Attraction to and repulsion from patches on the hypersphere and hyperplane for isotropic d-dimensional $\alpha$-stable processes with index in $\alpha \in (0, 1]$ and $d \geq 2$.

Andreas Kyprianou Joint work with:
Tsogzolmaa Saizmaa (National University of Mongolia)
Sandra Palau (National Autonomous University of Mexico)
Matthias Kwasniki (Wroclaw Technical University)
(ξ_t, t ≥ 0) is a Lévy process if it has stationary and independents with RCLL paths.

Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula

\[ E[e^{i\theta \cdot \xi_t}] = e^{-\Psi(\theta)t}, \quad \theta \in \mathbb{R}^d, \]

where,

\[ \Psi(\theta) = ia \cdot \theta + \frac{1}{2} \theta \cdot A \theta + \int_{\mathbb{R}^d} (1 - e^{i\theta \cdot x} + i(\theta \cdot x)1_{(|x|<1)})\Pi(dx), \]

where \( a \in \mathbb{R}, A \) is a \( d \times d \) Gaussian covariance matrix and \( \Pi \) is a measure satisfying \( \int_{\mathbb{R}^d} (1 \wedge |x|^2)\Pi(dx) < \infty \). Think of \( \Pi \) as the intensity of jumps in the sense of

\[ P(X \text{ has jump at time } t \text{ of size } dx) = \Pi(dx)dt + o(dt). \]

Stationary and independent increments gives the Strong Markov Property and the probabilities \( P_x(\cdot) = P(\cdot|X_0 = x) \) such that \( (X, P_x) \) is equal in law to \( (x + X, P) \).
Suppose that \((\xi_t, t \geq 0)\) is a one dimensional Lévy process without monotone paths.

Excluding the cases that \(\xi\) has monotone paths and assuming that \(\xi\) oscillates so that \(\xi\) fluctuates upwards and downwards and visits \((-\infty, 0)\) with probability 1:

\[
\mathbb{P}^\uparrow_x(A) = \lim_{s \to \infty} \mathbb{P}_x(A \mid \xi_{t+s} \geq 0)
\]

\[
= \lim_{s \to \infty} \mathbb{E}_x \left[ 1_{(A, \xi_t \geq 0)} \frac{\mathbb{P}_{\xi_t}(\xi_s \geq 0)}{\mathbb{P}_x(\xi_{t+s} \geq 0)} \right]
\]

\[
= \mathbb{E}_x \left[ 1_{(A, \xi_t \geq 0)} \frac{h^\uparrow(\xi_t)}{h^\uparrow(x)} \right] A \in \sigma(\xi_u : u \leq t)
\]

Boils down to understanding: \(\mathbb{P}_y(\xi_t \geq 0) \sim h^\uparrow(y)f(t)\) as \(s \to \infty\)

As it happens, \(h^\uparrow(x)\) is the descending ladder potential and has the harmonic property that

\[
h^\uparrow(\xi_t)1_{(\xi_t \geq 0)}
\]

is a martingale.

Under additional assumptions, can demonstrate \(\exists \lim_{x \downarrow 0} \mathbb{P}_x^\uparrow =: \mathbb{P}_0^\uparrow\) on the Skorokhod space.

---

1Bertoin 1993, Chaumont 1996, Chaumont-Doney 2005
Lévy processes conditioned to stay non-negative\(^2\)

\[ \begin{align*}
\text{Chaumont 1996}
\end{align*} \]
LÉVY PROCESSES CONDITIONED TO APPROACH THE ORIGIN CONTINUOUSLY FROM ABOVE\textsuperscript{3}

- A different type of conditioning, needs the introduction of a death time $\zeta$ at which paths go to a cemetery state

\[
\mathcal{P}_x^\uparrow(A, t < \zeta) = \lim_{\beta \to 0} \lim_{\varepsilon \to 0} \mathcal{P}_x(A, \xi_t > \beta \mid \xi_\infty \in [0, \varepsilon])
\]

\[
= \lim_{\beta \to 0} \lim_{\varepsilon \to 0} \mathbb{E}_x \left[ 1_{(A, \xi_t \geq \beta)} \frac{\mathcal{P}_{\xi_t}(\xi_\infty \in [0, \varepsilon])}{\mathcal{P}_x(\xi_\infty \in [0, \varepsilon])} \right]
\]

\[
= \mathbb{E}_x \left[ 1_{(A, \xi_t \geq 0)} \frac{h^\uparrow(\xi_t)}{h^\downarrow(x)} \right] \quad A \in \sigma(\xi_u : u \leq t),
\]

- It turns out that

\[
h^\downarrow(x) = \frac{d}{dx} h^\uparrow(x), \quad x \geq 0.
\]

and is superharmonic, i.e. $h^\downarrow(\xi_t) 1_{(\xi_t \geq 0)}$ is a supermartingale.

\textsuperscript{3}Chaumont 1996
Williams type decomposition\textsuperscript{4} for \((\xi, \mathbb{P}_x)\)

\[ Q(\infty) = \sup \left\{ t \geq 0 : \frac{\tilde{3}_t}{\tilde{3}_\infty} = \frac{1_{n t}}{\xi^*} \right\} \]

\[ X_{Q(\infty)} \sim \frac{h^\uparrow(x - dy)}{h^\uparrow(x)} \]

\textsuperscript{4}Chaumont 1996
For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a $d$-dimensional isotropic stable process.

- $X$ has stationary and independent increments (it is a Lévy process).
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(e^{i\theta \cdot X_1})$ satisfies
  \[
  \Psi(\theta) = |\theta|^\alpha, \quad \theta \in \mathbb{R}.
  \]
- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,
  \[
  \Pi(B) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \, dy.
  \]
- $X$ is Markovian with probabilities denoted by $\mathbb{P}_x, x \in \mathbb{R}^d$.
Sample path, $\alpha = 1.2$
Sample path, $\alpha = 0.9$
CONDITIONING TO HIT A PATCH ON A UNIT SPHERE FROM OUTSIDE
Recall $d \geq 2$, the process $(X, \mathbb{P})$ is transient in the sense that $\lim_{t \to \infty} |X_t| = \infty$ almost surely.

Define

$$G(t) := \sup\{s \leq t : |X_s| = \inf_{u \leq s} |X_u|\}, \quad t \geq 0,$$

Transience of $(X, \mathbb{P})$ means $G(\infty) := \lim_{t \to \infty} G(t)$ describes the point of closest reach to the origin in the range of $X$.

$A_\varepsilon = \{r\theta : r \in (1, 1+\varepsilon), \theta \in S\}$ and $B_\varepsilon = \{r\theta : r \in (1-\varepsilon, 1), \theta \in S\}$, for $0 < \varepsilon < 1$
We are interested in the asymptotic conditioning

\[ \mathbb{P}_x^S(A, t < \zeta) = \lim_{\varepsilon \to 0} \mathbb{P}_x(A, t < \tau_1^\oplus | C_\varepsilon^S), \quad A \in \sigma(\xi_u: u \leq t), \]

where \( \tau_1^\oplus = \inf\{t > 0: |X_t| < 1\} \) and \( C_\varepsilon^S := \{X_G(\infty) \in A_\varepsilon\} \).

Works equally well if we replace \( C_\varepsilon^S := \{X_G(\infty) \in A_\varepsilon\} \) by \( C_\varepsilon^S := \{X_{\tau_1^\oplus} \in B_\varepsilon\} \), or indeed \( C_\varepsilon^S = \{X_{\tau_1^\oplus -} \in A_\varepsilon\} \).
Recent work: For $|x| > |z| > 0$,

$$
\mathbb{P}_x(X_{G(\infty)} \in d\,z) = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2) \Gamma(\alpha/2)} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^{\alpha}} |x-z|^{-d} \, d\,z,
$$

---

$^5$K. Rivero, Satitkanitkul 2020
CONDITIONING TO CONTINUOUSLY HIT $S \subseteq \mathbb{S}^{d-1}$ FROM OUTSIDE

- Remember $C^S_{\varepsilon} := \{X_{G(\infty)} \in A_\varepsilon \}$, switch to generalised polar coordinates and estimate

$$
\lim_{\varepsilon \to 0} \varepsilon^{\alpha-d} \mathbb{P}_x(C^S_{\varepsilon}) = c_{\alpha,d} \int_S (|x|^2 - 1)^{\alpha/2}|x - \theta|^{-d} \sigma_1(d\theta),
$$

where $c_{\alpha,d}$ does not depend on $x$ or $S$ and $\sigma_1$ is the unit surface measure on $\mathbb{S}^{d-1}$.

- Use

$$
\mathbb{P}_x(A, t < \tau^+_\beta | C^S_{\varepsilon}) = \mathbb{E}_x \left[ \mathbf{1}_{\{A, t < \tau^+_\beta\}} \frac{\mathbb{P}_{X_t}(C^S_{\varepsilon})}{\mathbb{P}_x(C^S_{\varepsilon})} \right], \quad A \in \sigma(\xi_u : u \leq t),
$$

pass the limit through the expectation on the RHS (carefully with DCT!) to get

$$
\left. \frac{d\mathbb{P}_x^S}{d\mathbb{P}_x} \right|_{\mathcal{F}_t} = \mathbf{1}_{(t < \tau^+_1)} \frac{M_S(X_t)}{M_S(x)}, \quad \text{if } x \in \mathcal{B}^c_d
$$

with

$$
M_S(x) = \left\{ \begin{array}{ll}
\int_S |\theta - x|^{-d}||x|^2 - 1|^{\alpha/2} \sigma_1(d\theta) & \text{if } \sigma_1(S) > 0 \\
|\theta - x|^{-d}||x|^2 - 1|^{\alpha/2} & \text{if } S = \{\vartheta\},
\end{array} \right.
$$

which is a superharmonic function.
Suppose $\zeta$ is the lifetime of $(X, \mathbb{P}^S)$. Let $S'$ be an open subset of $S$. Then for any $x \in \mathbb{R}^d \setminus \overline{B}_d$, we have

$$\mathbb{P}^S_x(X_{\zeta_-} \in S') = \frac{\int_{S'} |\theta - x|^{-d}\sigma_1(d\theta)}{\int_S |\theta - x|^{-d}\sigma_1(d\theta)},$$

Hence, for $\theta \in S$,

$$\mathbb{P}^S_x(A|X_{\zeta_-} = \theta) = \mathbb{E}^S_x \left[ 1_{\{X_{\zeta_-} = \theta\}} \frac{\mathbb{P}^S_{X_t}(X_{\zeta_-} = \theta)}{\mathbb{P}^S_x(X_{\zeta_-} = \theta)} \right]$$

$$= \mathbb{E}_x \left[ 1_{(A, t < \tau_1^\oplus)} \frac{M_S(X_t)}{M_S(x)} \frac{M\{\theta\}(X_t)}{M\{\theta\}(x)} \frac{M_S(x)}{M\{\theta\}(x)} \right]$$

$$= \mathbb{E}_x \left[ 1_{(A, t < \tau_1^\oplus)} \frac{M\{\theta\}(X_t)}{M\{\theta\}(x)} \right]$$

$$= \mathbb{P}^S_x\{\theta\}(A), \quad A \in \sigma(\xi_u : u \leq t)$$

So

$$\mathbb{P}^S_x(A) = \int_{S'} \mathbb{P}^S_x\{\theta\}(A) \frac{|\theta - x|^{-d}\sigma_1(d\theta)}{\int_S |\vartheta - x|^{-d}\sigma_1(d\vartheta)}.$$
Now define

\[ P^S_x(A, t < \zeta) = \lim_{\varepsilon \to 0} P_x(A \mid \tau_{S_\varepsilon} < \infty), \]

where

\[ \tau_{S_\varepsilon} = \inf\{t > 0 : X_t \in S_\varepsilon\} \quad \text{and} \quad S_\varepsilon := A_\varepsilon \cup B_\varepsilon. \]

Note: need to insist on \( \alpha \in (0, 1] \) because \( P_x(\tau_S < \infty) = 1 \) if \( \alpha \in (1, 2). \)
Theorem
Suppose that $\alpha \in (0, 1]$ and the closed set $S \subseteq S^{d-1}$ is such that $\sigma_1(S) > 0$. For $\alpha \in (0, 1]$, the process $(X, \mathbb{P}^S)$ is well defined such that

$$\frac{d \mathbb{P}^S_x}{d \mathbb{P}_x}|_{\mathcal{F}_t} = \frac{H_S(X_t)}{H_S(x)}, \quad t \geq 0, x \not\in S,$$

(1)

where

$$H_S(x) = \int_S |x - \theta|^\alpha \sigma_1(d \theta), \quad x \not\in S.$$

Note, if $S = \{\theta\}$ then it was previously understood$^6$ that

$$H_S(x) = |x - \theta|^\alpha, \quad x \not\in S.$$

So it is still the case for a genera $S$ that

$$\mathbb{P}^S_x(A) = \int_S \mathbb{P}^{\{\theta\}}_x(A) \frac{|x - \theta|^\alpha \sigma_1(d \theta)}{\int_S |x - \vartheta|^\alpha \sigma_1(d \vartheta)}.$$

"pick a target uniformly in $S$ with the terminal strike distribution and condition to hit it."

---

$^6$K. Rivero, Statitkanitkul 2019
Conditioning to continuously hit $S \subseteq S^{d-1}$ from either side

**Theorem**
Let $S \subseteq S^{d-1}$ be a closed subset such that $\sigma_1(S) > 0$.

(i) Suppose $\alpha \in (0, 1)$. For $x \notin S$,

$$
\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \Pr_x(\tau_{S_\varepsilon} < \infty) = 2^{1-2\alpha} \frac{\Gamma((d + \alpha - 2)/2)}{\pi^{d/2} \Gamma(1 - \alpha)} \frac{\Gamma((2 - \alpha)/2)}{\Gamma(2 - \alpha)} \mathcal{H}_S(x).
$$

(ii) When $\alpha = 1$, we have that, for $x \notin S$,

$$
\lim_{\varepsilon \to 0} |\log \varepsilon| \Pr_x(\tau_{S_\varepsilon} < \infty) = \frac{\Gamma((d - 1)/2)}{\pi^{(d-1)/2}} \mathcal{H}_S(x).
$$
HEURISTIC FOR PROOF OF THEOREM 2

- The potential of the isotropic stable process satisfies $\mathbb{E} \left[ \int_0^\infty \mathbb{1}_{(X_t \in d\,y)} \, d\,t \right] = |y|^{\alpha - d}$.

- Let $\mu_\varepsilon$ be a finite measure supported on $S_\varepsilon$, which is absolutely continuous with respect to Lebesgue measure $\ell_d$ with density $m_\varepsilon$ and define its potential by

  $$U_{\mu_\varepsilon}(x) := \int_A |x - y|^{\alpha - d} \mu_\varepsilon(d\,y) = \int_{S_\varepsilon} |x - y|^{\alpha - d} m_\varepsilon(y) \ell_d(d\,y) \quad x \in \mathbb{R}^d,$$

- As $m_\varepsilon(y) = 0$ for all $y \notin A$. As such, the Strong Markov Property tells us that

  $$U_{\mu_\varepsilon}(x) = \mathbb{E}_x \left[ \mathbb{1}_{\{\tau_{S_\varepsilon} < \infty\}} \int_{\tau_{S_\varepsilon}}^\infty m_\varepsilon(X_t) \, d\,t \right] = \mathbb{E}_x \left[ U_{\mu_\varepsilon}(X_{\tau_{S_\varepsilon}}) \mathbb{1}_{\{\tau_{S_\varepsilon} < \infty\}} \right], \quad x \notin S_\varepsilon.$$  

  (2)

  Note, the above equality is also true when $x \in S_\varepsilon$ as, in that case, $\tau_{S_\varepsilon} = 0$.

- Let us now suppose that $\mu_\varepsilon$ can be chosen in such a way that, for all $x \in A$, $U_{\mu}(x) = 1$. Then

  $$\mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = U_{\mu_\varepsilon}(x), \quad x \notin S_\varepsilon.$$

- Strategy: ‘guess’ the measure, $\mu_\varepsilon$, by verifying

  $$U_{\mu_\varepsilon}(x) = 1 + o(1), \quad x \in S_\varepsilon \text{ as } \varepsilon \to 0,$$

  so that

  $$(1 + o(1))\mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = U_{\mu_\varepsilon}(x), \quad x \notin S_\varepsilon,$$

- Draw out the the leading order decay in $\varepsilon$ from $U_{\mu_\varepsilon}(x)$. 
HEURISTIC FOR PROOF OF THEOREM 2: FLAT EARTH THEORY

➤ Believing in a flat Earth is helpful
➤ In one dimension, it is known\(^7\) that for a one-dimensional symmetric stable process,
\[
\int_{-1}^{1} |x - y|^{\alpha-1} (1 - y)^{-\alpha/2} (1 + y)^{-\alpha/2} \, dy = 1, \quad x \in [-1, 1].
\]

➤ Writing \(X = |X| \arg(X)\), when \(X\) begins in the neighbourhood of \(S\), then \(|X|\) begins in the neighbourhood of 1 and \(\arg(X)\), essentially, from within \(S\).
➤ Flat earth theory would imply
\[
\mu_\varepsilon(dy) = m_\varepsilon(y) \ell_d(dy) 1_{(y \in S_\varepsilon)},
\]
with
\[
m_\varepsilon(y) = c_{\alpha,d,\varepsilon}(|y| - (1 - \varepsilon))^{-\alpha/2} (1 + \varepsilon - |y|)^{-\alpha/2}
\]

where \(\ell_d\) is \(d\)-dimensional Lebesgue measure and \(c_{\alpha,d,\varepsilon}\) is a constant to be determined so that
\[
U \mu_\varepsilon(x) = 1 + o(1) \quad x \in S_\varepsilon
\]

\(^7\)Profeta and Simon 2016
THE ASYMPTOTIC DOES NOT DEPEND ON $S$

So far we are guessing:

$$
\mu_\varepsilon(dy) = m_\varepsilon(y) \ell_d(dy) 1_{(y \in S_\varepsilon)},
$$

with $m_\varepsilon(y) = c_{\alpha,d,\varepsilon}(|y| - (1 - \varepsilon))^{-\alpha/2} (1 + \varepsilon - |y|)^{-\alpha/2}$

where $\ell_d$ is $d$-dimensional Lebesgue measure and $c_{\alpha,d,\varepsilon}$ is a constant to be determined so that

$$
U\mu_\varepsilon(x) = 1 + o(1) \quad x \in S_\varepsilon
$$

We don’t think that the restriction to $S_\varepsilon$ is important so we are going to write

$$
\mu_\varepsilon(dy) = \mu_\varepsilon^{(1)}(dy) - \mu_\varepsilon^{(2)}(dy)
$$

with $\mu_\varepsilon^{(1)}(dy) = m_\varepsilon(y) \ell_d(dy)$ and $\mu_\varepsilon^{(2)}(dy) = 1_{(y \in S_{d-1} \setminus S_\varepsilon)} m_\varepsilon(y) \ell_d(dy)$

where $S_{d-1} = \{ x \in \mathbb{R}^d : 1 - \varepsilon \leq |x| \leq 1 + \varepsilon \}$. 
NASTY CALCULATIONS: $\alpha \in (0, 1)$

For $x \in S_{\varepsilon}^{d-1}$,

\[
U_{\mu_{\varepsilon}^{(1)}}(x) = c_{\alpha, d} \int_{S_{\varepsilon}^{d-1}} |x - y|^{\alpha - d} (|y| - (1 - \varepsilon))^{\alpha/2} (1 + \varepsilon - |y|)^{-\alpha/2} \ell_d(dy)
\]

\[
= \frac{2c_{\alpha, d} \pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_{1-\varepsilon}^{1+\varepsilon} \frac{r^{d-1}}{(r - (1 - \varepsilon))^\alpha/2 (1 + \varepsilon - r)^\alpha/2} dr \int_0^\pi \sin^{d-2} \theta d \theta \frac{1}{(|x|^2 - 2|x| r \cos \theta + r^2)^{(d-\alpha)/2}}
\]

\[
= \frac{2c_{\alpha, d} \pi^{d/2}}{\Gamma(d/2)} |x|^{\alpha-d} \int_{1-\varepsilon}^{1+\varepsilon} \frac{2F_1 \left( \frac{d-\alpha}{2}, 1 - \frac{\alpha}{2} ; \frac{d}{2} ; (r/|x|)^2 \right) r^{d-1}}{(r - (1 - \varepsilon))^\alpha/2 (1 + \varepsilon - r)^\alpha/2} dr
\]

\[
+ \frac{2c_{\alpha, d} \pi^{d/2}}{\Gamma(d/2)} \int_{|x|}^{1+\varepsilon} \frac{2F_1 \left( \frac{d-\alpha}{2}, 1 - \frac{\alpha}{2} ; \frac{d}{2} ; (|x|/r)^2 \right) r^{\alpha-1}}{(r - (1 - \varepsilon))^\alpha/2 (1 + \varepsilon - r)^\alpha/2} dr.
\]

\[
= \cdots \cdots \cdots
\]

Turns out

\[
\frac{2^\alpha c_{\alpha, d \varepsilon} \pi^{d/2} \Gamma(1 - \alpha) \Gamma((2 - \alpha)/2)}{\Gamma((d + \alpha - 2)/2)} = 1
\]
The same concept works with a plane

Theorem
Suppose that $\alpha \in (0, 1]$ and the closed and bounded set $S \subseteq \mathbb{H}^{d-1}$ is such that $0 < \ell_{d-1}(S) < \infty$, where we recall that $\ell_{d-1}$ is $(d-1)$-dimensional Lebesgue measure.

(i) Suppose $\alpha \in (0, 1)$. For $x \not\in S$,

$$
\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \mathbb{P}_x (\tau_{S \varepsilon} < \infty) = 2^{1-\alpha} \pi^{-(d-2)/2} \frac{\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-\alpha}{2}\right) \Gamma\left(\frac{2-\alpha}{2}\right)^2}{\Gamma\left(\frac{1-\alpha}{2}\right) \Gamma\left(\frac{d-1}{2}\right) \Gamma(2-\alpha)} K_S(x),
$$
where

$$
K_S(x) = \int_S |x - y|^{\alpha-d} \ell_{d-1}(dy), \quad x \not\in S.
$$

(ii) Suppose $\alpha = 1$. For $x \not\in S$,

$$
\lim_{\varepsilon \to 0} |\log \varepsilon| \mathbb{P}_x (\tau_{S \varepsilon} < \infty) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{(d-2)/2}} K_S(x),
$$

(iii) The process $(X, \mathbb{P}^S)$ is well defined such that

$$
\frac{d \mathbb{P}^S_x}{d \mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{K_S(X_t)}{K_S(x)}, \quad t \geq 0, x \not\in S.
$$
Consider the case $\alpha \in (0, 1)$.

Recall for conditioning a continuous approach to the patch on the sphere from outside we had a scaling with index $\alpha - d$:

$$
\lim_{\varepsilon \to 0} \varepsilon^{\alpha - d} \mathbb{P}_x (X_{G}(\infty) \in A_\varepsilon) = c_{\alpha,d} \int_S (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(d \theta),
$$

Where conditioning a continuous approach to the patch from either side, we had scaling index $\alpha - 1$:

$$
\lim_{\varepsilon \to 0} \varepsilon^{\alpha - 1} \mathbb{P}_x (\tau_{S_\varepsilon} < \infty) = 2^{1-2\alpha} \frac{\Gamma((d + \alpha - 2)/2)}{\pi^{d/2} \Gamma(1 - \alpha)} \frac{\Gamma((2 - \alpha)/2)}{\Gamma(2 - \alpha)} H_S(x).
$$

In the first case, the conditioned path needs to be observant of the entire sphere. In the second case the conditioned path needs only a localised consideration of $S$, which appears flat in close proximity.
Thank you!