k-cut Model for the Brownian Continuum Random Tree*

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*Based on arXiv 2007.11080
Cutting down random trees

Meir & Moon ’70s

- Imagine a network $T_n$ (rooted tree with $n$ nodes)
- At rate 1, a uniform node is attacked. It is then removed from $T_n$ along with the subtree above.
- Iterate on the remaining tree until nothing left.
- $X(T_n) = \text{total number of attacks } \leq n$

Let $T_n$ be a uniform tree with $n$ nodes. Then Panholzer 06, Janson 06 show that

$$\frac{X(T_n)}{\sqrt{n}} \xrightarrow{(d)} Z \sim \text{Rayleigh dist.}$$
An invariance principle

- [Janson ’06] Holds more generally as $T_n$ can be replaced by a conditioned Galton–Watson tree with finite variance.


- **Question**: Does the previous cutting process of $T_n$ converge to a “cutting” of the CRT, so that $Z =$ functional of the CRT?

- Yes, according to Addario-Berry, Broutin & Holmgren ’15, Bertoin & Miermont ’13, Abraham & Delmas ’13
Cutting down resilient random trees

Cai, Devroye, Holmgren & Skerman 2019

- Imagine a resilient network $T_n$ (rooted tree with $n$ nodes)
- At rate 1, a uniform node is attacked. It is then removed from $T_n$ after $k$ attacks.
- Iterate on the remaining tree until nothing left.
- $X_k(T_n) =$ total number of attacks

Let $T_n$ be a conditioned Galton–Watson tree with variance $\sigma^2$. Then Berzunza, Cai & Holmgren ’20 show that

$$\frac{X_k(T_n)}{\sigma^k n^{1-\frac{1}{2k}}} \xrightarrow{(d)} Z_k$$

Question: Write $Z_k$ as a functional of the CRT?
Overview

- Continuum Random Tree
- Cutting down Continuum Random Tree
- Scaling limit of $X_k(T_n)$
Continuum Random Tree

\[ T_n \]

\[ T = \text{G-W tree with offspring dist. } (p_k)_{k \geq 0} \]

\[ \sum_k k p_k = 1 \]

\[ \sigma^2 = \sum_k (k^2 - k) p_k < \infty \]

\[ T_n \overset{(d)}{=} T \text{ cond. on } \#T = n. \]
Continuum Random Tree
Continuum Random Tree

$T_n$

$2(n-1)$
Continuum Random Tree

\[ d(a,b) = d(a,c) + d(c,b) \]
\[ = h(a) - h(c) + h(b) - h(c) \]
\[ = h(s) + h(t) - 2 \min_{u \in [s,t]} h(u) \]

Brownian excursion

\[ s \sim t \iff d(s,t) = 0 \]

\[ T = \left( \frac{[0,1]}{\sqrt{n}}, d \right) \]
Continuum Random Tree

$n \to \infty$

Brownian excursion

Continuum Random Tree

$\sqrt{n}/\sigma$

$2n$

$T_n$
Some properties of CRT

- **Tree-like:** loop-free and unique geodesic.
- **Of fractal dimension 2:** inherited from BM.
- **Countable number of branch points:** In bijection with local minima of Br. exc.; each one of degree 3.
- **Leaves are dense everywhere:** Define $\mu$ as the pushforward of unif. measure on $[0, 1]$. Sample $U \sim \mu$; then $U$ is a leaf a.s.
Cutting down CRT

Alternative formulation of cutting $T_n$

- Write $\mu_n = \text{unif. measure on vertex set of } T_n$.
- Launch a Poisson point proc. $\{(t_i, x_i) : i \geq 1\}$ on $T_n$ with intensity $n \cdot \mu_n$.
- At time $t_i$, attack $x_i$. This attack is counted in the tally $X_k(T_n)$ iff $x_i$ is connected to the root at $t_i$.
- If a vertex has been attacked $k$ times, remove it along with the subtree above.

**Extend to the CRT?** Look at spanning trees.

\[
\text{Skeleton} = T_n \setminus \{ \text{leaves} \} = \bigcup \{ \text{spanning trees} \}.
\]
Scaling limit of spanning trees

Let $\mathcal{R}_m^n = $ subtree of $T_n$ spanned by $m$ uniform vertices $V_1, \ldots, V_m$. Rescale the edge-length of $\mathcal{R}_m^n$ by $\frac{\sigma}{\sqrt{n}}$. Then,

$$ \frac{\sigma}{\sqrt{n}} \mathcal{R}_m^n \xrightarrow{(d)} \mathcal{R}_m $$

where $\mathcal{R}_m$ is the subtree of CRT spanned by $m$ uniform points.
Cutting down CRT

- Rank the vertices of $T_n$ in the order of their removal: $v_1, v_2, \ldots, v_n$ and let $\tau_1 < \tau_2 < \cdots$ be their corresponding removal times. Note that $(v_i)$ is a uniform permutation and $(\tau_i)$ is the order statistics of $n$ i.i.d. Gamma($k, 1$).

- Consider the sub-collection $\{(\tau_i, v_i) : v_i \in \mathcal{R}_m^n\}$. We have

$$
\left(\frac{1}{\sigma_k} n^{1-\frac{1}{2k}\tau_i}, v_i\right)_{i \geq 1} \xrightarrow{(d)} \left((k!t_i)^{\frac{1}{k}}, x_i\right)_{i \geq 1}
$$

in an appropriate sense, where $((t_i, x_i))_{i \geq 1}$ is a Poisson point process of unit rate on $\mathcal{R}_m$.

- As $m$ increases, $\mathcal{R}_m \nearrow$ skeleton of CRT; we can then extend the previous Poisson point proc. to the CRT and use the Poisson proc. to cut it down.
Understand the scaling...

\[ \mathbb{P}(\tau(k,1) \leq t) = \mathbb{P}(E_1 + \cdots + E_k \leq t) = \mathbb{P}(\text{Pois}(t) \geq k) \]

\[ = \sum_{j=k}^{\infty} e^{-t} \frac{t^j}{j!} \quad \text{as} \quad t \to 0+ \]

\[ \approx e^{-t} \frac{t^k}{k!} \]

\[ \mathbb{E}[\# \{ \text{removed vertices in } \mathbb{R}^n_m \text{ in } [0,t] \}] \]

\[ = \# \{ \text{vertices in } \mathbb{R}^n_m \} \cdot \mathbb{P}(\tau(k,1) \leq t) \]

\[ \geq m \cdot \frac{\sqrt{n}}{\sigma} \cdot \frac{t^k}{k!} \]

\[ \sqrt{n} \cdot t^k = O(1) \quad \Rightarrow \quad t = o\left(n^{-\frac{1}{2k}}\right) \]
Records & Number of cuts

- The $r$-th attack at a vertex $v$ is called a $r$-record if $v$ is still connected to the root when the attack occurs, $1 \leq r \leq k$, so that

$$X_k(T_n) = \# \{1\text{-records}\} + \cdots + \# \{k\text{-records}\}.$$  

$$\sim \# \{1\text{-records}\}.$$  

- We have

$$\mathbb{E}[\# \{r\text{-records}\}] = \mathcal{O}(n^{1-\frac{r}{2k}}).$$

- So it suffices to look at the asymptotic of 1-records.
Asymptotic of 1-records

- Let $S_n(t) =$ remaining part of $T_n$ at time $t$. Denote
  \[ a_n(t) = \# \{ \text{vertices in } S_n(t) \text{ which have received no attack at time } t \} . \]

- Since 1-records arrive at $\text{Exp}(1)$, we have
  \[
  \mathbf{E} \left[ \# \{1\text{-records arriving in } [t, t + dt]\} \mid a_n(t) \right] = a_n(t) dt
  \]
  A second moment argument then implies
  \[
  \# \{1\text{-records} \} \sim \int_0^\infty a_n(t) dt \text{ in prob.}
  \]
  \[
  \sim \sigma \frac{1}{k} n^{-\frac{1}{2k}} \int_0^\infty \mu_n \left(S_n(n^{-\frac{1}{2k}} t)\right) dt
  \]

- Given $\#S_n(t) = n \cdot \mu_n(S_n(t))$, we have $a_n(t) \sim \text{Binom}(\#S_n(t), e^{-t})$. Then,
  \[
  \frac{1}{n} a_n(\sigma \frac{1}{k} n^{-\frac{1}{2k}} t) \sim \mu_n(S_n(n^{-\frac{1}{2k}} t)) \text{ in prob.}
  \]
Scaling limit of $X_k(T_n)$

- Let $\mathcal{P} = \{(t_i, x_i) : i \geq 1\}$ be a Poisson point proc. of unit rate on the skeleton of CRT. Remove $x_i$ and the subtree above at time $(k! t_i)^{1/k}$.
- Let $\mathcal{S}(t)$ be the remaining part of the CRT at time $t$ and define

  $$Z_k = \int_0^\infty \mu(\mathcal{S}(t)) \, dt.$$  

**Theorem**

As $n \to \infty$, we have

$$\left( \frac{\sigma}{\sqrt{n}} T_n, \frac{X_k(T_n)}{\sigma \frac{1}{k} n^{1-\frac{1}{2k}}} \right) \xrightarrow{(d)} (\mathcal{T}, Z_k),$$
Some final remarks

• For $k = 1$, we recover the construction in Addario-Berry, Broutin & Holmgren, Bertoin & Miermont, Abraham & Delmas.

• From the previous construction of $Z_k$, we deduce
  • comparison between $(Z_k)_{k \geq 1}$; in particular,
    \[ k \cdot (k!)^{-\frac{1}{k}} Z_k \leq k + Z_1. \]
  • direct computations of $E[Z_k^i \mid \text{CRT}]$.

Thank you!