Quantitative two-scale stabilization on the Poisson space

Joint work with R. Lachièze-Rey and G. Peccati.

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Fluctuation of Poisson functionals

Let η be a Poisson point process on $\mathbb{R}^d$ with intensity $\lambda(dx)$. The fluctuation of a generic functional F is governed by some principles

- Poincaré inequality

$$\boxed{\mathbb{V}\mathrm{ar}[F] \leq \int \mathbb{E}[|D_x F|^2]\lambda(dx).}$$

  where $D_x F = F(\eta + \delta_x) - F(\eta)$ is the "add-one-cost".

- Second-order Poincaré inequality

$$\boxed{d_{\mathrm{W}}(F, N) \lesssim \text{integrated moments of } D^2_{x,y} F}$$

  where $D^2 = DD$ is the iterated add-one-cost, cf. **Chatterjee** ('09), **Nourdin, Peccati et Reinert** ('09), **Last, Schulte et Peccati** ('16), **Schulte et Yukich** ('19) ...

- The add-one-cost controls the variance, the iterated add-one-cost gives gaussianity.

Fluctuation of Poisson functionals

- Applications: Spatial networks, coverage processes, tessellations etc. useful objects in telecommunication, topological/geometrical data analysis, machine learning...
- This talk is concerned with a *principle alternative to 2nd order Poincaré*. What happens if the iterated add-one-cost is not tractable?
- We address this problem with a two-scale stabilisation theory, which is a quantified version of the stabilisation theory of **Penrose ('01), Penrose and Yukich ('01), Penrose ('05)**.
- This work is along the line of **Malliavin-Stein** methodology for normal approximation, combined with ideas from a quantitative CLT for the MST by **Chatterjee and Sen ('17)**

The iterated add-one-cost is not always tractable

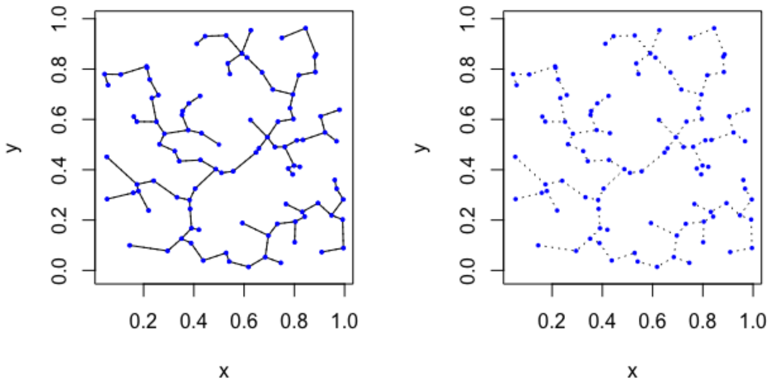

Figure 1: Right: MST. Left: MST after adding a point to the origin.

Setting

- Let η be a Poisson process with unit intensity on $\mathbb{R}^d$, identified with its support $\mathcal{P}$.
- For a Poisson functional $F = F(\eta)$ and $\mathsf{B} \in \mathcal{B}(\mathbb{R}^d)$, define the add-one-cost

$$D_x F(\mathsf{B}) = F((\eta + \delta_x)|_{\mathsf{B}}) - F(\eta|_{\mathsf{B}})$$

and the two-scale discrepancy

$$\boxed{\psi := \sup_{x \in \mathsf{B}} \mathbb{E}[|D_x F(\mathsf{B}) - D_x F(\mathsf{A}_x)|]}$$

- The set B represents the observation window growing to $\mathbb{R}^d$ and A_x is a local window of x with $\mathrm{Leb}(\mathsf{A}_x) \ll \mathrm{Leb}(\mathsf{B})$.
- In practice, $\mathsf{B} = \mathsf{B}_n$, $\mathsf{A}_x = \mathsf{B}_{b_n}(x) \cap \mathsf{B}$ with $b_n = o(n)$. In such case, the two-scale discrepancy is denoted by ψ_n. Define also

$$\boxed{\psi'_n = \sup_{x \in \mathsf{B}_{(n-b_n)}} \mathbb{E}[|D_x F(\mathsf{B}_n) - D_x F(\mathsf{A}_x)|].}$$

Main (user friendly) result

Theorem (Lachièze-Rey, Peccati and Y. ('20+))

Suppose that the following holds:

- *there exists* $p > 4$ *and* $C < \infty$ *such that for all* $n \in \mathbb{N}$

$$\sup_{x \in \mathsf{B}_n} \mathbb{E}[|D_x F(\mathsf{B}_n)|^p] + \mathbb{E}[|D_x F(\mathsf{A}_x)|^p] \leq C^p,$$

- *there exists* $c > 0$ *such that*

$$\mathbb{V}\mathrm{ar}[F(\mathsf{B}_n)] \geq c \cdot \mathrm{Leb}(\mathsf{B}_n) = cn^d.$$

Then there exists $c \in (0, \infty)$ *such that*

$$\frac{1}{c} d_{\mathrm{W}}\left(\frac{F(\mathsf{B}_n) - \mathbb{E}[F(\mathsf{B}_n)]}{\mathbb{V}\mathrm{ar}[F(\mathsf{B}_n)]^{1/2}}, N(0,1)\right) \leq \begin{cases} {\psi_n}^{\frac{1}{2}(1-\frac{4}{p})} + \left(\frac{b_n}{n}\right)^{d/2} \\ {\psi'_n}^{\frac{1}{2}(1-\frac{4}{p})} + \left(\frac{b_n}{n}\right)^{1/2} \end{cases}.$$

N.B. The choice of b_n is done by optimizing the final bound.

Stabilization theory: where it all began

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THE CENTRAL LIMIT THEOREM FOR WEIGHTED MINIMAL SPANNING TREES ON RANDOM POINTS

BY HARRY KESTEN AND SUNGCHUL LEE

Cornell University and National University of Singapore

Let $\{X_i,\ 1 \le i < \infty\}$ be i.i.d. with uniform distribution on $[0,1]^d$ and let $M(X_1,\dots,X_n;\alpha)$ be $\min\{\sum_{e\in T'} |e|^\alpha;\ T'$ a spanning tree on $\{X_1,\dots,X_n\}\}$. Then we show that for $\alpha > 0$,

$$\frac{M(X_1,\dots,X_n;\alpha) - EM(X_1,\dots,X_n;\alpha)}{n^{(d-2\alpha)/2d}} \to N(0,\sigma^2_{\alpha,d})$$

in distribution for some $\sigma^2_{\alpha,d} > 0$.

- **Strong stabilization**: $\exists$ a.s. finite random variable R_0 such that

$$D_0 F(\mathcal{P} \cap \mathsf{B}_{R_0}) = D_0 F((\mathcal{P} \cap \mathsf{B}_{R_0}) \cup \mathcal{U})$$

  for any finite $\mathcal{U} \subset (\mathsf{B}_{R_0})^c$.
- **Weak stabilization**: for **any** (E_n) with $\liminf \mathsf{E}_n = \mathbb{R}^d$, we have

$$D_0 F(\mathsf{E}_n) \to \delta_0(\infty) \text{ a.s.}$$

  for some random variable $\delta_0(\infty)$.

Theorem (Penrose and Yukich ('01))

Assume i) uniform 4th-moment condition; ii) weak stabilization at 0. Then

$$\frac{\mathbb{V}\mathrm{ar}[F(B_n)]}{n^d} \to \sigma^2 \in [0, \infty) \quad \textit{and} \quad \frac{F(\mathsf{B}_n) - \mathbb{E}[F(\mathsf{B}_n)]}{n^{d/2}} \xrightarrow{\mathcal{L}} N(0, \sigma^2).$$

If $\delta_0(\infty)$ is non-degenerate, then $\sigma^2 > 0$.

Relation with our bounds

- Corollary of our bound:

$$d_W\Big(\frac{F(\mathsf{B}_n)-\mathbb{E}[F(\mathsf{B}_n)]}{\mathbb{V}\mathrm{ar}[F(\mathsf{B}_n)]^{1/2}}, N(0,1)\Big) \leq c\Big[\sup_{x\in\mathsf{B}_n}\mathbb{P}[R_x \geq b_n]^{\frac{1}{2}(1-\frac{1}{p})(1-\frac{4}{p})} + \Big(\frac{b_n}{n}\Big)^{d/2}\Big],$$

  where R_x the radius of strong stabilization at x.

- Assume $F(\tau_x\mathcal{P}\cap\tau_x\mathsf{B}) = F(\mathcal{P}\cap\mathsf{B})$ and weak stabilization $\Rightarrow$

$$D_xF(\mathsf{E}_n)\to\delta_x(\infty)\quad a.s.$$

  for *any* $(E_n)\uparrow\mathbb{R}^d$. Therefore, the required condition

$$\psi_n' = \sup_{x\in\mathsf{B}_{n-b_n}}\mathbb{E}[|D_xF(\mathsf{B}_n)-\delta_x(\infty)+\delta_x(\infty)-D_xF(\mathsf{A}_x)|]\to 0$$

  is a uniform strengthening of weak stabilization. Note however that we do not require the existence of $\delta_0(\infty)$.

Far reach of the Penrose-Yukich theory (thus ours)

Weights, subgraph counts, components counts of

- k-nearest neighbor graphs
- sphere of influence graphs
- Voronoi tessellations
- minimal spanning trees

PY: (Multivariate) Gaussian approximation holds if strong/weak stabilisation holds for the functional of interest.

LrPY: To obtain rates, if suffices to compute ψ_n (or ψ'_n), or $\mathbb{P}[R_x \geq b_n]$.

Not always easy, here is an open problem
The optimal travelling salesman tour on Poisson points is believed to be stabilizing (implying CLT if proved).

Applications (in our paper)

- **Online NNG (S):** Mark $\mathcal{P} \cap \mathsf{B}_n$ with iid uniform $[0,1]$ representing the arrival time, each point is attached to its nearest neighbour prior to its arrival. We obtain n^{-c} for the rate of normal approximation of the weighted edge length.
- **Boolean model (S):** The number of connected components of the Boolean model
$$\mathsf{O}_u(\mathcal{P} \cap \mathsf{B}_n) = \bigcup_{x \in \mathcal{P} \cap \mathsf{B}_n} \mathsf{S}_u(x).$$
approaches normal with rate n^{-c} in $d = 2$ and $\log(n)^{-c}$ in $d \geq 3$.
- **Minimal spanning tree (W):** The total weighted edge length of MST approaches normal distribution with the same rate as the percolation example. In both cases, ψ'_n is bounded by the two arm events.
- **Excursion of heavy tail shot noise fields (W):** The intrinsic volumes of excursion sets $E_u = \{t \in \mathsf{B}_n : X(t) \geq u\}$ of heavy tail shot noise field X approaches normal with rate n^{-c}.

Component counts for the Boolean model (S)

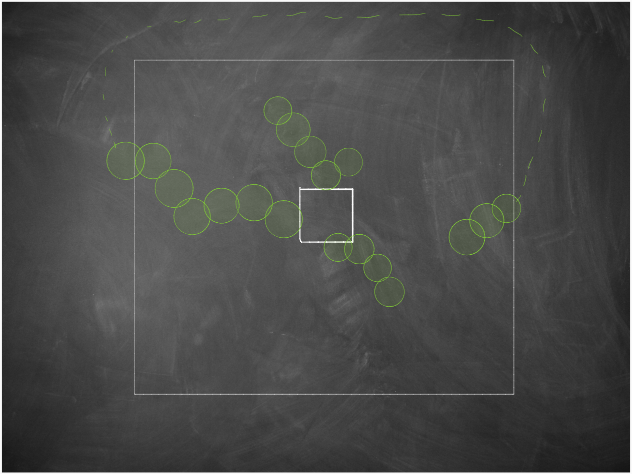

Figure 2: $R_x := \inf\{r > u : \text{at most 1 arm in } B_r(x) \setminus B_u(x)\}$ where 1 arm means that the Boolean model contains a path connecting the boundary of two boxes.

$$\{R_x > b_n\} \subset \{\text{at least 2 arms at distance } b_n\}.$$

Phase transition of occupied and vacant regions

$$u_c := \inf\{u : \mathbb{P}[0 \leftrightarrow \infty \text{ in } O_u] > 0\} \in (0, \infty),$$

$$u_c^* := \sup\{u : \mathbb{P}[0 \leftrightarrow \infty \text{ in } V_u] > 0\} \in (0, \infty),$$

and $u_c = u_c^*$ in dimension 2 by **Roy ('90)**, $u_c < u_c^*$ in dimension $d \geq 3$ by **Penrose ('96), Sarkar ('97)**.

- Subcritical phase $u < u_c$

$$\mathbb{P}[R_x > b_n] \leq \mathbb{P}[\text{at least 1 arm at distance } b_n] \leq e^{-cb_n}.$$

- Supercritical phase $u > u_c^*$

$$\mathbb{P}[R_x > b_n] \leq \mathbb{P}[\text{at least 1 vacant arm at distance } b_n] \leq e^{-cb_n}.$$

- Critical phase $u \in [u_c, u_c^*]$ Two-arm event decays as b_n^{-c} in $2D$ and $[\log(b_n)]^{-c}$ in $d \geq 3$ by a quantitative **Burton-Keane** argument of **Chatterjee-Sen ('17)**.

Minimal spanning tree (W)

- Minimal spanning tree over a finite point set $\mathcal{U}$

$$\mathrm{MST}(\mathcal{U}) = \mathrm{Argmin}\Big\{\sum_{e\in T} |e|,\ T \text{ connected with } \mathbb{V}(T) = \mathcal{U}\Big\}$$

- Functional of interest $\mathsf{M}(\mathsf{B}_n) \in \mathbb{R}^m$ given by

$$M(\varphi_i; \mathsf{B}_n) := \sum_{e\in \mathrm{MST}(\mathcal{P}|_{\mathsf{B}_n})} \varphi_i(|e|), \quad 1 \le i \le m.$$

- Suppose φ is given by $\varphi(x) = \psi(x)\mathbb{1}(x \le r)$ for some non-decreasing function ψ and some truncation level $r \in (0, \infty]$. If (and only if) $r = \infty$, suppose

$$\exists k \in \mathbb{N}, \quad \psi(x) \le (1+x)^k \text{ and } \int_0^\infty e^{-cu^d} d\psi(\sqrt{d}u) < \infty.$$

- Examples: power-weighted edge length $\varphi(x) = x^\alpha$ or empirical process $\varphi(x) = \mathbb{1}(x \le r)$.

Theorem (LrPY '20+)

Let $\mathrm{N} = \mathrm{N}(n)$ be a centered Gaussian vector with the same covariance matrix as

$$n^{-d/2}\mathrm{M}(\mathrm{B}_n).$$

Then, one has that

$$d_3(n^{-d/2}(\mathrm{M}(\mathrm{B}_n) - \mathbb{E}[\mathrm{M}(\mathrm{B}_n)]), \mathrm{N}) \leq \begin{cases} cn^{-\theta} & \text{if } d = 2, \\ c\exp(-c\log\log(n)) & \text{if } d \geq 3, \end{cases}$$

for some $0 < \theta < 1$. The above bound continues to hold for the distances d_2, d_c, if $\mathbb{C}\mathrm{ov}[n^{-d/2}\mathrm{M}(\mathrm{B}_n)] \to \Sigma_\infty > 0$.

- Two vertices $x, y \in \mathcal{P}$ form an edge of MST if and only if x and y belong to different component of $\mathrm{O}_{\frac{|x-y|}{2}}(\mathcal{P})$.
- In $d = 2$, consider $(\log(n))^a$ Boolean models with random radius and relate ψ'_n to the 2-arm estimates.

- **Stein**'s lemma

$$\mathbb{E}[f'(N)] = \mathbb{E}[Nf(N)].$$

if and only if $N \sim N(0,1)$.

- Heuristic: $F \approx N$ if and only if

$$\mathbb{E}[f'(F)] \approx \mathbb{E}[Ff(F)].$$

- Stein's equation

$$f'(x) - xf(x) = h(x) - \mathbb{E}[h(N)]$$

with $h \in \mathrm{Lip}_1$. Evaluate the expectation wrt $\mathbb{P} \circ F^{-1}$, then take sup over h gives

$$\begin{aligned} d_{\mathrm{W}}(F,N) &:= \sup_{h \in \mathrm{Lip}_1} |\mathbb{E}h(F) - \mathbb{E}h(N)| \\ &\leq \sup_{\|g'\|,\|g''\| \leq 1} |\mathbb{E}[Fg(F)] - \mathbb{E}[g'(F)]|. \end{aligned}$$

Proof of the general bound (ii) Integration by parts ('05 onwards)

- For $F = F(\mathsf{B})$ with $\mathbb{E}[F] = 0, \mathbb{E}[F^2] = 1$, we integrate by parts

$$\mathbb{E}[Fg(F)] = \mathbb{E}\Big[\int_{\mathsf{B}} D_x(g(F))\,(-D_x L^{-1} F)dx\Big]$$
$$\approx \mathbb{E}\Big[g'(F)\int_{\mathsf{B}} D_x F\,(-D_x L^{-1} F)dx\Big]$$

  where L^{-1} involves thinning and (independent) superposition.
- Proof of IBP by (birth and death) semigroup interpolation: in 1 dimension, $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{N}_0, \mathrm{Po}(1))$,

$$P_t f(k) = \mathbb{E}[f(\mathrm{Bin}(k, e^{-t}) + \mathrm{Po}(1 - e^{-t}))]$$

  and

$$Lf(k) = 1(f(k+1) - f(k)) - k(f(k) - f(k-1)).$$

  satisfying $-\mathbb{E}[fLg] = \mathbb{E}[DfDg]$ with $Df(k) = f(k+1) - f(k)$.

- Thus, interpolation and $-\mathbb{E}[FLG] = \mathbb{E}[\langle DF, DG\rangle]$ gives

$$\begin{aligned}\mathbb{E}[Fg(F)] &= \mathbb{E}[(P_0F - P_\infty F)g(F)] \\ &= -\int_0^\infty \mathbb{E}[(LP_tF)g(F)]dt \\ &= \int_0^\infty \mathbb{E}\Big[\int_B D_x(g(F))D_xP_tFdx\Big]dt \\ &= \mathbb{E}\Big[\int_{\mathrm{B}} D_x(g(F))\,(-D_xL^{-1}F)dx\Big]\end{aligned}$$

  by setting

$$-L^{-1} := \int_0^\infty P_t dt$$

- Combining Stein's bound, integration by parts, and Cauchy-Schwarz

$$\begin{aligned}d_{\mathrm{W}}(F,N) &\lesssim \mathbb{V}\mathrm{ar}\Big[\int_{\mathrm{B}} D_xF\,(-D_xL^{-1}F)dx\Big]^{1/2} \\ &= \Big(\iint_{\mathrm{B}^2} \mathbb{C}\mathrm{ov}[D_xFD_xL^{-1}F, D_yFD_yL^{-1}F]dxdy\Big)^{1/2}.\end{aligned}$$

Proof of the general bound (iii) two-scale stabilization

- When x and y are close i.e. $\mathsf{A}_x \cap \mathsf{A}_y \neq \emptyset$, bound the covariance by

$$\mathbb{E}[|D_x L^{-1} F(\mathsf{B})|^p] \leq \mathbb{E}[|D_x F(\mathsf{B})|^p] \leq C,$$

yielding a term $(\frac{b_n}{n})^{d/2}$.

- When they are far apart i.e. $\mathsf{A}_x \cap \mathsf{A}_y = \emptyset$, we replace everything by its local version

$$\mathbb{C}\text{ov}[D_x F(\mathsf{A}_x) D_x L^{-1} F(\mathsf{A}_x), D_y F(\mathsf{A}_y) D_y L^{-1} F(\mathsf{A}_y)] \tag{1}$$

with 4 error terms like

$$\mathbb{C}\text{ov}[(D_x F(\mathsf{B}) - D_x F(\mathsf{A}_x)) D_x L^{-1} F, D_y F D_y L^{-1} F]. \tag{2}$$

- By independence of Poisson points over non-overlapping regions, (1) $= 0$, we bound (2) by

$$\mathbb{E}[|D_x F(\mathsf{B}) - D_x F(A_x)||D_x L^{-1} F D_y F D_y L^{-1} F|].$$

- Applying Hölder's inequality and bounding the moments $\mathbb{E}[|D_x L^{-1} F(\mathsf{A}_x)|^p] \leq \mathbb{E}[|D_x F(\mathsf{A}_x)|^p] \leq C$ leads to the two-scale discrepancy ψ_n, ending the proof.

Extensions

Two-scale bounds of the type

$$(\psi_n)^{\frac{1}{2}(1-\frac{4}{p})} + \left(\frac{b_n}{n}\right)^{d/2}$$

holds for

- Kolmogorov distance

$$d_{\mathrm{K}}(F, N) = \sup_{x \in \mathbb{R}} |\mathbb{P}[F \leq x] - \mathbb{P}[N \leq x]|,$$

- probability metrics for multivariate normal approximation, including smooth ones d_2, d_3 (generalizing d_{W}), and the non-smooth convex distance (generalizing d_{K})

$$d_{\mathrm{c}}(\mathsf{F}, \mathsf{N}_\Sigma) = \sup_{\mathsf{E} \ \ convex} |\mathbb{P}[\mathsf{F} \in \mathsf{E}] - \mathbb{P}[\mathsf{N}_\Sigma \in \mathsf{E}]|$$

possibly subject to stronger moment conditions ($p > 6$).

Behind the scenes: a new Kolmogorov bound

Theorem (LrPY '20+)

Let $\widehat{F} = (F - \mathbb{E}[F])/\sigma$.

$$d_{\mathrm{K}}\left(\widehat{F}, N\right) \leq \left|1 - \frac{\mathbb{V}\mathrm{ar}[F]}{\sigma^2}\right| + \frac{1}{\sigma^2}\mathbb{E}\left[|\mathbb{V}\mathrm{ar}[F] - \langle DF, -DL^{-1}F\rangle|\right] + \frac{2}{\sigma^2}\mathbb{E}[|\delta(DF|DL^{-1}F|)|],$$

where δ is the Kabanov-Skorohod integral.

- Starting point of the two-scale bound in d_{K}.
- Two redundant terms in **Schulte ('16)** and **Eichelsbacher and Thäle ('14)** are removed.
- A good place to start if the 4th-moment assumption is not verified.

Final remarks

- Our theorem gives almost optimal rates $\log(n)^c n^{-d/2}$ in the case of exponential stabilization $\mathbb{P}[R(x) > t] \leq ce^{-c't}$.
- The second-order Poincaré estimates of **Last, Peccati and Schulte ('16)**, **Lachièze-Rey, Schulte and Yukich ('19)** and **Schulte and Yukich ('19)** is concerned with

$$\mathbb{P}[D^2_{x,y} F \neq 0],$$

  yielding Berry-Esseen bounds $n^{-d/2}$ for exponential stabilization.
- The upshot of our theorem is that we do not require knowledge on the iterated add-one-cost operators, which can be very hard to access quantitatively for not necessarily exponentially stabilizing functionals such as critical percolation models.

Thanks!