I Introduction

$G_{n,p}$

vertices: $1, 2, \ldots, n^2$

edges: keep each of the $n(n-1)$ possible directed edges independently with probability $p$. 
We're interested in strongly connected components (SCC)

x and y are in the same SCC if there are two paths: x \rightarrow y
y \rightarrow x

b) Phase Transition for the SCC

Same as the Erdős–Rényi graph:

p = \frac{c}{n} 
\cdot c > 1 \rightarrow unique giant component
\cdot c < 1 \rightarrow only small components

In fact, something more precise is known.


Write \( p = \frac{1}{n} + \frac{d_n}{n^{\frac{3}{2}}} \) and assume \( d_n = o(n^{\frac{3}{2}}) \)

\( d_n \rightarrow +\infty \) then: largest SCC has size \( \sim 4n^{\frac{3}{2}} d_n^2 \)

\( 2^{nd} largest \ has \ size \ O\left( n^{\frac{3}{2}} \sqrt{d_n} \right) \)

\( d_n \rightarrow -\infty \) then: largest SCC has size \( O\left( n^{\frac{3}{2}} \frac{1}{\sqrt{d_n}} \right) \)
Q: What happens if $d_n$ stays bounded or converges?

**Case of the Erdős–Rényi graph**

$$G(n, p) \quad p = \frac{1}{n} + \frac{d}{n^{4/3}} + o\left(\frac{1}{n^{4/3}}\right) \quad d \in \mathbb{R}$$

**Theorems:**

- Aldous (97): let $Z_n^h \geq Z_n^h \geq \cdots$

  be the sizes of the connected components of $G(n, p)$. Then

  $$\frac{Z_n^h}{n^{2/3}} \xrightarrow{d} \zeta_h$$

  (in $l^2$, as sequences)

- Addario-Berry, Broutin, Goldschmidt (72)

  Let $(C_i^h)$ be the cc of $G(n, p)$ with
\[ \# C_i^n = Z_i^n. \text{ Then} \]
\[ \frac{C_i^n}{n^{2/3}} \xrightarrow[n \to \infty]{} c_i \]

(l in Gromov–Hausdorff, as a sequence)

The main takeaway is that the distance between two typical points is of order \( n^{2/3} \).

(Side note: both the \( C_i^n \) and \( C_i \) have descriptions as “binary trees with a few additional edges”.)
Our results

\[ p = \frac{2}{n^2} + \frac{a}{n^{4/3}} + \Theta(\frac{1}{n^{3/2}}) \]

1) Informally

Within the SCC:

\[ \text{w.h.p. no vertices with degree } \geq 8 \]

\[ \text{number of vertices with degree} = 3 \text{ is } \leq 1 \]

\[ \text{number of vertices with degree} = 2 \text{ is } \leq n^{\frac{2}{3}} \]

\[ \sim a \cdot n^{\frac{1}{3}} \]

\[ \sim b \cdot n^{\frac{2}{3}} \]

b) Metric directed multigraphs (MDM)

Definition: \( \overline{G} \) is the set of finite multigraphs where each edge
has a direction and a length

- An isomorphism is a pair of bijections
  \( \phi : V(X) \to V(Y) \)
  \( \psi : E(X) \to E(Y) \)

which preserve the structure

examples: How many isomorphisms?

\[
\begin{align*}
\begin{tikzpicture}
\node (a) at (0,0) [circle,draw] {0};
\node (b) at (0,-0.5) [circle,draw] {0};
\draw (a) -- (b);
\end{tikzpicture}
\end{align*}
\quad \rightarrow \quad \begin{align*}
\begin{tikzpicture}
\node (a) at (0,0) [circle,draw] {0};
\node (b) at (0,-0.5) [circle,draw] {0};
\draw (a) -- (b);
\end{tikzpicture}
\end{align*}
\quad 0
\]

\[
\begin{align*}
\begin{tikzpicture}
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\end{align*}
\quad ?
\]

\[
\begin{align*}
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\end{tikzpicture}
\end{align*}
\quad 2
\]

\[
\text{distance: } d_{\mathcal{G}}(x, y) = \inf_{\phi \in \text{Iso}(X \to Y)} \sum_{e \in E(X)} |\ell_{\phi}(e) - \ell_{\psi}(\phi(e))|
\]
Remark: This metric is very rigid:

\[ d_{\hat{\mathcal{O}}} \left( \bigcup_{n=0}^{\infty} C_n, \bigcup_{n=0}^{\infty} C_n \right) = \infty \]

c) Main Theorem:

Let \( (C_i, n \in \mathbb{N}) \) be the SCC of \( \overline{G}(n, p) \), viewed as MDMs by removing the
\( d^0 = 2 \) vertices. Then:

\[
\left( \frac{C_i}{n^{2/3}}, i \in \mathbb{N} \right) \xrightarrow{d} (C_i, i \in \mathbb{N})
\]

where \( C \in \overline{G}^N \) s.t.
- Finitely many terms are 3-regular
- the rest are cycles

Topological tower sequences: $d(A_i, B_i) = \sum_{i=1}^{n} d_{\mathcal{G}}(A_i, B_i)$

IV Exploration and structure
A version of depth-first search
b) Edge classification

- tree edge (forward)
- surplus edge (forward)
- back edge (backward)

Interaction between forward and backward

\[ \Rightarrow \text{SCC} \]

\[ \square \] Limit behaviour of forest and surplus

\[ \text{prop: "Forward edges without arrows"} \]

\[ \binom{n}{2} \leq G(n,p) \leq \frac{Erdős-Rényi}{n} \]

So we know a lot about the trees and surplus edges. In particular:

- number of vertices in *tree* \( \sim n^{2/3} \)
- typical distances are \( \sim n^{1/3} \)
surplus edges are $O(n)$ in number.

It turns out, surplus edges don't count!

\textbf{prop:} \( \text{PC(scc of } G_{1,n,p} \text{ has a surplus edge)} \xrightarrow{n \to \infty} 0 \)

\[ \overline{PC(\cdot \text{ counts})} \]

\[ \text{VI Limit of the back edges} \]

\[ \hat{\varphi} \text{ problem?} \]

In a single tree, there are $\frac{k(k-1)}{2}$ possible back edges, appearing independently with probability $p \approx \frac{1}{n}$.

But $k \approx n^{3/4} \ldots$
$E[\# \text{ of back edges}] \approx \frac{1}{n} n^{\frac{4}{3}} \to \infty$ ??

b) Solution: Most of the BE don't matter!

How to find those which do matter:

- go around the tree
- keep the first ancestral BE
- keep any BE which is either ancestral or links into something we've already selected.

This turns out to converge to a PPP on the limiting continuum tree!
What we end up with:
Other works:

- Coulson (79): estimates for size of largest component

- De Panafieu, Dourgal, Ralairosysony, Rasendrahasina, Wagner (120): asymptotics for
  - probability of acyclic
    \[ \sim \frac{e^{-\lambda}}{\lambda} \]
  - only singleton and cyclic SCCs.

\( c \) and \( c' \) explicit