On the estimation of density and expected occupation density of Itô processes with irregular drift and diffusion coefficients

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Overview

General introduction

The analysis of **stochastic differential equations (SDEs)** and in particular the analysis of the **law** of their solutions has been a research topic of **great importance** .

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- Estimation

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- Estimation:
  - Global bounds & Explicit: Baños & Krühner [BK16a]
  - Local bounds & Sharpest & Implicit: Qian & Xu [ZX18]

Explicit local density bounds for Itô-processes with irregular drift

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Basic setting

Let X be a d-dimensional Itô-process such that

$$X(t) = x_0 + \int_0^t \beta(s)ds + \int_0^t \sigma(s)dW(s), \quad t \geq 0$$

where W is an n-dimensional standard Brownian motion and $x_0 \in \mathbb{R}^d$.

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Definition

We say that the X *has bounded drift while* X *is in some open set* $U \subseteq \mathbb{R}^d$ if there is a constant $C > 0$ such that

$$||\beta(t)||1_{\{X(t)\in U\}} \leq C$$

for any $t \geq 0$, P-a.s.

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We say that X has *locally bounded drift* if X has bounded drift on any bounded open set.
We say that X has *non-degenerate diffusion coefficient* if $\sigma(t)\sigma^\top(t)$ is positive definite for any $t \geq 0$.

Intuitive dialogue

Q: Roughly thinking (which means regardless of existence and other technical details for a moment), can we imagine the optimal (or ideal, best, extreme) situation that maximizes the ”density” $(\rho_t(x, y), y \in U)$?

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A: Sort of yes!

Doubly reflected Brownian motion

Definition (DRBM)

A stochastic process Z with continuous sample paths is a doubly reflected Brownian motion with drift $b \in \mathbb{R}$ on a compact interval J if for any $f \in C^2(\mathbb{R}, \mathbb{R})$ with $f'(x) = 0$ for any boundary point $x \in J$ we have

$$M^f(t) := f(Z(t)) - \int_0^t \left(\frac{1}{2} f''(Z(s)) + b f'(Z(s)) \right) ds, \quad t \geq 0.$$

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Let's start from the simple but intuitive 1-dim case!

- DRBM Z on $[0, l]$, $0 \leq l$
- with drift $-C$
- and with a simple $\sigma_t = 1$ for a moment

1-dim case with constant σ

Theorem (Comparison)

Let β be a progressively measurable $\mathbb{R}$-valued process with $||\beta(t)|| \leq C$ for any $t \geq 0$. Let $0 \leq z_0 \leq x_0$ and

$$X(t) := x_0 + \int_0^t \beta(s)ds + W(t) + R_X(t),$$
$$Z(t) := z_0 - Ct + W(t) + R_Z(t) - A(t),$$

where A is any continuous increasing progressively measurable process with $A(0) = 0$ and R_X resp. R_Z are the respective upward reflection terms at zero for X resp. Z, i.e.

$$R_X(t) := \sup\left\{\max\{0, -(x_0 + \int_0^u \beta(s)ds + W(u))\} : u \in [0, t]\right\},$$
$$R_Z(t) := \sup\left\{\max\{0, -(z_0 - Cu + W(u) - A(u))\} : u \in [0, t]\right\},$$

then $Z(t) \leq X(t)$ for any $t \geq 0$.

Remark

Proof.

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  - Higher dimensional

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- A good news!
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- Two possible direction to make it more general:
  - Higher dimensional
  - More flexible σ

Lemma (d-dim)

Let X be a d-dimensional Itô-process with diffusion coefficient constant equal to the identity matrix. Assume that the drift of X is bounded while X is in the set $B_{l,\infty}(x)$ where $l > 0$, $x \in \mathbb{R}^d$ and we denote the corresponding constant by $C \geq 0$. Let $Y_1, \ldots, Y_d$ be independent doubly reflected Brownian motions with drift $(-C)$ on $[0, l]$. Assume that $|Y_j(0)| \leq |X_j(0) - x_j|$ for any $j = 1, \ldots, d$.
Then

$$P(\|X(t) - x\| \leq a) \leq P(\|Y(t)\| \leq a), \quad a \in (0, l].$$

Veestraeten's result for DRBM

The following result is adopted from [Vee04, p. 193, formula (13)].

DRBM

Let p be the transition density of a doubly reflected Brownian motion. Then p is continuous in all its arguments and $p_{l,t}(x,0) \leq p_{l,t}(0,0)$. In particular, for all $x \in [0,l]$ and $t > 0$ we have

$$p_{l,t}(x,0) = \frac{2C}{1-\exp(-2Cl)} + e^{Cx-C^2t/2}\frac{2}{l}\sum_{n=1}^{\infty}\left[f_{t,x}(n\pi/l) - g_{t,x}(n\pi/l)\right],$$

where $f_{t,x}(z) := \frac{z^2\cos(zx)}{C^2+z^2}\exp(-tz^2/2)$ and $g_{t,x}(z) := \frac{Cz\sin(zx)}{C^2+z^2}\exp(-tz^2/2)$ for $z \in \mathbb{R}$.

Main result

Theorem: Existence

Let X be a d-dimensional Itô-process with constant, deterministic and non-degenerate diffusion coefficient. Assume that X has bounded drift while X is in some open set $U \subseteq \mathbb{R}^d$. Let $t > 0$.
Then

$$\rho_t(x) := \limsup_{\epsilon \to 0} \frac{P(||X(t) - x|| < \epsilon)}{\mathrm{vol}(B_{\epsilon,\infty}(0))}, \quad x \in U$$

is locally bounded.
Moreover, ρ_t is a version of the density of $X(t)$ on U, i.e. $P(X(t) \in A) = \int_A \rho_t(x)dx$ for any Borel-set $A \subseteq U$.
In particular, if X has locally bounded drift, then $X(t)$ has a locally bounded version of its density.

Main result

Corollary: Upper bound

Let X be an Itô-process where the diffusion coefficient is constant equal to the identity matrix on $\mathbb{R}^d$ and assume that the drift of X is bounded by C while X is in some open set $U \subseteq \mathbb{R}^d$. We define

$$\rho_t(x) := \limsup_{\epsilon \searrow 0} \frac{P(||X(t) - x|| < \epsilon)}{\mathrm{vol}(B_{\epsilon,\infty}(0))} \in [0, \infty]$$

for $t > 0$, $x \in \mathbb{R}^d$. Let $x \in U$ and $l > 0$ such that $B_{l,\infty}(x) \subseteq U$.
Then we have

$$\rho_t(x) \leq \prod_{j=1}^{d} \left(\frac{C \exp(-2Cl)}{1 - \exp(-2Cl)} + \frac{\phi(z_j)}{\sqrt{t}} + C\Phi(z_j) + e^{Ca_j - C^2 t/2} \frac{(3 + a_j C)^2}{ltC^2} \right)$$

where $a_j := \min\{l, |X_j(0) - x_j|\}$ and $z_j := \sqrt{t}C - a_j/\sqrt{t}$ for $j = 1, \dots, d$.

Limiting case

Pushing l to ∞, we just reproduce Baños & Krühner [BK16a]'s result:

Baños & Krühner's result

Let X be an Itô-process where the diffusion coefficient is constant equal to the identity matrix on $\mathbb{R}^d$ and assume that the drift of X is bounded by C. We define $\rho_t(x) := \limsup_{\epsilon\to 0} \frac{P(|X(t)-x|<\epsilon)}{\mathrm{vol}(B_\epsilon(0))} \in [0,\infty]$ for $t > 0$, $x \in \mathbb{R}^d$.

$$\rho_t(x) \le \prod_{j=1}^{d} \left(\frac{\phi(z_j)}{\sqrt{t}} + C\Phi(z_j) \right) \le \left(\frac{1}{\sqrt{2\pi t}} + C \right)^d$$

where $(z_j)_{j\le d}$ is given by $z_j := \sqrt{t}C - |X_j(0) - x_j|/\sqrt{t}$. In particular, if $d = 1$, then we have

$$\rho_t(x) \le \frac{\phi(z_1)}{\sqrt{t}} + C\Phi(z_1).$$

Conclusion of more general cases

We generalize our result as follows:

if	σ constant	σ Locally Lipschitz
$d = 1$	✓	✓
$d \geq 1$	✓	

A sharp upper bound for the expected occupation density of Itô processes with bounded irregular drift and diffusion coefficients

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Occupation density

Occupation densities have been surveyed by D. Geman and J. Horowitz [GH80]. Let $X(t) : \mathbb{R}^+ \mapsto \mathbb{R}$ be a measurable function. The occupation measure of X up to time $T \geq 0$ is

$$\mu_T(\Gamma) = m\{0 \leq s \leq T : X_s \in \Gamma\},$$

m being Lebesgue measure and Γ a Borel set on $\mathbb{R}$. It is the amount of time spent by X in the set Γ during $[0, T]$. And we say that X has an occupation density on $[0, T]$ if $\mu_T(\Gamma)$ is absolutely continuous with respect to the Lebesgue measure. In other words, $\mu_T(\Gamma)$ could be expressed as the sum of times spent by X at each $y \in \Gamma$ during $[0, T]$ in the following sense

$$\mu_T(\Gamma) = \int_{y \in \Gamma} \alpha_T(y) dy$$

where $\alpha_T(y) : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$. We then call $\alpha_T(y)$ an occupation density.

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We define $\mathcal{A}$ to be the set of progressively measurable process (β, σ) such that for any $t \geq 0$, $\sigma_t \in [a, b]$, $0 < a \leq b$ and $|\beta_t| \leq k\sigma_t^2$, for some $k \geq 0$. Note that when $k = 0$, Itô processes reduce to Brownian martingales with bounded diffusion coefficient away from 0.

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Let $\mathcal{C}$ be the class of stochastic process X such that there is $(\beta, \sigma) \in \mathcal{A}$ with $dX_t = \beta_t dt + \sigma_t dW_t$, where W_t is standard Brownian motion and the starting point X_0 is deterministic.

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Let $\mathcal{C}$ be the class of stochastic process X such that there is $(\beta, \sigma) \in \mathcal{A}$ with $dX_t = \beta_t dt + \sigma_t dW_t$, where W_t is standard Brownian motion and the starting point X_0 is deterministic.
We are interested in the quantity of the supremum expected occupation density at a given time $T \geq 0$ over all possible Itô processes $X \in \mathcal{C}$:

$$G(x, y, T) := \sup_{X \in \mathcal{C}, X_0 = x} \mathbb{E}[\alpha_T(y)]$$

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However, this is a priori meaningless since the existence and uniqueness of the occupation density is ambiguous.
Instead of using the implicit definition above, we use the following "approximation" version of definition:

$$G(x, y, T) := \sup_{X \in \mathcal{C}, X_0 = x} \left(\limsup_{N \to \infty} 2N\mathbb{E}\left[\int_0^T 1_{\{|X_s - y| \leq \frac{1}{N}\}} ds \right] \right). \quad (1)$$

Our result

Theorem [Main Result]

Let $(\beta, \sigma) \in \mathcal{A}$ and $X_t = x + \int_0^t \beta_s ds + \int_0^t \sigma_s dW_s$, $t \geq 0$, where W is a standard 1-dimensional Brownian motion, G is defined as above in (1). Then

$$G(x, y, T) = \int_0^T \left(\frac{b}{a^2\sqrt{t}} \phi(v(r,t)) + \frac{b^2 k}{a^2} \Phi(v(r,t)) \right) dt, x, y \in \mathbb{R}, T > 0$$

where $r := |x - y|$ and $v(r,t) := kb\sqrt{t} - \frac{r}{b\sqrt{t}}$, $t \geq 0$, $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$ for any $x \in \mathbb{R}$.

Our approach in a nutshell

We prove the main theorem above by studying the exponential stopped time case. Artificially given some special control, we compute an explicit and sharp solution of the associated Heath-Jarrow-Bellman (HJB) equation.

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We prove the main theorem above by studying the exponential stopped time case. Artificially given some special control, we compute an explicit and sharp solution of the associated Heath-Jarrow-Bellman (HJB) equation.
Although the control will not lead to the upper bound we got, we prove that the upper bound is optimal via a verification method. However, we found that the "optimal control" is outside our feasible set. In other words, we can always find a better control in the class $\mathcal{C}$ to get a better occupation density and limiting case is not Markovian anymore.

The "optimal" control

$$(\beta_s, \sigma_s) := \begin{cases} (-kb^2, b) & \text{if} \quad X_s > y \\ (0, a) & \text{if} \quad X_s = y \\ (kb^2, b) & \text{if} \quad X_s < y \end{cases} \tag{2}$$

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We define the exponentially stopped value function by

$$V_\lambda(x, y) := \sup_{(\beta,\sigma)\in\mathcal{A}, X_0=x} \limsup_{N\to\infty} 2N\mathbb{E}\left[\int_0^\infty \lambda e^{-\lambda t} \cdot 1_{\left\{\left|X_t^{\beta,\sigma}-y\right|\leq\frac{1}{N}\right\}} dt\right], \lambda > 0 \quad (3)$$

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We believe the HJB-equation of the optimal control problem (3) is:

$$-\lambda V_\lambda(x,y) + \sup_{(\beta,\sigma)\in\mathcal{A}}\left\{V_\lambda'(x,y)\beta + \frac{1}{2}V_\lambda''(x,y)\sigma^2\right\} = 0, \quad y \neq x \tag{4}$$

$$\sup_{(\beta,\sigma)\in\mathcal{A}}\left\{\delta_y(x) + \frac{1}{2}V_\lambda''(x,y)\sigma^2\right\} = 0, \quad y = x \tag{5}$$

Exponentially stopped case

Lemma

The solution for Equation (4) & (5) provided control as (2) is as follows:

$$Q_\lambda(x,y) = \frac{1}{\left(-k+\sqrt{k^2+\frac{2\lambda}{b^2}}\right)a^2} e^{\left(k-\sqrt{k^2+\frac{2\lambda}{b^2}}\right)|x-y|}, \quad \lambda > 0, x, y \in \mathbb{R}$$

Theorem

For the optimal control problem (3), $Q_\lambda(x,y) = V_\lambda(x,y)$ for any $x, y \in \mathbb{R}$.

Proof.

Proof by verification. □

Inverse Laplace transform

$$V_\lambda(r) = \int_0^\infty \lambda e^{-\lambda T} H_T(r) dT$$

$$\lambda \mathcal{L}[H_.(r)](\lambda) = V_\lambda(r) = \frac{1}{\left(-k + \sqrt{k^2 + \frac{2\lambda}{b^2}}\right) a^2} e^{\left(k - \sqrt{k^2 + \frac{2\lambda}{b^2}}\right)|x-y|}$$

Theorem

The inverse Laplace transform of V_λ is

$$H_T(r) = \int_0^T \left(\frac{b}{a^2\sqrt{t}} \phi(v(r,t)) + \frac{b^2 k}{a^2} \Phi(v(r,t)) \right) dt, \quad T > 0, t \geq 0$$

where $v(r,t) := kb\sqrt{t} - \frac{r}{b\sqrt{t}}$, $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ and $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$.

Convexity of H on r

Lemma

For any $r > 0$ and $T \geq 0$, we have

$$\frac{\partial H_T(r)}{\partial r} = \int_0^T \frac{1}{a^2\sqrt{2\pi t^3}} e^{-\left(\frac{r}{\sqrt{2b^2 t}} - k\sqrt{\frac{b^2 t}{2}}\right)^2} \left(-\frac{r}{b}\right) dt \quad (6)$$

$$\frac{\partial^2 H_T(r)}{\partial r^2} = \int_0^T \frac{1}{a^2 b\sqrt{2\pi t^3}} e^{-\left(\frac{r}{\sqrt{2b^2 t}} - k\sqrt{\frac{b^2 t}{2}}\right)^2} \left(\frac{r^2}{b^2 t} - kr - 1\right) dt \quad (7)$$

$$\frac{\partial H_T(r)}{\partial r} \leq 0$$

$$0 \leq \frac{\partial^2 H_T(r)}{\partial r^2}$$

and we also have for any $r > 0$ and $T > 0$

$$\lim_{r \to 0} \frac{\partial H_T(r)}{\partial r} = -\frac{1}{a^2}. \quad (8)$$

HJB of the original optimal control problem

We believe the HJB-equation of the original optimal control problem (1):

$$-\frac{\partial G(x,t)}{\partial t} + \sup_{(\beta,\sigma)\in\mathcal{A}} \left\{ \frac{\partial G(x,t)}{\partial x}\beta + \frac{1}{2}\frac{\partial^2 G(x,t)}{\partial x^2}\sigma^2 \right\} = 0, \quad x \neq y \tag{9}$$

$$\sup_{(\beta,\sigma)\in\mathcal{A}} \left\{ \delta_y(x) + \frac{1}{2}\frac{\partial^2 G(x,t)}{\partial x^2}\sigma^2 \right\} = 0, \quad y = x \tag{10}$$

$$G(x,0) = 0, \quad \forall x \in \mathbb{R} \tag{11}$$

Lemma

H solves the Hamilton-Jacobi-Bellman equation (9), (10) and (11) provided control as (2).

Proof by verification

Theorem: Main

For the optimal control problem (1), $H_T(x-y) = G(x,y,T)$ for any $T \geq 0$ and $x, y \in \mathbb{R}$.

Hint of proof.

Fix $M \in \mathbb{N}$ and we construct a control via choosing

$$\sigma_M(x) := a + (b-a)g_M(|x|)$$

where $g_M \in C(\mathbb{R}_+, [0,1])$ such that

$$g_M(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{1}{M}], \\ 1 & \text{if} \quad x \in [\frac{2}{M}, \infty], \end{cases}$$

The control is now specified as:

$$\beta_t := -k\sigma_M^2(X_t)\mathrm{sign}(X_t)dt, \quad \sigma_t := \sigma_M(X_t), \quad t \geq 0.$$

Literature review

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Similar techniche has been used in D. Banos and P. Kruhner [BK16b], see Lemma 3.4 and Propsition 3.5.

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Thanks for your attention!
Any questions?