unique vNM stable set at R. Indeed, one can check that at R, any subset of $\{x, y, z\}$ different from x violates either internal or external stability. A similar argument applies to R'.

To guarantee the property of "external stability" of $f : \mathcal{R} \longrightarrow Z$ at the profile R, for every outcome x different from $f(R)$, one agent needs to prefer strictly $f(R)$ to x. This property is captured by the following notion of von Neumann Morgenstern efficiency, hereafter vNM EFFICIENCY.

Definition 5 (vNM EFFICIENCY)**.** $f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY if there exists $Y \subseteq Z$ such that $f(\mathcal{R}) \subseteq Y$, and that for all $R \in \mathcal{R}$ and all $x \in Y$ with $x \neq f(R)$, it holds that $f(R)P_i x$ for some $i \in N$.

Theorem 1. *If $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set by a rights structure, then it satisfies* vNM EFFICIENCY*.*

It is straightforward to see that the SCF described in Example 1 satisfies vNM EFFICIENCY. However, vNM EFFICIENCY is not sufficient for the implementability in the vNM stable set. We show this in the example below.

Example 2. There are two agents $\{1, 2\}$, three outcomes $\{x, y, z\}$ and two preference profiles $R, R' \in \mathcal{R}$. The table below specifies agents' preferences.

R		R'	
1	2	1	2
y	x	y	x
x, z	z	x	y, z
	y	z	

The SCF $f : \{R, R'\} \longrightarrow \{x, y, z\}$ is such that $f(R) = x$ and $f(R') = y$. Note that the SCF satisfies vNM EFFICIENCY: Agent 2 strictly prefers $f(R)$ to y and to z at R and agent 1 strictly prefers $f(R')$ to x and to z at R'.

However, $f : \mathcal{R} \longrightarrow Z$ is not implementable in the vNM stable set. Indeed, if S^x is a vNM stable set at R and S^y a vNM stable set at R', then it has to be that S^x is also a vNM stable set at R'. To see it, note that S^x satisfies internal stability at any preference profile, including R'. Also, any rights structure implementing S^x at R must satisfy the following property. Agent 2 must be allowed to move from each $s' \in S^y$ to some $s \in S^x$ and from each $s'' \in S^z$ to some $s \in S^x$. Otherwise, external stability is not satisfied. Since this guarantees external stability at R' for the set S^x, it follows that S^x is a vNM stable set at R'.

Example 2 suggests that another property is required to rule out undesirable outcomes. In the particular case of Example 2, the planner wants to achieve x as the unique vNM stable set at R and y as the unique vNM stable set at R'. However, x happens to be a vNM stable set at R' because the agents strictly preferring x to y and x to z at R, namely agent 2, do the same at R'. In other words, from one side, agent 2 guarantees external stability of x at R; from another side, no other agent is breaking the external stability of x at R'.

An implementable SCF satisfies the von Neumann Morgenstern monotonicity. We abbreviate this condition as vNM MONOTONICITY. The condition captures and addresses the problem raised in the example above. It requires that for any x in the range of $f : \mathcal{R} \longrightarrow Z$ that is not optimal at some R', an outcome z acting as a breaking point of the vNM stability of x at R' exists. In particular, vNM MONOTONICITY requires that the set of agents preferring x to z at any profile where x is optimal differs from the set of agents preferring x to z at the profile where x is not optimal, and the same must hold for each image equivalent outcome to x at R'.

Definition 6 (vNM MONOTONICITY)**.** $f : \mathcal{R} \longrightarrow Z$ satisfies vNM MONOTONICITY if there exists $Y \subseteq Z$ such that $f(\mathcal{R}) \subseteq Y$, and that for all $(x, R') \in Z \times \mathcal{R}$ with $x \in f(\mathcal{R}) \backslash f(R')$, all $x^* \in I^f(x, R')$, and all $R \in f^{-1}(x^*)$, it holds that $K(R, x^*, z) \nsubseteq K(R', x^*, z)$ for some $z \in Y$.

Henceforth, we denote by $\mathcal{M}^f(x, R') \subseteq Y$ the maximal set of outcomes, like z, satisfying vNM MONOTONICITY at (x, R').

Theorem 2. *If $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set by a rights structure, then it satisfies* vNM MONOTONICITY*.*

The reader can check that, in Example 2, the SCF violates vNM MONOTONICITY: $x \in f(R) \backslash f(R')$, $\{2\} = K(R, x, y) = K(R', x, y)$ and $K(R, x, z) = K(R', x, z) = \{2\}$. Therefore, $\mathcal{M}^f(x, R') = \emptyset$.

Next, we show that vNM EFFICIENCY and vNM MONOTONICITY are not sufficient for the implementation of $f : \mathcal{R} \longrightarrow Z$. The following example makes the point.

Example 3. There are two agents $\{1, 2\}$, three outcomes $\{x, y, z\}$ and two preference profiles $R, R' \in \mathcal{R}$. The table below specifies agents' preferences.

R		R'	
1	2	1	2
y, z	x	y	x, z
x	y	x, z	y
	z		

Again, $f : \{R, R'\} \longrightarrow \{x, y, z\}$ is such that $f(R) = x$ and $f(R') = y$. Note that this (non-monotonic) SCF satisfies vNM EFFICIENCY: Agent 2 strictly prefers $x = f(R)$ to y and z at R and agent 1 strictly prefers $y = f(R')$ to x and z at R'. It also satisfies vNM MONOTONICITY because $\mathcal{M}^f(x, R') = z$ and $\mathcal{M}^f(y, R) = z$. However, this $f : \mathcal{R} \longrightarrow Z$ is not implementable. Indeed, if S^x were a vNM stable set at R and S^y a vNM stable set at R', then it would have to be that $S^x \cup S^z$ is a vNM stable set at R'. To see the latter point, note that any implementing rights structure where S^x is a vNM stable set at R must allow agent 1 to move from each $s^y \in S^y$ to an s^x and from each $s^z \in S^z$ to an s^x. Otherwise, external stability would not be satisfied for S^x. Then, the set $S^x \cup S^z$ satisfies external stability. Since agents are indifferent between x and z at R', it follows that $S^x \cup S^z$ is a vNM stable set at R'.

Example 3 suggests that vNM MONOTONICITY is too weak for ruling out all undesirable vNM stable sets. This suggestion does not come as a surprise. Suppose that x is optimal at R and that y is optimal at R'. The condition of vNM MONOTONICITY allows the planner to design a rights structure such that every s^x violates external stability at R'. However, vNM MONOTONICITY is silent on whether s^x could belong to a vNM stable set at R'. Therefore, we need another condition to rule out undesirable stable outcomes. The needed condition, called TEST CYCLE, builds over the notion of *odd cycle*. An odd cycle is a sequence of outcomes $z^k, z_1, z_2, ..., z^k$ where $k \in \mathbb{N}$ is odd and such that $z^k P'_{i_h} z^1 P'_{i_1} z^2 P'_{i_2} ... P'_{i_{h-1}} z^k$ holds for $i_1, i_2, ..., i_h \in N$.

When states and outcomes coincide, the literature has extensively shown (Richardson, 1946, 1953; Harary et al., 1966) that if there are no odd cycles, then a vNM stable set exists. This sufficient condition is undoubtedly relevant from a positive point of view. Nevertheless, its argument is helpful for our purposes as well. Indeed, we learn that a necessary condition for the non-existence of a vNM stable set is the presence of odd cycles. Pushing forward this idea, we claim the following. Suppose that x is optimal at some profile R but not at R', and that one agent strictly prefers x to $f(R')$ at R'. Suppose $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set via a rights structure. In that case, a specific odd cycle around x must exist so that each state $s^x \in S^x$ does not belong to any vNM stable at R'. Otherwise, internal stability would not be satisfied. Definition 7 and Theorem 3 formalize our claim.

Definition 7 (TEST CYCLE)**.** $f : \mathcal{R} \longrightarrow Z$ satisfies TEST CYCLE if for all $R' \in \mathcal{R}$ and all $x \in f(\mathcal{R}) \backslash \{f(R')\}$ such that $x P'_i f(R')$ for some $i \in N$, one of the following requirements holds:

(i) there exists $z \in \mathcal{M}^f(x, R')$ such that for all $x^* \in I^f(x, R')$, $x^* P'_i f(R') P'_j z P'_k x^*$ holds for some $j, k \in N$,

(ii) there exists an odd cycle with outcomes in $\mathcal{M}^f(x, R') \cup I^f(x, R')$,

(iii) $f(R') \in \mathcal{M}^f(x, R')$.

Theorem 3. *If $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set by a rights structure, then it satisfies* TEST CYCLE *.*

Theorem 1, Theorem 2 and Theorem 3 prove the following corollary.

Corollary 1 (Necessity)**.** *If $f : \mathcal{R} \longrightarrow Z$ is implementable in vNM stable set, then there exists $X \subseteq Z$ such that $f(\mathcal{R}) \subseteq X$ and that $f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY, vNM MONOTONICITY and TEST CYCLE with respect to X.*

Next, we show that vNM EFFICIENCY, vNM MONOTONICITY, and TEST CYCLE are also sufficient for the implementation in the vNM stable set.

Theorem 4 (Sufficiency)**.** *Let $X \subseteq Z$ be such that $f(\mathcal{R}) \subseteq X$. If $f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY, vNM MONOTONICITY and TEST CYCLE with respect to X, then $f : \mathcal{R} \longrightarrow Z$ is implementable in vNM stable set by a rights structure.*

We conclude this section by showing that if preferences are linear orders, then TEST CYCLE is redundant. A binary relation $R_i \subseteq Z \times Z$ is a linear order if it is reflexive, transitive and anti-symmetric. Let $\mathcal{L}$ be the domain of all profiles of linear orderings.

Proposition 1. *If $f : \mathcal{L} \longrightarrow Z$ satisfies* vNM EFFICIENCY *and vNM MONOTONICITY, then it satisfies TEST CYCLE.*

Since by Proposition 1 the TEST CYCLE condition is redundant, vNM EFFICIENCY and vNM MONOTONICITY fully characterize the class of implementable function in vNM stable set when agents' preference are linear orders. The following corollary establishes this point.

Corollary 2. *$f : \mathcal{L} \longrightarrow Z$ is implementable in vNM stable set via arights stricture if and only if there exists a set $X \subseteq Z$ such that $f : \mathcal{L} \longrightarrow Z$ satisfies vNM EFFICIENCY and vNM MONOTONICITY with respect to X.*

In Section 4, we show that a similar result applies in environments with transfers under mild conditions on agents' preferences.

A non-implementable SCF: The Vikrey auction rule

A seller has an indivisible item for sale. There are $n \geqslant 2$ buyers or bidders, with valuations for the item in the interval $[0, \infty)$, and the valuations are common knowledge. Bidder i's valuation is denoted by v_i. Each bidder i simultaneously submit a bid $b_i \in [0, \infty)$. The highest bidder wind the object and pays the second-highest bid (i.e., if he wins ($b_i > \max_{j \neq i} b_j$), the bidder i has a net utility of $v_i - \max_{j \neq i} b_j$), and the other bidders do not pay anything. If several bidders bid the highest bid, the item is allocated randomly among them. Let $Z = N \times [0, \infty)$ and let the Vickrey auction rule $f^V : [0, \infty)^N \to Z$ be defined by $f^V(b) = (i, p)$, where bidder i wins the item and pays a price $p = \max_{j \neq i} b_j$, where $b = (b_1, ..., b_n) \in [0, \infty)^N$.

The Vickrey auction rule is not implementable in vNM stable set because it violates vNM MONOTONICITY. Assume, to the contrary, that f^V satisfies vNM MONOTONICITY. Then, there exists $Y \subseteq Z$ such that $f^V\left([0, \infty)^N\right) \subseteq Y$. Suppose that bidders' profile of valuations v for the object is such that $v_1 > v_2 > .... > v_n$. Let the profile v' be such that $v' = (v_{-2}, v'_2)$, with $v_1 > v'_2 > v_2$. Note that $f^V(v) = (1, v_2)$ and $f^V(v') = (1, v'_2)$. Since $f^V(v) \neq f^V(v')$, it follows that $I^{f^V}\left(f^V(v'), v\right) = \left\{f^V(v')\right\}$. Fix any $z \in Y$ such that $z \neq f^V(v')$. Let us show that $K\left(v', f^V(v'), z\right) \subseteq K\left(v, f^V(v'), z\right)$. Suppose that $i \in K\left(v', f^V(v'), z\right)$. Let us proceed according to whether $i = 2$ or not. Suppose that $i \neq 2$. Since the utility of bidder i is unchanged when we move from v' to v, it follows that $i \in K\left(v, f^V(v'), z\right)$. Suppose that $i = 2$. Since bidder 2 obtains utility 0 at $\left(v', f^V(v')\right)$ and since $2 \in K\left(v', f^V(v'), z\right)$, it must be the case that $z = (2, p')$ and that bidder 2's utility at (v', z) is $v'_2 - p' < 0$. Recall that, by construction, $v' = (v_{-2}, v'_2)$ is such that $v_1 > v'_2 > v_2$. Since $z = (2, p')$ and since, moreover, $v'_2 > v_2$, it follows that bidder 2's utility at (v, z) is $v_2 - p' < 0$. Since bidder 2 obtains utility 0 at $\left(v, f^V(v')\right)$, it follows that $2 \in K\left(v, f^V(v'), z\right)$. Therefore, we conclude that $K\left(v', f^V(v'), z\right) \subseteq K\left(v, f^V(v'), z\right)$. Since the choice of $z \in Y$ was arbitrary, we conclude that $f^V(v) \neq f^V(v')$, $I^{f^V}\left(f^V(v'), v\right) = \left\{f^V(v')\right\}$, $v' \in f^{V-1}(f^V(v'))$ but $K\left(v', f^V(v'), z\right) \subseteq K\left(v, f^V(v'), z\right)$ for all $z \in Y$. Thus, the Vickrey auction rule f^V does not satisfy vNM MONOTONICITY.

4 Applications

In this section, we apply the above characterization result to three well-studied environments: An environment with transfers, a bilateral trading environment, a voting environment, and a facility location environment.

4.1 Environments with Transfers

Let D be a set of potential social decisions with typical element $d \in D$. A transfer of agent i is any real number $t_i \in \mathbb{R}$. As usual, we write $t_{-i} \equiv (t_i)_{i \in N \setminus \{i\}} \in \mathbb{R}^{n-1}$. In this environment, an outcome $z \in Z^{D,t} \equiv D \times \mathbb{R}^n$ consists of a social decision d together with a profile of transfers $t = (t_1, ..., t_n)$. For any $i \in N$, agent' preference relation R_i is defined over the pairs (d, t). An *environment with transfers* is a triplet $\langle N, Z^{D,t}, (R_i)_{i \in N} \rangle$.

We impose over R_i the following requirements:

Definition 8. Agent i 's preference relation R_i on $Z = D \times \mathbb{R}^n$ is *self-regarding* if i cares only about what he or she consumes.

Definition 9 (Money Monotonicity)**.** Agent i 's preference relation R_i is *money monotonic* if for all $d \in D$, all $t_{-i} \in \mathbb{R}^{n-1}$, and all $t_i, t'_i \in \mathbb{R}$,

$$t_i > t'_i \quad \Rightarrow \quad (d, (t_i, t_{-i})) P_i (d, (t'_i, t_{-i})).$$

The next proposition shows that in an environment with transfers where preferences satisfy some requirements, vNM MONOTONICITY implies TEST CYCLE. In light of this result and of Theorem 4, we obtain that in an environment with transfers where preferences are continuous, self-regarding and money monotonic and where the domain of preference profile is finite, vNM EFFICIENCY and vNM MONOTONICITY fully characterize the class of implementable functions.

Proposition 2. *Assume that preferences are self-regarding, continuous, and money monotonic and that the preference domain is finite. In an environment with transfers, if $f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY and vNM MONOTONICITY, then it satisfies TEST CYCLE.*

Corollary 3. *Assume that preferences are self-regarding, continuous, and money monotonic and that the preference domain is finite. Let the environment be an environment* *with transfers. $f : \mathcal{R} \longrightarrow Z$ is implementable in the vNM stable set by a rights structure if and only if it satisfies vNM EFFICIENCY and vNM MONOTONICITY in some set $Y \subseteq Z$ such that $f(\mathcal{R}) \subseteq Y$.*

4.2 Facility Location Problems

In facility location problems, a central planner has to determine the location of a public facility y in the interval $[0, x]$, representing a "linear" city that needs to serve $n \geqslant 2$ agents. Each agent i lives in location $p_i \in [0, x]$. Agent i's value of a facility location depends on its distance from p_i. Preferences of agent i are represented by the utility function

$$U_i(y) = -|y - p_i|,$$

The planner wants to build the facility to the location of the agent who lives closer to the location 0 (for whatever reason). Since knowing the location p_i of agent i is sufficient to specify his preferences completely, we can represent the minimum distance rule that the planner wants to implement as

$$f_{md}(p_1, p_2, \ldots, p_n) = \min\{p_1, p_2, \ldots, p_n\}.$$

Note that the minimum distance rule is not (Maskin) monotonic. For example, take two profiles of locations, $(p_1, p_2, \ldots, p_n)$ and $(p_1, p'_2, \ldots, p_n)$ such that $p_1 = \min\{p_1, p_2, \ldots, p_n\}$, $p_2 \neq p_1$, $p'_2 < p_1$, and $(p_2 - p_1) > (p_1 - p'_2)$. Monotonicity is violated because the location p_1 has not dropped in any customers' preferences when moving from $(p_1, p_2, \ldots, p_n)$ to $(p_1, p'_2, \ldots, p_n)$ but the social choice has changed from p_1 to p'_2.

Proposition 3. *In a facility location environment, the minimum distance rule f_{md} is implementable in the vNM stable set by a rights structure.*

4.3 A Bilateral Trading Environemnt

A basic model of bilateral trading (Myerson and Satterthwaite, 1983) consists of one indivisible object to be traded between agent 1 (the seller) and agent 2 (the buyer). The value of agent i is denoted by v_i. Both values lie in the interval $[a, b]$ and all value profiles $(v_1, v_2) \in [a, b]^2$ are admissible. The set of outcomes Z is the set of all possible trading prices $p \in \{0\} \cup [a, b]$ where 0 means that there is no trade and $p \in [a, b]$ means that agents trade with price p. Agents' utility functions are $u_1(p) = p - v_1$ and $u_2(p) = v_2 - p$.

f maps any profile of valuations (v_1, v_2) to a trading price $p \in [a, b]$, or to 0 if there is no trade. We require f to be individually rational $-$ both agents must benefit from trade when it takes place.

Fix any $p \in [a, b]$. f_p is a fixed-price rule if and only if $f_p(v_1, v_2) = p$ for $v_1 < p < v_2$, and $f_p(v_1, v_2) = 0$, otherwise.[3]

Proposition 4. *In a bilateral trading environment, for all $p \in [a, b]$, f is implementable in the vNM stable set by a rights structure if and only if $f = f_p$.*

4.4 A Voting Environment

The Condorcet winner is a fundamental concept in voting theory. An outcome is a Condorcet winner if it can beat any other outcome in a head-to-head comparison, i.e., $z \in Z$ is a *Condorcet winner* at R if $|K(R, z, x)| \geqslant \frac{n}{2}$ holds for all $x \in Z \backslash \{z\}$. However, it is well-known that pairwise ranking can lead to cycles, so a Condorcet winner can fail to exist. A well-studied domain restriction (see, for instance, Fishburn, 1997, Gaertner, 2001, Saari, 2009) is the so-called *Condorcet domain*. $\mathcal{R}$ is a Condorcet domain if a Condorcet winner exists at any preference profile $R \in \mathcal{R}$. The SCF $f : \mathcal{R} \longrightarrow Z$ that selects the Condorcet winner at each preference profile is called the *Condorcet rule*. Let us denote it by f_C. We have the following important application of our characterization result.

[3]This rule is not efficient. Sometimes trade would be Pareto improving but will not take place at the pre-specified price. This no-trade situation also happens under incomplete information (Myerson and Satterthwaite, 1983).

Proposition 5. *Assume that $\mathcal{R}$ is a Condorcet domain. f_C is implementable in the vNM stable set by a rights structure.*

5 A Simple Condition

The necessary and sufficient conditions presented above may be difficult to check. For this reason, in this section, we present a simple condition, which is sufficient for implementation when combined with vNM EFFICIENCY. This simple condition builds on the well-known *independence of irrelevant alternatives* condition. We also show that this simple condition is necessary for implementation in some well-studied domains.

The condition can be stated as follows.

Definition 10 (IIA)**.** $f : \mathcal{R} \longrightarrow Z$ satisfies *independence of irrelevant alternatives* (IIA) if for all $R, R' \in \mathcal{R}$, and all $z \in f(\mathcal{R}) \backslash f(R)$,

$$f(R) = x \text{ and } K(R, x, z) \subseteq K(R', x, z) \quad \Rightarrow \quad f(R') \neq z$$

In words, if those agents who prefer x to z at R when x socially optimal at R also prefer x to z at R', then z cannot be socially optimal at R'. Note that the condition does not require that x remains socially optimal at R'.

One can show[4] that the minimum distance rule in a facility location environment, the fixed price rule in a bilateral trading environment and the Condorcet rule in a voting environment, as defined in Section 4, satisfy IIA.

Our simple characterization can be stated as follows.

Theorem 5 (Sufficiency)**.** *$f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY and IIA, then it is implementable in vNM stable set by a rights structure.*

The proof of Theorem 5 builds on a simple rights structure. The set of states is the graph of $f : \mathcal{R} \longrightarrow Z$, and the range of the outcome function is the range of $f : \mathcal{R} \longrightarrow Z$. Finally, its code of rights is such that only coalitions of the type

[4]See Proposition 6, Proposition 7 and Proposition 8 in the Appendix.

$K(R, x, y)$ can move from the state (y, R') to (x, R). For further details, we refer the reader to the Appendix.

As already mentioned, mIIA is a necessary condition for implementation in some preference domains. Let us introduce these domains.

Definition 11 (Comprehensive preference domain)**.** A preference domain $\mathcal{R} = \times \mathcal{R}_i$ is called *comprehensive* if for all $i \in N$ and all $x, y \in Z$, there exists a preference relation $R_i \in \mathcal{R}_i$ such that x is ranked first (uniquely) and y is ranked second (uniquely).

Theorem 6 (Necessity)**.** *Assume that $\mathcal{R}$ is a comprehensive domain. $f : \mathcal{R} \to Z$ is implementable in vNM stable set by a rights structure, then it satisfies IIA.*

A similar result applies to single-crossing domains (Milgrom and Shannon, 1994; Gans and Smart, 1996; Athey, 2001).

Definition 12 (Single-crossing domain)**.** A preference profile R satisfies the *single-crossing condition* (SC) if there exists a strict ranking $\rhd'$ among the agents in N and a strict ranking $\rhd$ among the outcomes in Z such that

$$\forall\, z' \rhd z,\ \forall\, i \rhd' j: \quad z' R_j z \ \Rightarrow\ z' R_i z \ \text{ and } \ z' P_j z \ \Rightarrow\ z' P_i z.$$

A preference domain $\mathcal{R}$ is called a *single-crossing domain* if there exists a strict ranking $\rhd'$ among the agents in N and a strict ranking $\rhd$ among the outcomes in Z such that each preference profiles $R \in \mathcal{R}$ satisfy SC with respect to these rankings.

By renaming agents, we can assume that the ranking $\rhd'$ is the standard "greater than" relation among numbers.

Theorem 7 (Necessity)**.** *Assume that $\mathcal{R}$ is a single-crossing domain. $f : \mathcal{R} \to Z$ is implementable in vNM stable set by a rights structure, then it satisfy IIA in $f(\mathcal{R})$.*

We conclude this section by connecting the above result to the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975). Indeed, roughly speaking, Corollary 4 states that the only implementable SCF is the dictatorial one when the preferences are linear orderings and the preference domain is full. Corollary 4 builds on the following intermediate result.

Lemma 1. *If $f : \mathcal{L} \longrightarrow Z$ satisfies IIA and vNM EFFICIENCY w.r.t. Y, then it is Maskin monotonic w.r.t. Y.*

Denote the full domain of linear orderings by $\mathcal{L}^*$.

Corollary 4. *Assume that $|Z| \geqslant 3$ and that $f : \mathcal{L}^* \longrightarrow Z$ is onto. f is implementable in vNM via a rights structure if and only if it is dictatorial.*[5]

5.1 Robustness to Farsighted Reasoning

Harsanyi (1974) criticized the vNM stable set for being myopic. He argued that the predictions of the vNM stable set might be incorrect when agents are farsighted. The following example shows that Harsanyi's critique has a bite even in a normative environment, where it translates into a problem of robustness to farsighted reasoning.

Example 4. There are two agents $\{1, 2\}$, three outcomes $\{x, y, z\}$ and two preference profiles $R, R' \in \mathcal{R}$. The table in Figure 2 specifies agents' preferences. The SCF $f : \{R, R'\} \longrightarrow \{x, y, z\}$ is such that $f(R) = x$ and $f(R') = y$. The right-hand side of Figure 2 illustrates an implementing rights structure employed by the planner. Thus, x and y are respectively vNM stables sets at R and R'. In the spirit of von Neumann and Morgenstern (1944), z is unstable since agent 1 can profitably deviate from z to the stable state x. However, suppose that agents are farsighted. Then, the deviation of agent 1 is deterred by the fact that from x further deviations may occur. In particular, agent 2 may deviate from x to y and then from y to z, which makes the initial deviation of agent 1 ineffective. Therefore, the planner's design may fail if the status quo is z.

Example 4 illustrates how farsighted behavior is relevant to the design of institutions. A natural question arises: Can the planner design a rights structure

[5] f is dictatorial if there exists an agent $i \in N$ such that for all $L \in \mathcal{L}$, $x = f(L)$ if and only if xL_iy for all $y \in Z\backslash\{x\}$

R		R'	
1	2	1	2
x	z	x, y	z
z	x, y	z	y
y			x

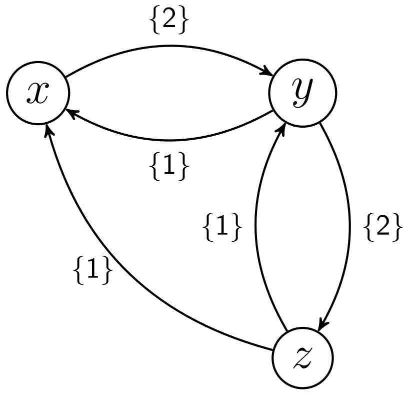

Figure 2: Example of non-farsightedly robustly implementing rights structure.

such that agents select the socially desirable alternatives, irrespective of whether they are farsighted? In the particular case of Example 4, this goal can be achieved by forbidding agent 2 to move from y to z. In what follows, we propose a solution to this problem.

In order to do this, we first formalize a notion of robustness to farsighted reasoning.

The Harsanyi's critique led to the introduction of the indirect dominance relation (Harsanyi, 1974; Chwe, 1994) to incorporate farsightedness in models with binding agreements. A state s' indirectly dominates s if there exists a path from s to s' such that every coalition effective on this path prefers the final state of the path s' to the state they replace.

Definition 13 (Indirect Dominance)**.** For a given social environment $\langle \Gamma, R \rangle$, a state s is indirectly dominated by s' under γ, denoted by $s' \gg_{(\Gamma,R)} s$, if there are states $s^0, s^1, ..., s^m$ and corresponding coalitions $K^1, ..., K^m$ where $s = s^0$ and $s' = s^m$ such that for all $\ell = 1, ..., m$ the following statements hold:

1. $K^\ell \in \gamma(s^{\ell-1}, s^\ell)$
2. $h(s') P_{K^\ell} h(s^{\ell-1})$

Several farsighted solutions concepts build over the indirect dominance relation (Chwe, 1994; Herings, Mauleon and Vannetelbosch, 2009; Ray and Vohra, 2015; Dutta and Vohra, 2017; Ray and Vohra, 2019; Karos and Robles, 2021), each of them captures different aspects of farsighted rationality. Among those, the Largest Consistent Set (LCS) (Chwe, 1994) has been criticized as being too permissive (Herings, Mauleon and Vannetelbosch, 2009; Dutta and Vohra, 2017). However, this critique does not apply to our framework. Indeed, the LCS allows for a broad spectrum of farsighted behaviors, making it a perfect candidate for our purposes.

Definition 14 (Consistent Set)**.** For a given social environment (Γ, R), a set $Y \subseteq S$ is consistent if the following statement holds: $s \in Y \iff \forall s' \in S, K \in \gamma(s, s')$ there is an $s'' \in Y$ such that either $(s'' = s')$ or $(s'' \gg_{(\Gamma,R)} s')$ and not $h(s'')P_K h(s)$.

A state s in the consistent set Y if and only if any deviation of a coalition K to any other state s' is deterred by an indirect dominance path leading to a state s'' in Y that makes at least one of the agents in K weakly worse off. The *largest consistent set*, hereafter LCS, denoted by $LCS(\Gamma, R)$, is the maximal consistent set with respect to set inclusion.

We frame model robustness to farsighted reasoning by requering implementation in the vNM stable set and the largest consistent set.

Definition 15 (Double Implementation)**.** We say that rights structure $\mathit{\Gamma} = (S, \gamma, h)$ implements a SCF $f : \mathcal{R} \longrightarrow Z$ in LCS and vNM stable set if $vNM(\mathit{\Gamma}, R) = LCS(\mathit{\Gamma}, R) = f(R)$ for all $R \in \mathcal{R}$.

Our implementing conditions relies on the following notion of *indirect independence of irrelevant alternatives*, which strengthens the notion of IIA.

Definition 16 (iIIA)**.** $f : \mathcal{R} \longrightarrow Z$ satisfies *indirect independence of irrelevant alternatives* (iIIA) if for all $R, R' \in \mathcal{R}$ and all $x, y, z \in f(\mathcal{R})$ such that $z \notin \{x, y\}$, we have

$$f(R) = x \ \text{ and } \ K(R, x, z) \subseteq K(R', y, z) \implies f(R') \neq z$$

In words, if those agents who prefer x to z at R when x is socially optimal also prefer y to z at R', then z cannot be socially optimal R'. The condition does not exclude that x is still optimal at R'. In IIA, the comparison between x and z is direct, while in iIIA the comparison can happen indirectly through a third outcome y. Note that iIIA implies IIA: the latter is a special case of the former whenever, in Definition 16, it holds that $x = y$. Examples[6] of SCFs satisfying iIIA are the fixed price rule in a bilateral trading environment and the Condorcet rule in a voting environemnt, as defined in Section 4.

Theorem 8 (Sufficiency). *If $f : \mathcal{R} \longrightarrow Z$ satisfies iIIA and vNM EFFICIENCY, then it is implementable in vNM stable set and in the LCS by a rights structure.*

If $f : \mathcal{R} \longrightarrow Z$ satisfies iIIA and vNM EFFICIENCY, then the implementing rights structure in the vNM stable set and the LCS also implements in (farsighted) core (Diamantoudi and Xue, 2003), farsighted stable set (Ray and Vohra, 2015), strong rational expectation farsighted stable set (Dutta and Vohra, 2017), absolutely maximal farsighted stable set (Ray and Vohra, 2019), farsighted stable set with heterogeneous expectations (Bloch and van den Nouweland, 2021) and equilibrium stable set (Karos and Robles, 2021). The result, stated in Corollary 5, builds on the fact that our theory employs single-payoff solutions.

Corollary 5. *If $f : \mathcal{R} \longrightarrow Z$ satisfies vNM EFFICIENCY and iIIA, then there exists a rights structure implementing $f : \mathcal{R} \longrightarrow Z$ in vNM stable set, largest consistent set, (farsighted) core, farsighted stable set, strong rational expectation farsighted stable set, absolutely maximal farsighted stable set, farsighted stable set with heterogeneous expectations and equilibrium stable set.*

Corollary 5 is in line with recent contributions studying dominance invariance in coalitional games (Mauleon, Molis, Vannetelbosch and Vergote, 2014; Kimya, 2022). A social environment satisfies dominance invariance if direct and indirect dominance are equivalent. Kimya (2022) shows that dominance invariant plays a fundamental role in eliminating differences among various farsighted solutions. Our result show that iIIA and vNM EFFICIENCY (see Lemma 4 in the Section 6) are sufficient for designing a rights structure that exhibits dominance invariance when it is restricted to the set of socially optimal states. This

[6]See Proposition 9 and Proposition 10 in the Appendix.

fact sheds new light on the role played by dominance invariance to harmonize different solutions, myopic and farsighted.

We conclude by pointing out that iIIA and vNM EFFICIENCY provide an almost full characterization. Indeed, it can be shown that these two conditions are necessary for implementation in comprehensive domains and single-crossing domains. We leave the investigation of vNM EFFICIENCY, iIIA, and IIA in other domains as an avenue for future research.

Theorem 9 (Necessity)**.** *Suppose that $\mathcal{R}$ is a comprehensive domain. If $f : \mathcal{R} \to Z$ is implementable in the vNM stable set and in the LCS via a rights structure, then it satisfies vNM EFFICIENCY and iIIA.*

Theorem 10 (Necessity)**.** *Suppose that $\mathcal{R}$ is a single-crossing domain. If $f : \mathcal{R} \to Z$ is implementable in the vNM stable set and in the LCS via a rights structure, then it satisfies vNM EFFICIENCY and iIIA.*

6 Conclusion

Despite the good number of applications in several environments, the notion of stable set is difficult to use. The stable set is, in general, not unique (Lucas, 1968) and may fail to exist (Lucas, 1992). Also, the problem of finding a vNM stable set is undecidable (Deng and Papadimitriou, 1994). These unpleasant features made Aumann (1987) write:

> "Finding stable sets involves a new tour de force of mathematical reasoning for each game or class of games that is considered. Other than a small number of elementary truisms [...] there is no theory, no tools, certainly no algorithms."

We contribute to the normative theory of the stable set in the realm of implementation theory. Methodologically, we adopt the rights structures (Koray and Yildiz, 2018) as implementing devices. The planner aims to induce the socially desirable alternative by (i) describing the available alternatives via a set of states and (ii) specifying which agents or groups of agents have the power to move from one state to another. This design amounts to adopting a "blocking" approach to implementation theory. An SCF is implementable in the vNM stable set via a rights structure if there exists a rights structure such that for each preference profile, the outcomes induced by any vNM stable set at this profile coincide with the outcome prescribed by the SCF at this profile.

Our conditions of vNM EFFICIENCY, vNM MONOTONICITY and TEST CYCLE are necessary and sufficient for the implementation in the vNM stable set via a rights structure. When preferences are continuous, money monotonic and self-regarding and when there are sides payments, or when preferences are linear orderings, TEST CYCLE is redundant. Under these requirement, vNM EFFICIENCY and vNM MONOTONICITY fully characterize the class of functions that are implementable in the vNM stable set via a rights structure.

As examples, the minimum distance rule in a facility location environment and the Condorcet rule in a voting environment are implementable SCFs. In a bilateral trading environment, the fixed price rule is the only SCF that can be implemented in the vNM stable set. We further characterize the implementation exercise by showing that INDEPENDENCE OF IRRELEVANT ALTERNATIVES (IIA), when combined with vNM EFFICIENCY, is sufficient for implementation. However, they are also necessary for implementation in comprehensive domains and single-crossing domains.

Finally, to take into account the Harsanyi'critique (Harsanyi, 1974), we study conditions for the design of implementing rights structures that are robust to farsighted reasoning. The robustness is achieved by designing a rights structure that doubly implements $f : \mathcal{R} \longrightarrow Z$ in the vNM stable set and in the largest consistent set (Chwe, 1994).

A monotonic transformation of INDEPENDENCE OF IRRELEVANT ALTERNATIVES, namely INDIRECT INDEPENDENCE OF IRRELEVANT ALTERNATIVES (iIIA), combined with vNM EFFICIENCY, is sufficient for the double implementation. These two conditions are also necessary when $f : \mathcal{R} \longrightarrow Z$ is defined on a comprehensive or single-crossing domain. The fixed price rule in a bilateral trading environment and the Condorcet rule in a voting environment are doubly implementable.

Appendix

Proof of Theorem 1. Suppose that $\Gamma = (S, h, \gamma)$ implements f in single-payoff vNM stable set. Let us show that f must satisfy vNM EFFICIENCY. Let $h(S) = Y \subseteq Z$, where $h(S) = \{h(s) \in Z | s \in S\}$. Assume, to the contrary, that the condition is violated. Then, there exist $R \in \mathcal{R}$ and $x \in h(S)$ such that $f(R) \neq x$ and $xR_i f(R)$ for all $i \in N$. Since Γ implements f and $f(R) \neq x$, it holds that $s \notin vNM(\Gamma, R)$ for all $s \in S$ such that $h(s) = x$. Fix any $s \in S$ such that $h(s) = x$. External stability and implementability of f imply that there exist $s' \in vNM(\Gamma, R)$ and $K \in \gamma(s, s')$ such that $K \subseteq K(R, h(s'), h(s)) = K(R, f(R), x)$, which is a contradiction. Thus, f satisfies vNM EFFICIENCY ■

The following lemma has been used in Theorem 2 and Theorem 3.

Lemma 2. *Suppose that f is implementable in vNM stable set via a rights structure. If $x \in f(\mathcal{R}) \backslash f(R')$ for some $R' \in \mathcal{R}$, then for all $x^* \in I^f(x, R')$ it holds that $x^* \notin f(R')$.*

Proof of Lemma 2. Fix any $x^* \in I^f(x, R')$. Suppose toward a contradiction that $x^* = f(R')$. Since $x \in f(\mathcal{R})$, it holds that $x \in I^f(x, R')$. Then, by definition of $I^f(x, R')$ we have that $xI'_N x^*$ or, in other terms, $K(R', x^*, x) = \varnothing$. Since f is implementable and since $x \neq f(R') = x^*$, we have that $V(\gamma, R')) = S^{x^*} \neq S^x$. Then, by external stability of S^{x^*}, for any $s \in S^x$ (where $h(s) = x$) and some $s^* \in S^{x^*}$ (where $h(s^*) = x^*$), it must be the case that $K(R', h(s^*), h(s)) \neq \varnothing$, which is a contradiction. ■

Proof of Theorem 2. Suppose that $\Gamma = (S, h, \gamma)$ implements f in vNM stable set. Let $h(S) = Y \subseteq Z$, where $h(S) = \{h(s) \in Z | s \in S\}$. Recall that, for all $y \in Y$, $S^y = \{s \in S | h(s) = y\}$ denotes the set of states where the outcome

y. Fix any $(x, R') \in Z \times \mathcal{R}$ with $x \in f(\mathcal{R}) \backslash f(R')$, any $x^* \in I^f(x, R')$, and any $R \in f^{-1}(x^*)$. Let us show that $K(R, x^*, z) \nsubseteq K(R', x^*, z)$ for some $z \in Y$. Since $f(R) = x^*$ and since $\Gamma = (S, h, \gamma)$ implements f in vNM stable set, it follows that $vNM(\Gamma, R) = S^{x^*}$. Let $S^{I^f(x,R')}$ be defined by $S^{I^f(x,R')} = \{s \in S \mid h(s) \in I^f(x, R')\}$. Since $x^* \in I^f(x, R')$, it follows that $S^{x^*} \subseteq S^{I^f(x,R')}$. Lemma 2 together with implementability of f implies that $S^{I^f(x,R')}$ is not a vNM stable set at R'.

Then, given that $S^{I^f(x,R')}$ is internally stable at R', it must violates external stability. Thus, there exists $s \in S \backslash S^{I^f(x,R')}$ such that for all $s' \in S^{I^f(x,R')}$, and all $K \subseteq K(R', h(s'), h(s))$, $K \notin \gamma(s, s')$ holds. Since S^{x^*} is a vNM stable set at R and since $S^{x^*} \subseteq S^{I^f(x,R')}$, it holds that $K(R, x^*, h(s)) \nsubseteq K(R', x^*, h(s))$ for some $h(s) \in Y$, with $s \in S \backslash S^{I^f(x,R')}$. Thus, f satisfies vNM MONOTONICITY. ■

The following lemma will be used in the proof of Theorem 3

Lemma 3. (Richardson, 1953) *If a vNM does not exist, then there is an odd cycle.*

Proof of Theorem 3. Suppose that Γ implements f in vNM stable set. Fix any $R' \in \mathcal{R}$ and any $x \in f(\mathcal{R}) \backslash \{f(R')\}$. Suppose that $xP'_i f(R')$ for some $i \in N$. Since $x \in f(\mathcal{R})$, it holds that $x \in I^f(x, R')$.

Let $S^{I^f(x,R')} = \left\{s \in S | h(s) \in I^f(x, R')\right\}$. Since $f(R') \neq x$, Lemma 2 implies that $V(\Gamma, R') \neq S^{I^f(x,R')}$. Moreover, let

$$S' = \left\{s' \in S \backslash S^{I^f(x,R')} | s \not>_{(\Gamma,R')} s' \text{ for all } s \in S^{I^f(x,R')}\right\}$$

Since $V(\Gamma, R') \neq S^{I^f(x,R')}$, and since $S^{I^f(x,R')}$ is internally stable at R', it must be that $S^{I^f(x,R')}$ violates external stability at R', and so $S' \neq \emptyset$. By construction, $h(S') \subseteq M^f(x, R')$ and $S^{I^f(x,R')} \cup S'$ is externally stable at R'. However, since, by implementability of f, $V(\Gamma, R') \neq S^{I^f(x,R')} \cup S'$, it follows that $S^{I^f(x,R')} \cup S'$ is not internally stable at R'.

Suppose that there exists $s' \in S'$ such that $s' >_{(\Gamma,R)} s$ for some $s \in S^{I^f(x,R')}$. Then, there exists $K \in \gamma(s, s')$ such that $h(s') P'_K h(s)$. Since $s \in S^{I^f(x,R')}$ and preferences are transitive, it holds that $h(s') P'_K x^*$ for all $x^* \in I^f(x, R')$. Fix any $l \in K$, so that $h(s')P'_l x^*$ for all $x^* \in I^f(x, R')$. Let us proceed according to whether $f(R') = h(s')$ or not.

Suppose that $f(R') = h(s')$. Since $s' \in S'$ and since $h(s') \in \mathcal{M}^f(x, R')$ it follows that $f(R') \in \mathcal{M}^f(x, R')$. This shows that part (iii) of the test-cycle condition is satisfied.

Suppose that $f(R') \neq h(s')$. Since f satisfies Pareto optimality without total indifference, there exists $j \in N$ such that $f(R')\, P'_j h(s')$. Since, by our initial assumption, there exists an agent $i \in N$ such that $xP'_i f(R')$ and agent i's preferences are transitive, it follows that $x^* P'_i f(R')$ for all $x^* \in I^f(x, R')$. Since $f(R')\, P'_j h(s')$ and since $h(s')P'_l x^*$ for all $x^* \in I^f(x, R')$, we have that for all $x^* \in I^f(x, R')$, $x^* P'_i f(R')\, P'_j h(s')P'_l x^*$ some $i, j, l \in N$. This shows that part (i) of the test-cycle condition is satisfied.

Otherwise, suppose that there does not exist any $s' \in S'$ such that $s' >_{(\Gamma,R)} s$ for some $s \in S^{I^f(x,R')}$. Then it has to be that S' is not internally stable at R'. Hence, by definition of S', $S^{I^f(x,R')} \cup S'$ is not internally stable at R' because S' is not internally stable at R'. Given a rights structure Γ, a restriction of Γ to $S' \subseteq S$, denoted by $\Gamma_{|S'} = \left(S', h_{|S'}, \gamma_{|S'}\right)$, is a rights structure such that for all $s \in S'$, $h_{|S'}(s) = h(s)$, and for all $s, s' \in S'$, $\gamma_{|S'}(s, s) = \gamma(s, s')$. Suppose that $V\left(\Gamma_{|S'}, R'\right) \neq \varnothing$. Then, $S^{I^f(x,R')} \cup V\left(\Gamma_{|S'}, R'\right) = V(\Gamma, R')$, which is a contradiction. Then it must be that $V\left(\Gamma_{|S'}, R'\right) = \varnothing$. Lemma 3 implies that there exists a sequence of states $(s^1, ..., s^k)$ in S', such that the outcomes yield an odd cycle at R'. Since $h(S')$ is contained in $\mathcal{M}^f(x, R')$, this shows that part (ii) of the test-cycle condition is satisfied. ■

Proof of Proposition 1. Fix any $R' \in \mathcal{L}$ and suppose that $x \in f(\mathcal{L})\backslash f(R')$ and that $xP'_i f(R')$ for some $i \in N$. Since preferences are linear orders, $I^f(x, R') = \{x\}$. By vNM MONOTONICITY, for all $R \in f^{-1}(x)$, we have that $K(R, x, z) \nsubseteq K(R', x, z)$ for some outcome $z \in \mathcal{M}^f(x, R')$. If $z = f(R')$, then requirement (iii) of the TEST CYCLE condition is satisfied. In what follows, let $z \neq f(R')$. Since $K(R, x, z)$ is non empty, take any $j \in K(R, x, z)\backslash K(R', x, z)$. Then, $xP_j z$ and $zR'_j x$. Since $xP_j z$, it follows that $x \neq z$. Since R'_j is a linear order, it follows that zP'_jx. Since, by our initial supposition, $xP'_if(R')$ for some $i \in N$, we have that $zP'_jxP'_if(R')$ for some $i \in N$, and some $j \in K(R,x,z)\backslash K(R',x,z)$. Since $f(R') \neq z$ and since f is vNM efficient, it follows that there exists $k \in N$ such that $f(R')P'_kz$. We have established that $zP'_jxP'_if(R')P'_kz$ for some $i,j,k \in N$ and some $z \in \mathcal{M}^f(x,R')$. Thus, f satisfies requirement (i) of TEST CYCLE. ■

Proof of Theorem 4. Let us construct a rights structure that implements f under the given conditions. We will denote outcome z in condition (i) of TEST CYCLE by $z(x,R)$, and outcome z^h in condition (ii) of TEST CYCLE by $z^h(x,R)$. Thus, whenever we speak of $z(x,R)$, we mean that for the pair (x,R) it is condition (i) of TEST CYCLE that is satisfied. Furthermore, we will denote the agent who prefer $z^k(x,R)$ to $z^{k+1}(x,R)$ at R in condition (ii) by $j(x,R,k,k+1)$ modulo k.

Let f satisfy conditions (i)-(iii) with respect to $Y \subseteq Z$ such that $f(\mathcal{R}) \subseteq Y$. In what follows we construct an implementing Γ. Let $\bar{S}$ be defined by

$$\bar{S} = \bigcup_{R\in\mathcal{R}} \bigcup_{x\in f(\mathcal{R})} \left\{ \left(y, I^f(x,R)\right) | y \in I^f(x,R) \text{ and } f(R) \neq x \right\} \cup Gr(f)$$

Furthermore, fix any $x \in f(\mathcal{R})$ and any $R \in \mathcal{R}$ such that $f(R) \neq x$. If either (i), (ii) or (iii) holds then we say that there exists a test cycle for (x,R).

Suppose that there exists a test cycle for (x,R). Let us define the following sets of states according to whether condition (i), condition (ii), or condition (iii) applies:

$$S((x,R),\text{i}) = \left\{ (u,(x,R),\text{i}) \;\middle|\; \begin{array}{l} \text{the test cycle for } (x,R) \\ \text{satisfies condition (i) and} \\ u \in \{f(R), z(x,R)\} \cup I^f(x,R) \end{array} \right\}$$

$$S\left((x,R),\text{ii}\right)=\left\{\left(z^{h}\left(x,R\right),(x,R),\text{ii}\right)\;\middle|\;\begin{array}{l}\text{the test cycle for }(x,R)\\ \text{satisfies condition (ii) and}\\ h=1,...,k\end{array}\right\}$$

$$S\left((x,R),\text{iii}\right)=\left\{\left(f\left(R\right),(x,R),\text{iii}\right)\;\middle|\;\begin{array}{l}\text{the test cycle for }(x,R)\\ \text{satisfies condition (iii).}\end{array}\right\}$$

Let us define the set of states S by

$$S=\bar{S}\cup\left\{\bigcup_{R\in\mathcal{R}}\bigcup_{x\in f(\mathcal{R})\backslash\{f(R)\}}\left(S\left((x,R),\text{i}\right)\cup S\left((x,R),\text{ii}\right)\cup S\left((x,R),\text{iii}\right)\right)\right\}.$$

Then, for all $s\in S$, let us defined the outome function h by $h\left(s\right)=s_1$, where s_1 is the outcome of the first entry of the tuple s. Finally, let us the code of rights γ be defined by, for all $s,s'\in S$ and all $i\in N$,

RULE 1: If $s,s'\in\bar{S}$, then:

(a) if $s,s'\in Gr(f)$, then $\{i\}\in\gamma\left(s,s'\right)$.

(b) if $s\in\bar{S}\backslash Gr(f)$ and $s'\in Gr(f)$, then $\{i\}\in\gamma\left(s,s'\right)$.

RULE 2: If $s,s'\in S\left((x,R),\text{i}\right)$, then:

(a) if $s=\left(f\left(R\right),(x,R),\text{i}\right)$ and $s'=\left(x^{*},(x,R),\text{i}\right)$, then $\{i\}\in\gamma\left(s,s'\right)$.

(b) if $s=\left(x^{*},(x,R),\text{i}\right)$ and $s'=\left(z\left(x,R\right),(x,R),\text{i}\right)$, then $\{\text{i}\}\in\gamma\left(s,s'\right)$.

(c) if $s=\left(z\left(x,R\right),(x,R),\text{i}\right)$ and $s'=\left(f\left(R\right),(x,R),\text{i}\right)$, then $\{i\}\in\gamma\left(s,s'\right)$.

RULE 3: If $s\in S\left((x,R),\text{i}\right)$ and $s'=\left(R',y\right)\in Gr\left(f\right)$, then:

(a) if $s=\left(x^{*},(x,R),\text{i}\right)$, then $K\left(R',y,x^{*}\right)\in\gamma\left(s,s'\right)$.

(b) if $s=\left(z\left(x,R\right),(x,R),\text{i}\right)$, then $K\left(R',y,z\left(x,R\right)\right)\in\gamma\left(s,s'\right)$.

RULE 4: If $s = (f(R), (x,R), \text{i}) \in S((x,R), \text{i})$, $s' = (R', y) \in Gr(f)$ and $y \notin I(x,R)$, then $K(R', y, f(R)) \in \gamma(s, s')$.

RULE 5: If $s, s' \in S((x,R), \text{ii})$, $s = \left(z^{h+1}(x,R), (x,R), \text{ii}\right)$ and $s' = \left(z^{h}(x,R), (x,R), \text{ii}\right)$ for some $h = 1, ..., k$ and $z^h(x,R)\, P_i z^{h+1}(x,R)$, then $\{i\} \in \gamma(s, s')$, where $z^{k+1}(x,R) = z^1(x,R)$.

RULE 6: If $s \in S((x,R), \text{ii})$ and $s' = (R', y) \in Gr(f)$, then $K(R', y, h(s)) \in \gamma(s, s')$.

RULE 7: If $s \in S((x,R), \text{iii})$ and $s' = (R', y) \in Gr(f)$, then $K(R', y, h(s)) \in \gamma(s, s')$.

RULE 8: If $s \in \{(y, I(x,R)) \,|\, y \in I(x,R) \text{ and } f(R) \neq x\}$ and $s' \in S((x,R), \text{iii})$, then $\{i\} \in \gamma(s, s')$.

RULE 9: Otherwise, $\gamma(s, s') = \varnothing$.

By construction, Γ is a rights structure. Let us show that Γ implements f in single-payoff vNM stable set. To this end, suppose that R is the true preference profile, and let $f(R) = \{x\}$. We show that $S^x \equiv \{s \in S \mid h(s) = x\}$ is the unique vNM stable set of (Γ, R).

Clearly, S^x satisfies internal stability, Then, let us show that S^x satisfies external stability.

To this end, note that vNM efficiency implies that for all $z \in Y$, xP_iz for some $i \in N$. Thus, by construction of Γ, (x,R) dominates all states in $\bar{S}\backslash S^x$ by RULE 1, all states in $S((y,R'), \text{ii})\backslash S^x$ by RULE 6, and all states in $S((y,R'), \text{iii})\backslash S^x$ by RULE 7. The set $S(R', y, i)$ needs a more careful examination.

Suppose that $S((y,R'), \text{i}) \neq \varnothing$. Suppose that $f(R) \neq y$. By vNM efficiency, we have that $f(R)P_iy$ for some $i \in N$. RULE 3 implies that (x,R) dominates all states $s \in S((y,R'), \text{i})\backslash S^x$ such that $h(s) \in \{z(R', y)\} \cup I(y,R')$. Suppose that $f(R) = f(R')$. Then, $(f(R'), (y,R'), \text{i}) \in S^x$. Suppose that $f(R) \neq f(R')$. Suppose that $f(R) \subseteq I(y,R')$. Then, (x,R) dominates $(f(R'), (y,R'), \text{i})$ via RULE 2. Thus, let $f(R) \neq f(R')$ and $f(R) \cap I(y,R') = \varnothing$. Since $K(R, f(R), f(R')) \neq \varnothing$ and since $f(R) \cap I(y,R') = \varnothing$, we have that (x,R) dominates $(f(R'), (u,R'), \text{i})$ via Rule 4.

Suppose that $f(R) = y$. Then, $f(R) \neq f(R')$, $f(R) \cap I(y, R') \neq \varnothing$ and $(f(R), (y, R'), \text{i}) \in S^x$. By vNM efficiency, we have that $f(R) P_i w$ for some $i \in N$ if $w \neq f(R)$. Since for all $s \in S((y, R'), \text{i})$ such that $h(s) = f(R)$, it holds that $s \in S^x$, we need to focus only on the cases that both $z(R', y) \neq y$ and $y^* \neq y$. Since $K(R, f(R), z(R')) \neq \varnothing$ and since $K(R, f(R), y^*)) \neq \varnothing$, it follows that (x, R) dominates any state $s \in S((y, R'), \text{i})$ such that either $h(s) = z(R', y)$ or $h(s) = y^*$ via Rule 3. Thus, we are left to show that $(f(R'), (y, R'), \text{i}))$ is dominated by a state in S^x. To this end, note that $f(R) \neq f(R')$, and so vNM efficiency implies that $f(R) P_i f(R')$ for some $i \in N$. Since $(f(R), (y, R'), \text{i}) \in S^x$ and since $f(R) P_i f(R')$ for some $i \in N$, it follows from Rule 2-a that agent i has the power and incentive to move from $(f(R'), (y, R'), \text{i})$ to $(f(R), (y, R'), \text{i})$. Thus, a state in S^x dominates $(f(R'), (y, R'), \text{i})$.

We conclude that S^x is externally stable, and so S^x is a vNM stable set of (Γ, R).

Next, We show that this is the only stable set at R. Assume, to the contrary, that there exists a nonempty set $S^* \subseteq S$ that is a vNM stable set of (Γ, R) such that $S^x \neq S^*$.

Note that at least one state of $\bar{S}$ must be in S^* by external stability. The reason is that the rights structure Γ does not allow any move from states inside the $Gr(f)$ to states outside of $\bar{S}$. Moreover, RULE 1 implies that if $s \in S^* \cap Gr(f)$ and $h(s) = z$, then $\{(z, R') \mid R' \in \mathcal{R},\ z = f(R')\} \subseteq S^*$. Given that S^* is externally stable and since $S^* \cap \bar{S} \neq \varnothing$, it follows from Rule 1 that $s \in S^* \cap Gr(f)$. Fix any $s \in S^* \cap Gr(f)$.

We proceed according to whether $h(s) = x$ or not.

Suppose that $h(s) = x$. Thus, $(x, R) \in S^*$, and so $S^x \subseteq S^*$. Since we have already shown that S^x is a vNM stable set of (Γ, R) and since S^* is a vNM stable set of (Γ, R), it follows that $S^x = S^*$, yielding a contradiction.

Suppose that $h(s) = y \neq x$. Since S^* is internally stable and since f is vNM efficient, it follows from RULE 1 that $\{(z, R') \big| R' \in \mathcal{R},\ f(R') = z \in I(y, R)\} = S^* \cap Gr(f)$.

We proceed according to whether $f(R)R_N y$ or not.

Suppose that $f(R)R_N y$. Since f is vNM efficient, there exists $i \in N$ such that $f(R)P_i y$. Since agent i has the power to move from s to (x, R) via Rule 1-a and since S^* is internally stable, it follows that $(x, R) \notin S^*$. Since $f(R)R_N y$, it follows that no agent has incentive to move from (x, R) to any state $\bar{s} \in S^* \cap Gr(f)$, thought they have the power to do so via Rule 1-a. Therefore, S^* is not externally stable, which is a contradiction.

Suppose that $yP_i f(R)$ for some agent $i \in N$. Since $y \in f(\mathcal{R})\backslash f(R)$, since $yP_i f(R)$ for some agent $i \in N$ and since, moreover, f satisfies the test-cycle property, it follows that a test cycle for (y, R) exists. There are three cases to be considered according to whether the test cycle for (y, R) is given either by condition (i), or by (ii), or by (iii).

Case 1: The test cycle is given by condition (i). Then, for some $i, j, k \in K$, it holds that $y^* P_i f(R) P_j z(y, R) P_k y^*$ for some $z(y, R) \in \mathcal{M}^f(y, R)$ and all $y^* \in I(y, R)$. Moreover, aggregate monotonicity implies that for all $y^* \in I^f(y, R)$ and all $R'' \in f^{-1}(y^*)$, it holds that $K(R'', y^*, z(y, R)) \subsetneq K(R, y^*, z(y, R))$. Suppose that $(z(y, R), (y^*, R), \text{i}) \notin S^*$ for some $y^* \in I^f(y, R)$. Since S^* satisfies external stability, it follows that there exists $K \in \gamma((z(y, R), (y^*, R), \text{i}), t)$ for some $t \in S^*$. By construction of Γ, since K can move only to a state in $S^* \cap Gr(f)$ via RULE 3, we have that $t = (R'', z) \in S^*$ for some $z \in Y$ and $R'' \in f^{-1}(z)$ and $K = K(R'', z, z(y, R))$. Since $S^* \cap Gr(f) = \{(R', z) \,|\, f(R') = z \in I^f(y, R)\}$, it follows that $t = (R'', z)$ is such that $z \in I^f(y, R)$. Since $z \in I^f(y, R)$ and since, for all $y^* \in I^f(y, R)$ and all $R'' \in f^{-1}(y^*)$, $K(R'', y^*, z(y, R)) \subsetneq K(R, y^*, z(y, R))$, it follows that S^* violates external stability at R, which is a contradiction. Therefore, it must be the case that $(z(y, R), (y^*, R), \text{i}) \in S^*$ for all $y^* \in I^f(y, R)$.

Suppose that $(y^*, (y, R), \text{i}) \notin S^*$ for some $y^* \in I^f(y, R)$. Again, since S^* satisfies external stability, there exists a coalition K such that $K \in \gamma((y^*, (y, R), \text{i}), t)$ for some $t \in S^*$. Since $y^* \in I^f(y, R)$, it follows from Γ that K can move only to a state in $S^* \cap Gr(f)$ via RULE 3. This implies that $t = (R'', z) \in S^*$ for some $z \in Y$ and $R'' \in f^{-1}(z)$ and that $K = K(R'', z, y^*)$. Again, since $S^* \cap Gr(f) = \{(R', z) \mid f(R') = z \in I^f(y, R)\}$, it follows that $t = (R'', z)$ is such that $z \in I^f(y, R)$. Since $z \in I^f(y, R)$, we have that the state $t = (R'', z) \in S^*$ cannot dominate at R the state $(y^*, (y, R), \text{i})$, in violation of the external stability of S^*. We conclude that $(y^*, (y, R), \text{i}) \in S^*$ for all $y^* \in I^f(y, R)$. Fix any $y^* \in I^f(y, R)$. Then, $(y^*, (y, R), \text{i}) \in S^*$ and $(z(y, R), (y^*, R), \text{i}) \in S^*$. Since, by condition (i) of TEST CYCLE, there exists $k \in N$ such that $z(y, R) P_k y^*$ and since Rule (2-b) implies that $k \in \gamma((y^*, (y, R), \text{i}), (z(y, R), (y^*, R), \text{i}))$, it follows that S^* violates internal stability at R, which is a contradiction.

Case 2: The test cycle is given by condition (ii). The states that are designed as a test cycle for (y, R) are

$$(z^1(y, R), (y, R), \text{ii}), (z^2(y, R), (y, R), \text{ii}), \ldots, (z^k(y, R), (y, R), \text{ii}).$$

Note that, by construction, if $(z^h(y, R), (y, R), \text{ii}) \notin S^*$, then we can move only to states of the type $(z, R') \in S^*$ with $z \in I^f(y, R)$. Fix any $h = 1, ..., k$. Suppose that $(z^h(y, R), (y, R), \text{ii}) \notin S^*$. Then, aggregate monotonicity implies that $K(R'', y^*, z^h(y, R)) \nsubseteq K(R, y^*, z^h(y, R)$ for all $y^* \in I^f(y, R)$ and all $R'' \in f^{-1}(y^*)$. This implies that RULE 6 can never applies. This contradicts our assumption that S^* is externally stable. Therefore, it must be the case that

$$(z^1(y, R), (y, R), \text{ii}), (z^2(y, R), (y, R), \text{ii}), \ldots, (z^k(y, R), (y, R), \text{ii}) \in S^*.$$

Since condition (ii) of test cycle implies that there is a cycle at R of odd length among the outcomes $z^1(y, R)$, ...$z^k(y, R)$, it follows from RULE 5 that S^* is not internally stable, which is a contradiction.

Case 3: The test cycle is given by condition (iii). Then, $f(R) \in \mathcal{M}^f(y, R')$. By definition of the rights structure Γ, only states in $\bar{S}$ can dominate the state $(f(R), (y, R), \text{iii})$ (via Rule 7). Since $f(R) \in \mathcal{M}^f(y, R')$, no state in $S^* \cap \bar{S}$

dominates the state $(f(R), (y, R), \text{iii})$ by aggregate monotonicity; the reason is that $f(R) \in \mathcal{M}^f(y, R')$, and so $K(R'', y^*, f(R)) \nsubseteq K(R, y^*, f(R))$ for all $y^* \in I^f(y, R)$ and all $R'' \in f^{-1}(y^*)$. Thus, it must be the case that $(f(R), (y, R), \text{iii}) \in S^*$. Since $s \in S^* \cap Gr(f)$ and $h(s) = y$, it follows that $(y, I^f(y, R)) \in S^*$. Since $(y, I^f(y, R)) \in S^*$ and since, by vNM efficiency, there exists a agent i such that $f(R)P_i y$, it follows that the internal stability of S^* is violated because agent i has the incentive and the power (via Rule 8) to move from $(y, I^f(y, R))$ to $(f(R), (y, R), \text{iii})$.

Since the choice of state $s \in S^* \cap Gr(f)$ was arbitrary, we conclude that S^* is not a vNM stable set of (Γ, R), which is a contradiction. ■

Proof of Proposition 2. Suppose preferences are self-regarding, continuous, money monotone, and that the preference domain $\mathcal{R}$ is finite. Suppose that f satisfies vNM EFFICIENCY and vNM MONOTONICITY with respect to Y. We show that f satisfies TEST CYCLE.

Fix any $R' \in \mathcal{R}$ and any $x \in Y$. Suppose that $x \in f(\mathcal{R}) \backslash f(R')$ and that $x P'_i f(R')$ for some $i \in N$. Since f satisfies vNM MONOTONICITY, it follows that for all $R \in f^{-1}(x)$ and for all $x^* \in I^f(x, R')$, $K(R, x^*, z) \nsubseteq K(R', x^*, z)$ for some outcome $z \in Y$. Thus, $z \in \mathcal{M}^f(x, R')$. We proceed according to whether $z = f(R')$ or not.

Suppose that $z = f(R')$. Then, requirement (iii) of the TEST CYCLE property is satisfied.

Suppose that $z \neq f(R')$. Let $z = (d, t)$. Since agents' preferences are continuous and self-regarding, it follows that there exists $\hat{\varepsilon} > 0$ such that for all $R \in f^{-1}(x)$, all $x^* \in I^f(x, R')$ and all $i \in K(R, x^*, z) \backslash K(R', x^*, z)$, it holds that $x_i^* P_i (d, t_i + \hat{\varepsilon})$. Moreover, since preferences are money monotonic and transitive, we have that for all $R \in f^{-1}(x)$, all $x^* \in I^f(x, R')$ and all $i \in K(R, x^*, z) \backslash K(R', x^*, z)$, $(d, t_i + \hat{\varepsilon}) P'_i x_i^*$. Therefore, for all $R \in f^{-1}(x)$, all $x^* \in I^f(x, R')$ and all $i \in K(R, x^*, z) \backslash K(R', x^*, z)$, it holds that $x_i^* P_i (d, t_i + \hat{\varepsilon})$ and $(d, t_i + \hat{\varepsilon}) P'_i x_i^*$.

Since $\mathcal{R}$ is finite and since, moreover, f is vNM efficient and agents' preferences are continuous and self-regarding, it follows that there exists $\varepsilon' > 0$ such that for all $\bar{R} \in \mathcal{R}$ and all $i \in N$ such that $f_i\left(\bar{R}\right)\bar{P}_i z_i$, it holds that $f_i\left(\bar{R}\right)\bar{P}_i\left(d, t_i + \varepsilon'\right)$.

Let $\varepsilon = \frac{\min\{\hat{\varepsilon}, \varepsilon'\}}{2}$. By construction, we have that:

1. For all $R \in f^{-1}(x)$, all $x^* \in I^f(x, R')$ and all $i \in K(R, x^*, z) \backslash K(R', x^*, z)$, it holds that $x^*_i P_i(d, t_i + \varepsilon)$ and $(d, t_i + \varepsilon) P'_i x^*_i$.

2. For all $\bar{R} \in \mathcal{R}$ and all $i \in N$ such that $f_i\left(\bar{R}\right)\bar{P}_i(d, t_i)$, it holds that $f_i\left(\bar{R}\right)\bar{P}_i(d, t_i + \varepsilon)$.

Let us define z' by

$$z'_i = \begin{cases} (d, t_i + \varepsilon_i) & \text{if } i \in \cup_{R \in f^{-1}(x)} \cup_{x^* \in I^f(x,R')} K(R, x^*, z) \\ z_i & \text{otherwise.} \end{cases}$$

By construction of z', we have that if $i \in \cup_{R \in f^{-1}(x)} \cup_{x^* \in I^f(x,R')} K(R, x^*, z)$, then $i \in K(R, x^*, z') \cap K(R', z', x^*)$ and that for all $\bar{R} \in \mathcal{R}$ and all $i \in N$ such that $f_i\left(\bar{R}\right)\bar{P}_i z_i$, it holds that $f_i\left(\bar{R}\right)\bar{P}_i z'_i$. Moreover, by construction, we also have that f is vNM efficient and vNM monotonic with respect to $Y \cup \{z'\}$, that $z' \in \mathcal{M}^f(x, R')$ and that $z' \neq f(R')$.

Since $z' \neq f(R')$ and since f is vNM efficient with respect to $Y \cup \{z'\}$, it follows that $f(R') P'_k z'$ for some $k \in N$. Since, by our initial supposition, $x P'_i f(R')$ for some $i \in N$, and since R'_i is transitive, we have that for all $x^* \in I^f(x, R')$, $x^* P'_i f(R')$ for some $i \in N$. Thus, we have that for all $x^* \in I^f(x, R')$, $x^* P'_i f(R') P'_k z'$ for some $i, k \in N$, with $z' \in \mathcal{M}^f(x, R')$. Fix any $j \in K(R, x^*, z) \backslash K(R', x^*, z)$ for some $R \in f^{-1}(x)$ and some $x^* \in I^f(x, R')$. Then, by construction, $j \in K(R, x^*, z') \cap K(R', z', x^*)$, and so $z' P'_j x^*$. Therefore, we have that $x^* P'_i f(R') P'_k z' P'_j x^*$ for some $i, j, k \in N$, with $z' \in \mathcal{M}^f(x, R')$. Since the previous argument holds for all $x^* \in I^f(x, R')$, we have that there exists $z' \in \mathcal{M}^f(x, R')$ such that for all $x^* \in I^f(x, R')$, there exists $i, j, k \in N$ such that $x^* P'_i f(R') P'_k z' P'_j x^*$. Thus, f satisfies requirement (i) of TEST CYCLE.

Since the above arguments hold for any $(R', x) \in \mathcal{R} \times F(\mathcal{R})$ such that $f(R') \neq \{x\}$ and $x P'_i f(R')$ for some $i \in N$, it follows that we can construct a set Y', with $Y \subseteq Y'$, such that f is vNM efficient and vNM monotonic with respect to Y', and so f satisfies TEST CYCLE with respect to Y'. ■

Proof of Proposition 3. f_{md} obviously satisfies vNM EFFICIENCY. To show vNM MONOTONICITY, fix any profile of locations $p \equiv (p_1, p_2, \ldots, p_n)$ and assume without loss of generality that p_1 be the smallest location. Select any $z \in [0, x] \backslash \{p_1\}$. Note that vNM MONOTONICITY is trivially satisfied for any location $y \in [0, p_1)$ since $K(p; y, p_1) = \varnothing$ (we can select $z = p_{min}$). For any location $y \in (p_1, x]$, and any profile of peaks p' where y is the smallest location, we have $K(p'; y, p_1) = N$. Since $1 \notin K(p; y, p_1)$, we can again select $z = p_1$ to show that vNM MONOTONICITY holds. This also shows that TEST CYCLE is satisfied because condition (iii) of the definition of TEST CYCLEis always satisfied. ■

Proof of Proposition 4. First we show that if f is implementable in vNM stable set, then f is a fixed price rule. For any $v_1 \in [a, b]$, let f be a SCF implementable in vNM stable set. Suppose toward a contradiction that f is not a fixed price rule, that is for some v_2 and v_2' with $v_2 > v_2'$, f is such that $f(v_1, v_2) \neq f(v_1, v_2') \equiv p' > 0$. Take any $z \in Z$. Note that if trading with price p' is more profitable than z to a buyer of type v_2', then it is more profitable to a buyer of type v_2 too. Since the argument holds for any $z \in Z$, we have that $K((v_1, v_2'), p', z) \subseteq K((v_1, v_2), p', z)$ holds for all $z \in Z$ which contradicts vNM MONOTONICITY of f. Therefore, if f is implementable in vNM stable set, then for any $v_1 \in [a, b]$, the function $f_{v_1} \equiv f(v_1, \cdot)$ must be a fixed price rule, conditionally on v_1. To complete the proof, it remains to show that f is a fixed price rule unconditionally on v_1.

Fix any $v_1 \in [a, b)$,[7] such that the price p of the fixed price rule $f(v_1, x)$ satisfies $b > p > 0$. Notice that by individual rationality this implies $p > v_1$. If v_1 does not exist, then f must be the zero price rule f_0, a particular case of a fixed price rule. To show that f is the fixed price rule f_p (unconditionally on v_1), we must verify that $f_{v_1'}$ is the fixed price rule with a price p for any $v_1' < p$, and the zero price rule for any $v_1' \geqslant p$. We study the two cases separately.

Suppose towards a contradiction that $v_1' < p$ but $f_{v_1'}$ is not the fixed price

[7]The rule f_b is equivalent to the zero price rule since trade never takes place.

rule with a price p. Take any value of the buyer $v_2 \in [p, b)$. Recall that f is implementable in vNM, hence it must satisfies vNM MONOTONICITY by Theorem 2. However, it is straightforward to see that $K((v_1, v_2), p, z) \subseteq K((v_1', v_2), p, z)$ holds for all $z \in Z$- a contradiction. Hence $f_{v_1'}$ is indeed the fixed price rule with a price p for any $v_1' < p$ as claimed.

Next, suppose towards a contradiction that $v_1' \geqslant p$, but $f_{v_1'}$ is not the zero price rule. One can easily see that this case follows directly from the previous case. Let $p' > 0$ be the fixed price of the rule $f_{v_1'}$. By individual rationality $p' > v_1' \geqslant p$. However, by the previous argumentation, f_{v_1} must be a fixed price rule with a price p' too - a contradiction.

Finally, it is easy to see that any fixed price rule f_p is implementable in vNM stable set. A simple rights structure (code of rights) $\Gamma = (S, \gamma)$, where $S = \{0, p\}$, $\gamma(0, p) = \{\{1, 2\}\}$ (trade must be accepted by both), and $\gamma(p, 0) = \{\{1\}, \{2\}\}$ (trade can be rejected by either), implements it. ■

Proof of Proposition 5. A rights structure where the consent of majority is needed to replace the status quo implements it. Let us verify that it satisfies our conditions. The Condorcet rule f_C clearly satisfies vNM EFFICIENCY in a Condorcet domain. To verify vNM MONOTONICITY, select any $R \in \mathcal{R}$, and any $x \in f(\mathcal{R}) \backslash \{f(R)\}$. Take $z = f(R)$. We are going to show that $z \in \mathcal{M}^{f_C}(x, R)$. For the sake of contradiction, suppose that $K(R'', x^*, z) \subseteq K(R, x^*, z)$ holds for some $x^* \in I^f(x, R)$ and some $R'' \in \{R' \in \mathcal{R} \mid f(R') = x\}$. Since x^* is the Condorcet winner at R'', this implies that z cannot be the Condorcet winner at R (otherwise majority would prefer x^* to it), which is a contradiction. Therefore, $M^{f_C}(x, R)$ is non-empty, and f_C satisfies vNM MONOTONICITY.

Condorcet rule satisfies also the TEST CYCLE condition, since according to the previous analysis, $f(R) \in M^{f_C}(x, R)$ holds for all $R \in \mathcal{R}$ and all $x \in f(\mathcal{R}) \backslash \{f(R)\}$. ■

Proof of Theorem 5. Let us construct a rights structure that implements f under the given conditions. The set of states is

$$S = \{(x, R) \mid R \in \mathcal{R},\ x = f(R)\},$$

define the outcome function $h : S \to X$ as $h(x, R) = x$ for all $(x, R) \in S$, and the code of rights $\gamma : S \times S \rightrightarrows N$ by the following rules:

RULE 1 $K(R, x, y) \in \gamma((y, R'), (x, R))$ for all $(x, R), (y, R') \in S$.

RULE 2 $\gamma(s, s') = \varnothing$ for all other $s, s' \in S$.

Suppose that $R \in \mathcal{R}$ is the true profile and $\{x\} = f(R)$. Let us first show that $S^x = \{s \in S \mid h(s) = x\}$ is a vNM stable set at R. First of all, this set is obviously internally stable. To verify external stability, choose any state $s \in S \backslash S^x$. Denote $h(s) = y \neq x$. Since by vNM EFFICIENCY it must be that $K(R, x, y) \neq \varnothing$, and $K(R, x, y) \in \gamma(s, (x, R))$ by RULE 1, the state $(x, R) \in S^x$ dominates s. This implies that S^x is externally stable, and hence S^x is a vNM stable set at R. Next we show that S^x is the unique vNM stable set at R. Suppose toward a contradiction that there exits a set $S' \subseteq S$ with $S' \neq S^x$ that is a vNM stable set at R. We have that either $(x, R) \in S'$ or not. Suppose not. Then some state inside S' must dominate (x, R) by external stability. Let $s \in S'$ be this state. Denote $s = (y, R')$. Clearly $y \neq x$. By rules (1) and (2), there is exactly one coalition K in $\gamma((x, R), (y, R'))$, namely $K = K(R', y, x)$. This implies $K(R', y, x) \subseteq K(R, y, x)$. Therefore, by IIA, we must have $f(R) \neq x$ – a contradiction. Previous argument requires that $(x, R) \in S'$. Then, the set S' must contains all the states that are related to x, otherwise external stability of S' at R would be violated, i.e., $S^x \subseteq S'$. Finally, we show the contradiction by proving that set inclusion relation cannot be strict. Suppose towards a contradiction that $S^x \subset S'$. Then, by external stability of S^x there is a state $s^x \in S^x$ and a state $s' \in S' \backslash S^x$ such that $s^x >_{(\Gamma, R)} s'$. Since $s', s^x \in S'$, then internal stability of S' is violated. Therefore, it has to be that $S^x = S'$, a contradiction.

■

Proposition 6. *In facility location problems, the minimum distance rule f_{md} satisfies IIA*

Proof of Proposition 6 Fix any profile of locations $p \equiv (p_1, p_2, \ldots, p_n)$ and assume without loss of generality that p_1 is the smallest location. Then, $p_1 = f(p)$. Fix any other profile of locations p' and any allocation z with $z = f(\mathcal{R}) \backslash f(p)$. Let us assume that $K(p, p_1, z) \subseteq K(p', p_1, z)$. We show that $z \neq f(p')$. First, note that, $p_1 = f(p)$ implies $K(p, p_1, z) \neq \varnothing$. Since p_1 is the smallest location at p, it holds that for all $i \in K(p, p_1, z)$, $p_1 \leqslant p_i < (z - p_1)/2$. Therefore, $p_1 < z$. Then, either p_1 is a location at p' or not. In the former case, z is not the smallest inhabited location at p', and $z \neq f_{md}(p')$. In the latter case, by the fact that $K(p', p_1, z) \neq \varnothing$, there exists at least an agent i located at some $0 \leqslant p_i < (z - p_1)/2$. Since $p_i < z$, z is not the smallest inhabited location at p' and $z \neq f(p')$. ■

Proposition 7. *In bilateral trades, the fixed price rule f_p satisfies IIA*

Proof of Proposition 7. Let us verify that the fixed price rule f_p satisfies IIA. Observe that $f_p(\mathcal{R}) = \{0; p\}$, that is at any preference profile, either there is no trade or there is a trade with price p. Take any $R \in \mathcal{R}$ such that $f_p(R) = 0$. Thus, $K(R, 0, p) \in \{\{1\}, \{2\}, \{1, 2\}\}$. Suppose, $K(R, 0, p) \subseteq K(R', 0, p)$. Then, at R', at least one of the agents does not want to trade with price p and hence $f_p(R') = 0 \neq p$, as required by IIA.

As the second case, take any $R \in \mathcal{R}$, such that $f_p(R) = p$. Thus, $K(R, p, 0) \in \{\{1, 2\}\}$. Suppose $K(R, p, 0) \subseteq K(R', p, 0)$, then both agents want to trade with price p at R', and hence $f_p(R') = p \neq 0$, as required by IIA. ■

Proposition 8. *Given a voting environment, the Condorcet rule f_C satisfies IIA*

Proof of Proposition 8. Let us verify that the Condorcet rule f_C satisfies IIA. Take any $R, R' \in \mathcal{R}$, and any $z, x \in Z$, such that $f(R) = x$ and $K(R, x, z) \subseteq K(R', x, z)$. Since x is a Condorcet winner at R, we have

$$|K(R', x, z)| \geqslant |K(R, x, z)| > \frac{n}{2}.$$

This implies that $|K(R', z, x)| < \frac{n}{2}$, and therefore, z cannot be a Condorcet winner at R'. Hence $f_C(R') \neq z$ as required by IIA. ■

Proof of Theorem 6. Let $\Gamma = (S, \gamma, h)$ be a rights structure implementing f in vNM stable set. Suppose that for some $R, R' \in \mathcal{R}$, and $x, z \in f(\mathcal{R})$ we have that $f(R) = x$ and $K(R, x, z) \subseteq K(R', x, z)$. We need to verify that $f(R') \neq z$ holds. Suppose towards a contradiction that $f(R') = z$. Let construct a preference profile R'' as follows. If $xP'_i z$ then R''_i is such that x is ranked first (uniquely) and z is ranked second (uniquely), and if $zR'_i x$ then R''_i is such that z is ranked first (uniquely) and x is ranked second (uniquely). By comprehensiveness of $\mathcal{R}$ such a R'' exists, i.e. $R'' \in \mathcal{R}$.

In the remaining part of the proof we show that there are two vNM stable sets at R'' – $V(\Gamma, R'') = S^z$ and $V'(\Gamma, R'') = S^x$. The fact that for all $s \in V(\Gamma, R'')$ and $s' \in V'(\Gamma, R'')$ it holds that $z = h(s) \neq h(s') = x$ violates the implementability of f.

First we show that S^z is a vNM stable set at R''. Since the set S^z trivially satisfies internal stability we only need to prove its external stability. Recall that by implementability of f, S^z is a vNM at R'. Then, for any $s \in S \backslash S^z$ there is some $K \subseteq K(R', z, h(s))$ such that $K \in \gamma(s, s^z)$ for some $s^z \in S^z$. We consider two cases: either $s = s^x$ or $s \neq s^x$. (i) if $s = s^x$, then, by construction of R'', it holds that $K(R', z, x) \subseteq K(R'', z, x)$. (ii) if $s \neq s^x$, then, by construction of R'' it holds that $K(R'', z, h(s)) = N$. Therefore, $K(R', z, x) \subseteq K(R'', z, x)$. In both cases, the ability and the incentive to move from s^x to s^z at R' of any coalition is preserved also at R''. This proves external stability for S^z.

Finally, by similar argument, we show that S^x is a vNM stable set at R''. As before, the set S^z trivially satisfies internal stability, thus we only need to prove its external stability. Recall that by implementability of f, S^x is a vNM at R. Then, for any $s \in S \backslash S^x$ there is some $K \subseteq K(R, x, h(s))$ such that $K \in \gamma(s, s^x)$ for some $s^x \in S^x$. As before, we have that either $s = s^z$ or $s \neq s^z$. (i) if $s = s^z$, then, the construction of R'' together with the assumption implies that

$K(R,x,z) \subseteq K(R',x,z) \subseteq K(R'',x,z)$. (ii) if $s \neq s^z$, then, by construction of R'' it holds that $K(R'',x,h(s)) = N$. Again, the ability and the incentive to move from s to s^x at R of any coalition is preserved also at R''. This proves external stability for S^x. ∎

Proof of Theorem 7. Suppose that rights structure $\Gamma = (S,\gamma,h)$ implements f in vNM stable set. Suppose that for some $R, R' \in \mathcal{R}$, and $x,y,z \in f(\mathcal{R})$, such that $z \notin \{x,y\}$, we have $f(R) = x$ and $K(R,x,z) \subseteq K(R',y,z)$. We need to verify that $f(R') \neq z$ holds. To complete the proof we only need to show that it is possible to construct the preference profile R'' in Theorem 6 so that it is included in the single-crossing domain. We divide the proof in two cases; either (1) $y \triangleright z$ or (2) $z \triangleright y$ ($\triangleright$ is the relation on Z given by SC). If (1) holds, then $K(R',y,z) = \{n,\dots,k\}$ and $N \backslash K(R',y,z) = \{k,k-1,\dots,1\}$ for some k. Now construct the preference profile R'' in the following way: Agents $\{n,\dots,k\}$ rank outcome y as the best (uniquely) and outcome z as the second best (uniquely), while agents $\{k,k-1,\dots,1\}$ rank outcome z as the best (uniquely) and outcome y as the second best (uniquely). All other outcomes $Z\backslash\{y,z\}$ are rank according to $\triangleright$ by everyone. It is easy to check that this preference profile belong to the single-crossing domain $\mathcal{R}$. Analogous argument holds in the case (2). ∎

Proof of Lemma 1. Suppose that $f : \mathcal{L} \longrightarrow Z$ satisfies vNM EFFICIENCY w.r.t $Y \subseteq Z$ and IIA. Take any $R, R' \in \mathcal{R}$, denote $f(R) = x$, and assume that $L_i(x,R) \subseteq L_i(x,R')$ holds for all $i \in N$. We need to show that x is selected at R' to verify the claim. For any $z \in Y\backslash\{x\}$, there exists at least one agent $j \in N$ such that $z \in L_j(x,R)$ holds by vNM EFFICIENCY, and $K(R,x,z) \subseteq K(R',x,z)$ holds by the nestedness of the lower contour sets. But then, by IIA, $f(R') \neq z$. Since the choice of $z \in Y\backslash\{x\}$ was arbitrarly, it follows that none of the alternatives in $Y\backslash\{x\}$ can be selected at R'. Therefore, $f(R') = x$ must hold, and f is Maskin monotonic w.r.t. Y.

∎

Proof of Corollary 4. Suppose that $|Z| \geqslant 3$ and $f : \mathcal{L}^* \to Z$ is onto. Suppose that f is implementable vNM stable set via a rights structure. Then, f satisfies vNM EFFICIENCY w.r.t Y. Since f is onto then $Y = Z$. Since $\mathcal{L}^*$ is a full domain, then, by Theorem 6, f satisfies IIA. This together with Lemma 1 implies that f is Maskin monotonic. Moreover, since f is vNM EFFICIENCY and since $R \in \mathcal{L}^*$ it also satisfies unanimity.[8] Then, Muller-Satterthwaite theorem (Muller and Satterthwaite, 1977) implies that f is dictatorial. It remains to show the only if part. Suppose that $f : \mathcal{L}^* \to Z$ is dictatorial. We proceed by proving that f satisfies IIA and vNM EFFICIENCY. For some $R \in \mathcal{L}^*$, fix any $x \in Z$ such that $x = f(R)$. Suppose that i is the dictator. Then, $i \in K(R, x, z)$ for some $z \neq x$. For any $R' \neq R$, if $K(R, x, z) \subseteq K(R', x, z)$ then $i \in K(R', x, z)$. This implies that $z \neq f(R')$. This together with the fact that the choice of R and R' was arbitrarily proves IIA for f. Furthermore, since the dictator i strictly prefer x to any $z \in Z \backslash \{x\}$ it holds that f satisfies vNM EFFICIENCY. Then, by Theorem 4, f is implementable in vNM stable set via a rights structure.

■

Proof of Theorem 8. We construct a rights structure $\Gamma = (S, h, \gamma)$ that double implements f in vNM stable set and in LCS. Let the set of states S be defined by

$$S = \{(x, R) \mid R \in \mathcal{R},\ x = f(R)\} = Gr(f).$$

Let us define the outcome function $h : S \to Z$ by $h(x, R) = x$ for all $(x, R) \in S$, and the code of rights $\gamma : S \times S \to \mathcal{N}$ by the following rules:

RULE 1: $K(R, x, y) \in \gamma((y, R'), (x, R))$ for all $(y, R'), (x, R) \in S$.

RULE 2: $\gamma(s, s') = \varnothing$ for all other $s, s' \in S$.

Suppose that $R \in \mathcal{R}$ is the true preference profile and suppose $x = f(R)$. To

[8] f satisfies unanimity if for all $R \in \mathcal{L}^*$ and all $x \in Z$, if $Z \subseteq L_i(x, R)$ for all $i \in N$ then $f(R) = x$.

verify the claim, let us show that $S^x \equiv \{s \in S \mid h(s) = x\} \subseteq S$ is the unique vNM stable set and the LCS of (Γ, R).

The proof is based on the following lemma.

Lemma 4. *Any state in S^x is neither directly nor indirectly dominated by any state in S.*

Proof. Fix any $(x, \bar{R}) \in S^x$. Firstly, assume, to the contrary, that $(z, R') \in S \backslash S^x$ directly dominates $(x, \bar{R})$ under (Γ, R), i.e., $(z, R') >_{(\Gamma,R)} (x, \bar{R})$. RULE 1 implies that $K(R', z, x) = \gamma((x, \bar{R}), (z, R'))$. Since $(z, R') >_{(\Gamma,R)} (x, \bar{R})$, it follows that $K(R', z, x) \subseteq K(R, z, x)$. Since $z, x \in F(\mathcal{R})$, $f(R') = z$, and $K(R', z, x) \subseteq K(R, z, x)$, IIA implies that $f(R) \neq x$, which is a contradiction.

Secondly, assume, to the contrary, $(z, R'') \in S \backslash S^x$ indirectly dominates $(x, \bar{R})$ under (Γ, R); that is, $(z, R'') \gg_{(\Gamma,R)} (x, \bar{R})$. Since the domination cannot be direct, the indirect domination path from (x, R) to (z, R'') is such that $(x, R), (w, \widehat{R}), \ldots, (z, R'')$. Note that $x \notin \{w, z\}$. Also, note that $w, x, z \in F(\mathcal{R})$. Since $(z, R'') \gg_{(\Gamma,R)} (x, \bar{R})$, RULE 1 implies that $K(\widehat{R}, w, x) = \gamma((x, R), (w, \widehat{R}))$. Since $(z, R'') \gg_{(\Gamma,R)} (x, \bar{R})$ and $K(\widehat{R}, w, x) = \gamma((x, R), (w, \widehat{R}))$, it holds that $K(\widehat{R}, w, x) \subseteq K(R, z, x)$. Since $w, x, z \in F(\mathcal{R})$, with $x \notin \{w, z\}$, and since $f(\widehat{R}) = w$ and $K(\widehat{R}, w, x) \subseteq K(R, z, x)$, iIIA implies that $f(R) \neq x$, which is a contradiction. $\square$

Let us show that S^x is a vNM stable set at (Γ, R). By construction, the set S^x satisfies internal stability. To prove external stability of S^x at (Γ, R), fix any $(z, R') \in S \backslash S^x$. Since $z \notin f(R)$, vNM EFFICIENCY implies that $x P_i z$ for some $i \in N$, and so $K(R, x, z) \neq \emptyset$. Since RULE 1 implies $K(R, x, z) = \gamma((z, R'), (x, R))$, it follows that $(x, R) >_{(\Gamma,R)} (z, R')$. Since the choice of $(z, R') \in S \backslash S^x$ was arbitrary, we have that S^x satisfies external stability at (Γ, R).

Next, let us show that S^x is the unique vNM stable set at (Γ, R). Assume, to the contrary, that there exists $S' \subseteq S$, with $S' \neq S^x$, such that S' is a vNM stable set at (Γ, R). Suppose that there exists $s \in S^x$ such that $s \notin S'$. Lemma 4 implies that S' does not satisfies external stability at (Γ, R), which is a contradiction. Thus, suppose that $S^x \subseteq S'$. Since $S^x \neq S'$, there exists $(z, R') \in S'$ such that $(z, R') \notin S^x$. Since $z \notin f(R)$, vNM EFFICIENCY implies that $x P_i z$ for some $i \in N$, and so $K(R,x,z) \neq \varnothing$. Since RULE 1 implies $K(R,x,z) = \gamma((z,R'),(x,R))$, it follows that $(x,R) >_{(\Gamma,R)} (z,R')$. This implies that S' does not satisfy internal stability at (Γ,R). Thus, S^x is the unique vNM stable set at (Γ,R).

Finally, let us show that S^x is the largest consistent set at (Γ,R). Let us first show that S^x is a consistent set at (Γ,R). Fix any $s \in S$. Suppose that there exists $K \in \mathcal{N}_0$ such that $K \in \gamma(s^x,s)$ for some $s^x \in S^x$. We proceed according to whether $s \in S^x$ or not. Since, by the construction of γ, it holds that $\varnothing = \gamma(s',s'')$ for all $s',s'' \in S^x$, it follows that $s \in S \backslash S^x$. Without loss of generality, let $s = (z,R')$ and $s^x = (x,\bar{R})$. Moreover, RULE 1 implies that $K = K(R',z,x) = \gamma((x,\bar{R}),(z,R'))$. When we proved above that S^x satisfies external stability at (Γ,R), we proved that $(x,R) >_{(\Gamma,R)} (w,R'')$ for all $(w,R'') \in S\backslash S^x$. Therefore, since $(x,R) >_{(\Gamma,R)} (z,R')$, we have that there exists $(x,R) \in S^x$ such that $(x,R) >_{(\Gamma,R)} (z,R')$ and not $h(x,R)P_{K(R',z,x)}h(x,\bar{R})$. Since the choice of $s \in S$ was arbitrary, we have that S^x is a consistent set at (Γ,R).

Finally, let us show that S^x is the largest consistent set at (Γ,R). Assume, to the contrary, that there exists a consistent set S' such that $S' \supseteq S^x$ and $S' \neq S^x$. Fix any $(y,R') \in S'\backslash S^x$. Let us consider the state $(x,R) \in S^x$. vNM EFFICIENCY implies that xP_iy for some $i \in N$, and so $K(R,x,y) \neq \varnothing$. RULE 1 implies that $K(R,x,y) = \gamma((y,R'),(x,R))$. Since $(y,R') \in S'$ and S' is a consistent set at (Γ,R), it follows that there exists $\bar{s} \in S'$ such that either $\bar{s} = (x,R)$ or $\bar{s} \gg_{(\Gamma,R)} (x,R)$ and not $h(\bar{s})P_{K(R,x,y)}y$. Since Lemma 4 implies that it cannot be that $\bar{s} \gg_{(\Gamma,R)} (x,R)$, it follows that $\bar{s} = (x,R)$. Since $h(\bar{s}) = x$, it holds that not $xP_{K(R,x,y)}y$, which is a contradiction. Thus, S^x is the largest consistent set at (Γ,R). ■

Proof of Corollary 5. Consider the implementing rights structure designed in the proof of Theorem 8. Suppose that the true preference profile is R and that $f(R) = x$. Then, the vNM stable set and the largest consistent set coincides with S^x. According to Lemma 4, S^x can be equivalently defined as the set of states that are not directly dominated by any state in S, i.e., $S^x \equiv S - Dom_{(\Gamma,R,>)}(S)$.

Therefore, S^x is the core of (Γ, R). It follows that Γ implements in f in core.

Lemma 4 also implies that S^x can be equivalently defined as the set of states that are not indirectly dominated by any state in S, i.e., $S^x \equiv S - Dom_{(\Gamma,R,\gg)}(S)$, where $Dom_{(\Gamma,R,\gg)}(A) \equiv \{s \in S | \exists s' \in A : s' \gg_{(\Gamma,R)} s\}$ for all $A \subseteq S$. Therefore, S^x is the farsighted core of (Γ, R). It follows that Γ implements in f in farsighted core.

Next, let us show that Γ implements f in farsighted stable set. Recall that the farsighted stable set extends the vNM to the indirect dominance relation. Hence, the farsighted stable set at (Γ, R), denoted by $F(\Gamma, R)$, is defined as the set of states that are not indirectly dominated by any states in the farsighted stable set, i.e., $F(\Gamma, R) \equiv F(\Gamma, R) - Dom_{(\Gamma,R,\gg)}(F(\Gamma, R))$. It is well known that if the farsighted stable set exists, then it is weakly contained in the LCS (Chwe, 1994). Note that the farsighted stable set at (Γ, R) cannot be a strict subset of S^x; otherwise, any subset of S^x has x an outcome, the indirect external stability of the farsighted stable set is violated. Then, $F(\Gamma, R) = S^x$. Therefore, Γ, implements f in farsighted stable set.

By the fact that strong rational expectation farsighted stable set (Dutta and Vohra, 2017, Theorem 1, p.1203) and absolutely maximal farsighted stable set (Ray and Vohra, 2019, Theorem 1, p.1769) coincides with the farsighted stable set when the latter is a single-payoff, we conclude that Γ implements f in both strong rational expectation farsighted stable and absolutely maximal farsighted stable set. Finally, since farsighted stable set with heterogeneous expectations (Bloch and van den Nouweland, 2021, Proposition 3.9, p.38) and equilibrium stable set (Karos and Robles, 2021, Theorem 4.4, p.414) coincides with the strong rational expectation farsighted stable when it is single-payoff, then Γ implements f in both farsighted stable set with heterogeneous expectations and equilibrium stable set.

Proposition 9. *In bilateral trading, the fixed price rule f_p satisfies* iIIA

Proof of Proposition 9 Recall that by Proposition 7, any f_p satisfies IIA. To show that f_p satisfies iIIA we prove that iIIA reduces to IIA for fixed price rules. Note that $f_p(\mathcal{R} = \{0, p\}$. Suppose that for all $R, R' \in \mathcal{R}$ and $x, y, z \in \{0, p\}$, such that $z \notin \{x, y\}$ it holds that $f(R) = x$ and $K(R, x, z) \subseteq K(R', y, z)$. Note that it must be that $x \neq z$ since preference are strict for agents in $K(R, x, z)$. This together with the fact that e the range of f has cardinally equals to two implies that $x = y$. Therefore, for fixed price rules, the premises in the definition of iIIA equals those in the definition of IIA. The fact that both definitions conclude that $f(R') \neq z$ implies that iIIA reduces to IIA for fixed price rules.

■

Proposition 10. *Given a voting environment, the Condorcet rule f_C satisfies iIIA*

Proof of Proposition 10. Let us verify that the Condorcet rule f_C satisfies iIIA. Take any $R, R' \in \mathscr{R}$, and any $x, y, z \in Z$, such that $f(R) = x$ and $K(R, x, z) \subseteq K(R', y, z)$. Since x is a Condorcet winner at R, we have

$$|K(R', x, z)| \geqslant |K(R, y, z)| > \frac{n}{2}.$$

This implies that $|K(R', z, y)| < \frac{n}{2}$, and therefore, z cannot be a Condorcet winner at R'. Hence $f_C(R') \neq z$ as required by indirect iIIA. ■

The following is a corollary to Theorem 1 and it will be used in the proofs of Theorem 9 and Theorem 10.

Corollary 6. *If SCF f is double implementable in vNM and LCS by a rights structure, then it satisfies vNM EFFICIENCY*

Proof of Theorem 9 Suppose that rights structure $\Gamma = (S, \gamma, h)$ double implements f in vNM stable set and in LCS. Since Corollary 6, we only need to show that f satisfies iIIA. Suppose that for some $R, R' \in \mathcal{R}$, and $x, y, z \in f(\mathcal{R})$, such that $z \notin \{x, y\}$, we have $f(R) = x$ and $K(R, x, z) \subseteq K(R', y, z)$. We need to verify that $f(R') \neq z$ holds. Suppose towards a contradiction that $f(R') = z$. Let us derive a contradiction by showing that the set $S^z \cup S^y$ is a consistent set at preference profile R'' defined in the following way: If $yP'_i z$ (i.e., $i \in K(R', y, z)$), then R''_i is such that y is ranked first (uniquely) and z is ranked second (uniquely), and if $zR'_i y$ (i.e., $i \notin K(R'', y, z)$), then R''_i is such that z is ranked first (uniquely) and y is ranked second (uniquely).

There are three different type of deviations from the set $S^z \cup S^y$ that we need to consider:

(1) *Some coalition deviates from a state in S^z to a state in $S \backslash S^z$*: Since by counter assumption S^z is a vNM stable set at R', it is a vNM stable set also at R'' by construction. Therefore, some coalition has the ability, and incentives, to deviate back in S^z by external stability.

(2) *Some coalition deviates from a state in S^y to a state in $S \backslash (S^z \cup S^y)$*: Suppose this coalition deviates to $s \in S \backslash (S^z \cup S^y)$. Thus $h(s) \notin \{y, z\}$. Since $y \in f(\mathcal{R})$, there exists a preference profile $R^* \in \mathcal{R}$ such that $f(R^*) = y$. By implementability S^y is a vNM stable set at R^*, and hence by external stability, some coalition $K \subseteq K(R^*, y, h(s))$ has the ability to move from s to some state in S^y. By the definition of R'', and the fact that $h(s) \notin \{y, z\}$, it must be that also $K \subseteq K(R'', y, h(s))$ holds. Therefore, coalition K has the ability, as well as incentive, to move back in S^y.

(3) *Some coalition deviates from a state in S^y to a state in S^z*: In this last case, we use the assumption $f(R) = x$ and $K(R, x, z) \subseteq K(R', y, z)$. Notice first that by the definition of R'' this implies $K(R, x, z) \subseteq K(R'', y, z)$. Now suppose that $s \in S^z$ is the state where the coalition deviates. Since $f(R) = x$, we know that S^x is a vNM stable set at R. Hence, by external stability there exists a state $s' \in S^x$, and a coalition $K \in K(R, x, z)$, such that $K \in \gamma(s, s')$. Therefore, since $K(R, x, z) \subseteq K(R', y, z)$ holds, there exists an indirect domination path from state s to some state in S^y as long as some coalition has the ability, as well as incentive, to move from state s' to some state in S^y. This coalition must exist by the argument used in case (2).

This shows that $S^z \cup S^y$ is a consistent set at R'' – a contradiction with the fact that f is a function. ■

Proof of Theorem 10. Suppose that rights structure $\Gamma = (S, \gamma, h)$ double implements f in vNM stable set and LCS. Since Corollary 6, we only need to show that f satisfies iIIA. Suppose that for some $R, R' \in \mathcal{R}$, and $x, y, z \in f(\mathcal{R})$, such that $z \notin \{x, y\}$, we have $f(R) = x$ and $K(R, x, z) \subseteq K(R', y, z)$. We need to verify that $f(R') \neq z$ holds. To complete the proof, we only need to show that it is possible to construct the preference profile R'' in Theorem 4 so that it is included in the single-crossing domain. We divide the proof in two cases; either (1) $y > z$ or (2) $z > y$ ($>$ is the relation on Z given by SC). If (1) holds, then $K(R', y, z) = \{n, \ldots, k\}$ and $N \backslash K(R', y, z) = \{k, k-1, \ldots, 1\}$ for some k. Now construct the preference profile R'' in the following way: Agents $\{n, \ldots, k\}$ rank outcome y as the best (uniquely) and outcome z as the second best (uniquely), while agents $\{k, k-1, \ldots, 1\}$ rank outcome z as the best (uniquely) and outcome y as the second best (uniquely). All other outcomes $Z \backslash \{y, z\}$ are rank according to $>$ by everyone. It is easy to check that this preference profile belong to the single-crossing domain $\mathcal{R}$. Analogous argument holds in the case (2). ■

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