Pure Mathematics and Dynamical Systems at Liverpool

Lasse Rempe

Department of Mathematical Sciences,
University of Liverpool

AIMS-Liverpool Joint Postgraduate Conference
June 2023

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena ...
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena . . .
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena …
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena . . .
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena . . .
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics

"Pure" Mathematics studies the intrinsic properties of abstract structures and concepts – e.g. *number* or *space*.

- Focus on establishing *general principles*, and achieving a *deeper understanding* of *fundamental phenomena*.
- Establishes *rigorous* results by mathematical proof.
- Not primarily concerned with *accurate modelling* of the physical world.
- Nonetheless, often *motivated* by "real-world" phenomena . . .
- and *applicable* (usually in the long, rather than short, term).

Pure Mathematics at Liverpool

Pure Mathematics research at Liverpool tends to have a strong *geometrical flavour*.

The research interests of this group are grouped into three areas:

- Algebraic Geometry
- Geometry & Topology
- Dynamical Systems

Pure Mathematics at Liverpool

Pure Mathematics research at Liverpool tends to have a strong *geometrical flavour*.

The research interests of this group are grouped into three areas:

- Algebraic Geometry
- Geometry & Topology
- Dynamical Systems

Pure Mathematics at Liverpool

Pure Mathematics research at Liverpool tends to have a strong *geometrical flavour*.

The research interests of this group are grouped into three areas:

- Algebraic Geometry
- Geometry & Topology
- Dynamical Systems

Pure Mathematics at Liverpool

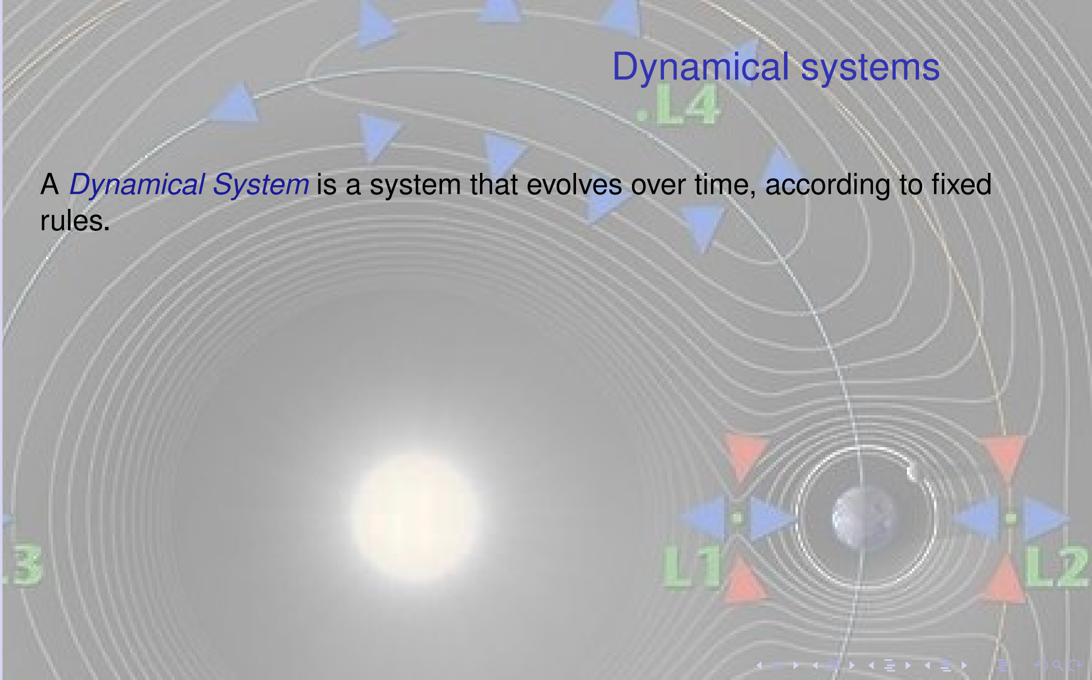
Pure Mathematics research at Liverpool tends to have a strong *geometrical flavour*.

The research interests of this group are grouped into three areas:

- Algebraic Geometry
- Geometry & Topology
- Dynamical Systems

Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.



Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;

Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;
- a system of *planetary bodies* (governed by gravitational attraction);

Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;
- a system of *planetary bodies* (governed by gravitational attraction);
- one or several populations in a closed *ecosystem*;

Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;
- a system of *planetary bodies* (governed by gravitational attraction);
- one or several populations in a closed *ecosystem*;
- the earth's *weather*;

Dynamical systems

A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;
- a system of *planetary bodies* (governed by gravitational attraction);
- one or several populations in a closed *ecosystem*;
- the earth's *weather*;
- mathematical models for the *stock market*;

Dynamical systems

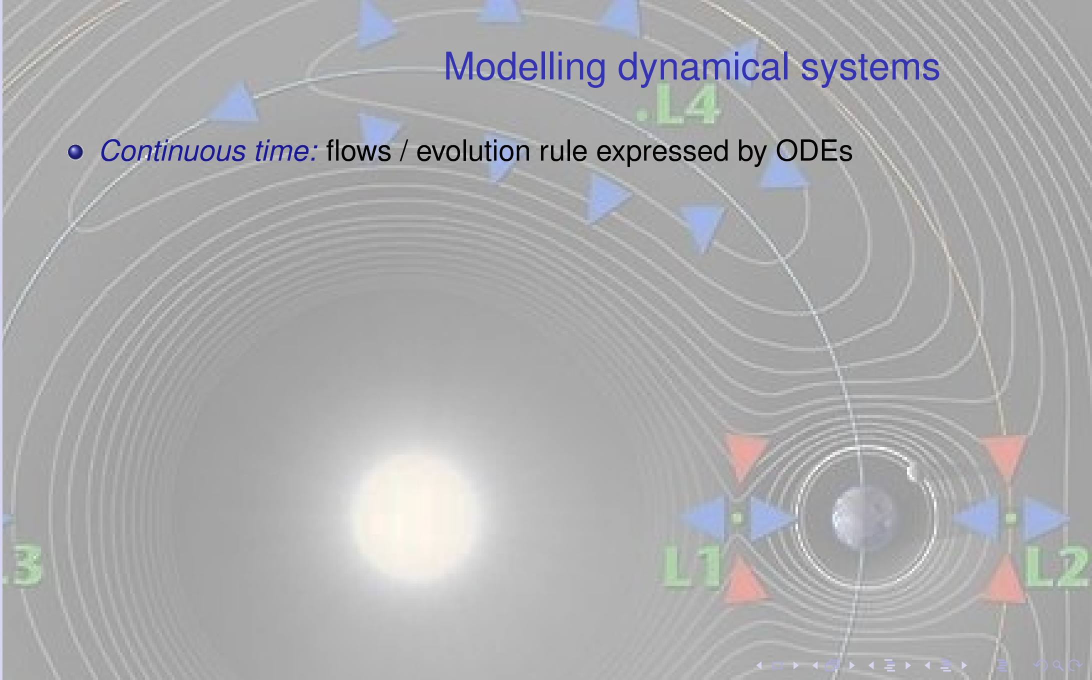
A *Dynamical System* is a system that evolves over time, according to fixed rules.

E.g.,

- a *pendulum* or a double pendulum;
- a system of *planetary bodies* (governed by gravitational attraction);
- one or several populations in a closed *ecosystem*;
- the earth's *weather*;
- mathematical models for the *stock market*;
- the *universe*.

Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs



Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.

Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.
- X: *state space*

Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.
- X: *state space*
- $f : X \to X$ *evolution rule*

Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.
- X: *state space*
- $f : X \to X$ *evolution rule*
- x_0 starting value $\rightsquigarrow$ *orbit* of x_0:

$$x_{n+1} = f(x_n)$$

Modelling dynamical systems

- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.
- X: *state space*
- $f: X \to X$ *evolution rule*
- x_0 starting value $\rightsquigarrow$ *orbit* of x_0:

$$x_{n+1} = f(x_n) = \underbrace{f(\dots f(x_0)\dots)}_{n+1 \text{ times}} := f^{n+1}(x_0)$$

Modelling dynamical systems

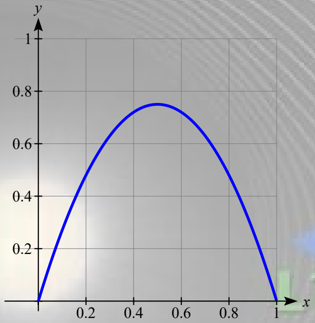
- *Continuous time:* flows / evolution rule expressed by ODEs
- *Discrete time:* iteration of an evolution rule expressed by a self-map of the state space.
- X: *state space*
- $f\colon X \to X$ *evolution rule*
- x_0 starting value $\rightsquigarrow$ *orbit* of x_0:

$$x_{n+1} = f(x_n) = \underbrace{f(\dots f(x_0)\dots)}_{n+1 \text{ times}} := f^{n+1}(x_0)$$

Typical questions: What is the *long-term* behaviour of orbits? How does it change under *perturbations* of the starting value / the system?

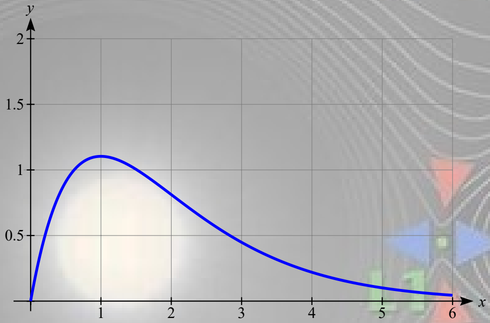
Simple models for population dynamics

$$f_\lambda\colon [0,1] \to [0,1]; \quad x \mapsto \lambda x(1-x) \qquad (\lambda \in [1,4].)$$

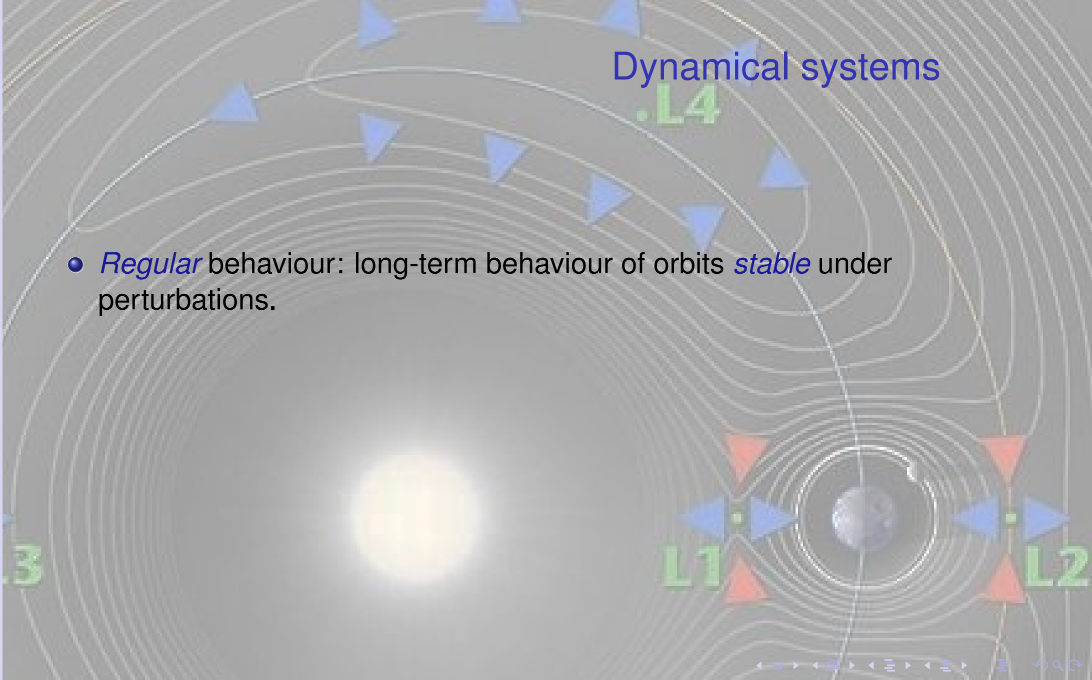


Simple models for population dynamics

$$f_\lambda\colon [0,\infty) \to [0,\infty); \quad x \mapsto \lambda x e^{-x} \qquad (\lambda \geq 1)$$



Dynamical systems



- *Regular* behaviour: long-term behaviour of orbits *stable* under perturbations.

Dynamical systems

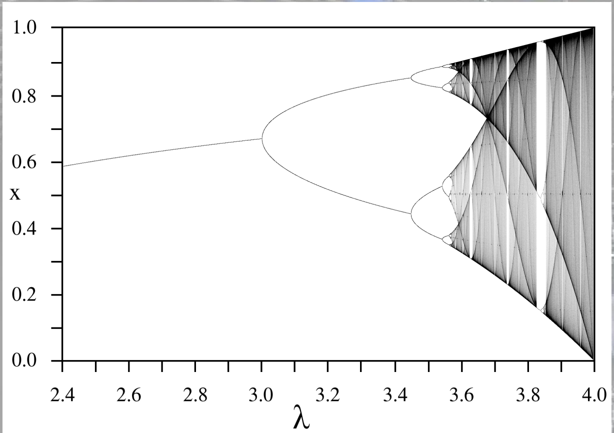
- *Regular* behaviour: long-term behaviour of orbits *stable* under perturbations.
- *Unstable / chaotic* behaviour: long-term behaviour of orbits *unstable* under arbitrarily small perturbations.

Dynamical systems

- *Regular* behaviour: long-term behaviour of orbits *stable* under perturbations.
- *Unstable / chaotic* behaviour: long-term behaviour of orbits *unstable* under arbitrarily small perturbations.

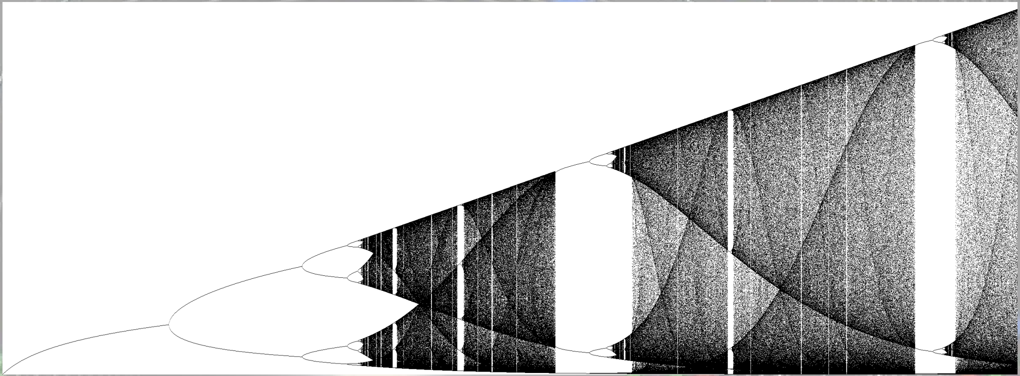
Key observation: Even *simple* rules may lead to *chaotic* behaviour.

Bifurcations in population models



$$\lambda x(1-x)$$

Bifurcations in population models



$\lambda x e^{-x}$

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Dynamical systems at Liverpool

- *Permanent group members*: Hall, Martí-Pete, Meyer, Nair, Rempe.
- *Postdocs*: Reinke, Ferreira.
- *PhD students*: Brown, Münch.

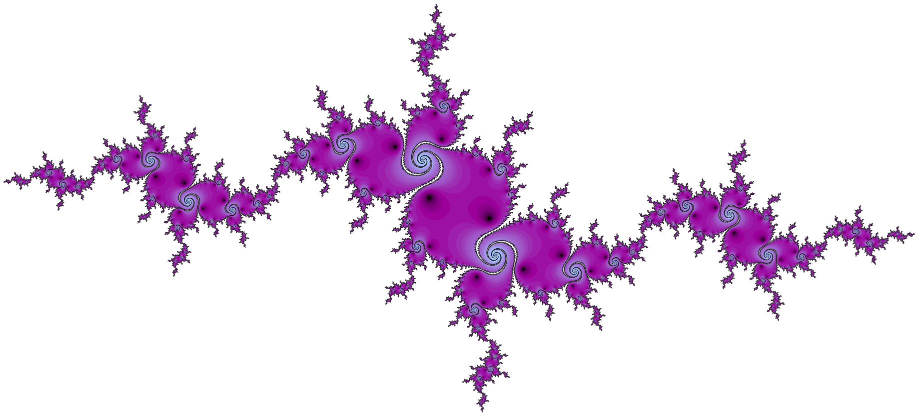
(Pure) dynamical systems research at Liverpool tends to study *low-dimensional dynamics*:

- Ergodic theory and number theory (Nair).
- Topological dynamics; surface homeomorphisms (Hall)
- One-dimensional holomorphic dynamics (Martí-Pete, Meyer, Rempe);

Holomorphic dynamics

Iteration of a function of *one complex variable*.

$$f : \mathbb{C} \to \mathbb{C} \quad \text{analytic.}$$

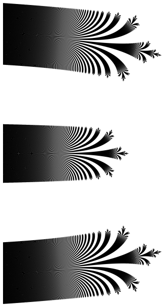


$$f(z) = \lambda z(1 - z)$$

Holomorphic dynamics

Iteration of a function of *one complex variable*.

$$f\colon \mathbb{C} \to \mathbb{C} \quad \text{analytic.}$$

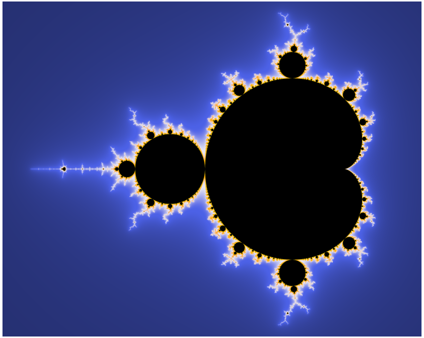


$$f(z) = z \exp(-z)$$

Holomorphic dynamics

Iteration of a function of *one complex variable*.

$f : \mathbb{C} \to \mathbb{C}$ analytic.



$$f(z) = z^2 + c$$

Transcendental dynamics

$f : \mathbb{C} \to \mathbb{C}$ transcendental (not polynomial).

- First studied by *Fatou* in 1926.
- Among the examples Fatou studied was the map

$$z \mapsto \frac{\sin(z)}{2}.$$

- *Observation* (Fatou, 1926): There is a collection of *infinitely many curves* on which orbits *tend to infinity*.

Transcendental dynamics

$f : \mathbb{C} \to \mathbb{C}$ transcendental (not polynomial).

- First studied by *Fatou* in 1926.
- Among the examples Fatou studied was the map

$$z \mapsto \frac{\sin(z)}{2}.$$

- *Observation* (Fatou, 1926): There is a collection of *infinitely many curves* on which orbits *tend to infinity*.

Transcendental dynamics

$f : \mathbb{C} \to \mathbb{C}$ transcendental (not polynomial).

- First studied by *Fatou* in 1926.
- Among the examples Fatou studied was the map

$$z \mapsto \frac{\sin(z)}{2}.$$

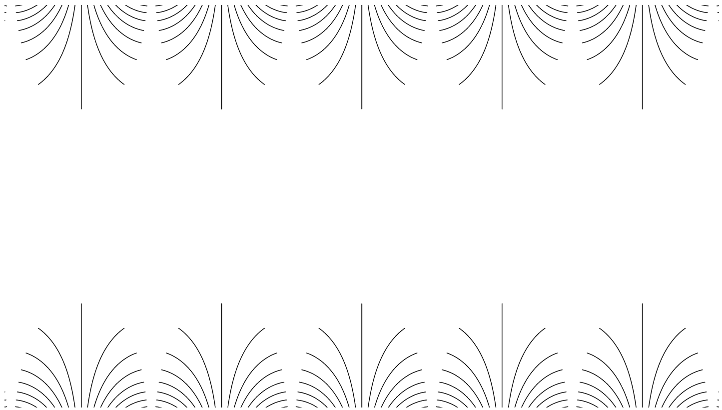
- *Observation* (Fatou, 1926): There is a collection of *infinitely many curves* on which orbits *tend to infinity*.

Curves in the escaping set

$$f_0(z) = \sin(z)/2$$

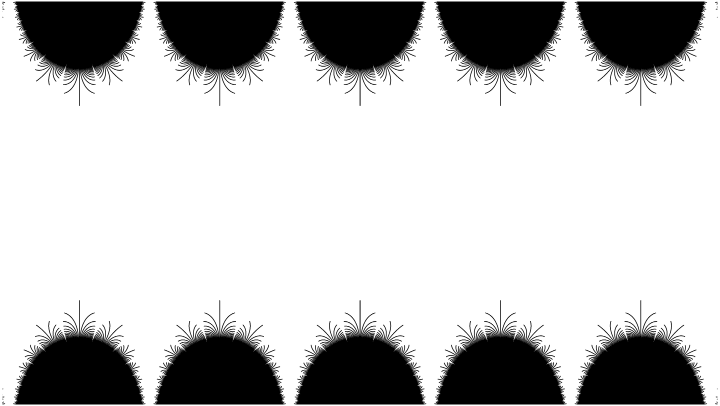
Curves in the escaping set

$$f_0(z) = \sin(z)/2$$



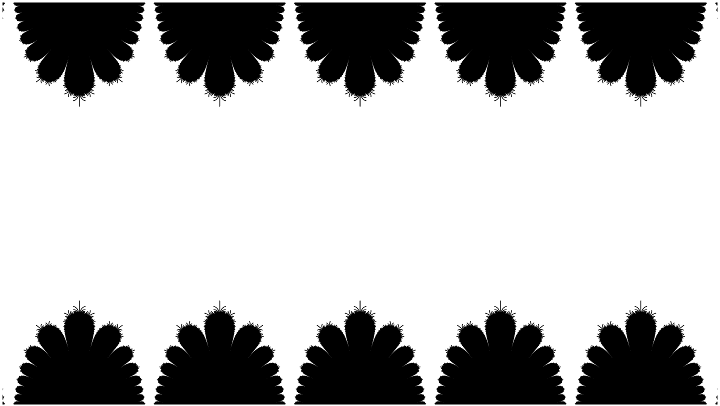
Curves in the escaping set

$$f_0(z) = \sin(z)/2$$



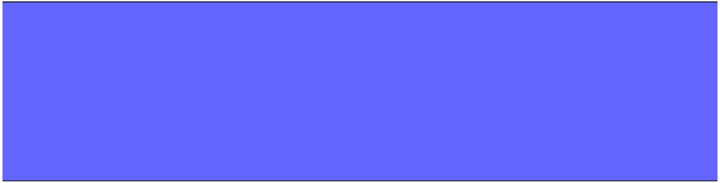
Curves in the escaping set

$$f_0(z) = \sin(z)/2$$



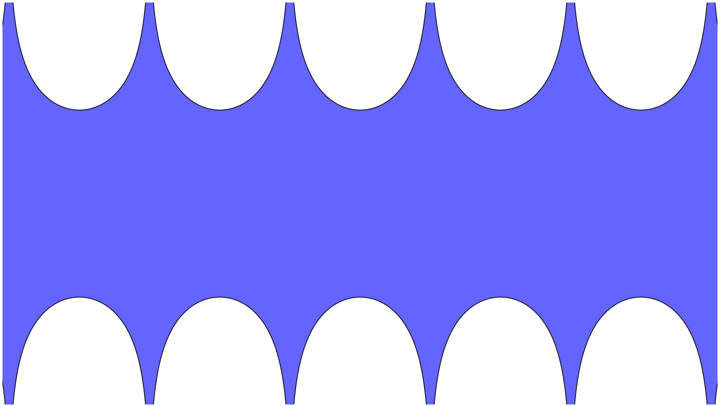
The basin of 0

$$f_0(z) = \sin(z)/2$$



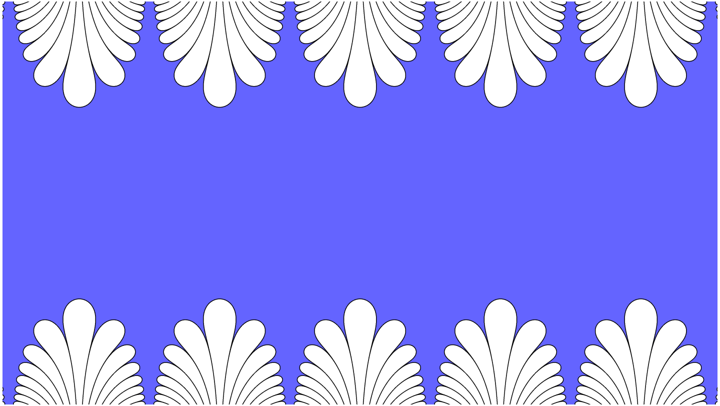
The basin of 0

$$f_0(z) = \sin(z)/2$$



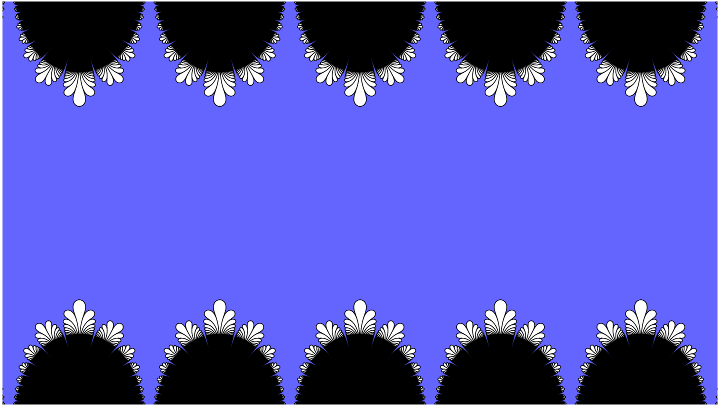
The basin of 0

$$f_0(z) = \sin(z)/2$$



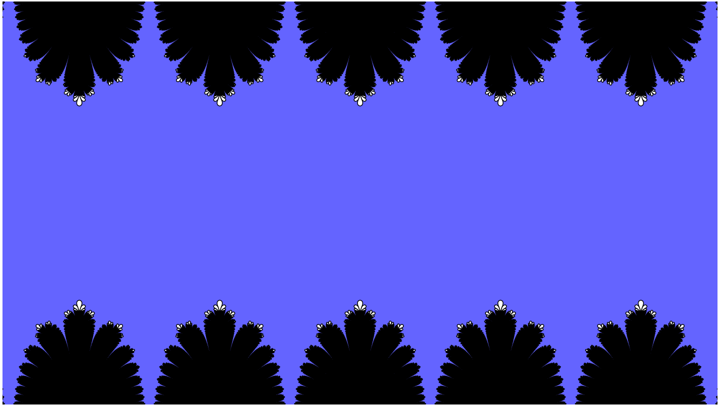
The basin of 0

$$f_0(z) = \sin(z)/2$$



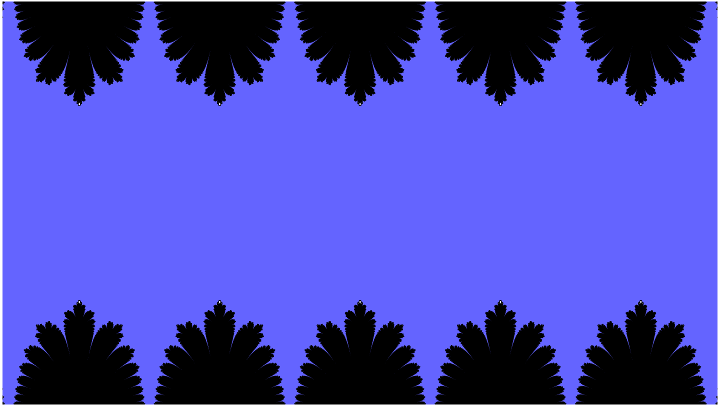
The basin of 0

$$f_0(z) = \sin(z)/2$$



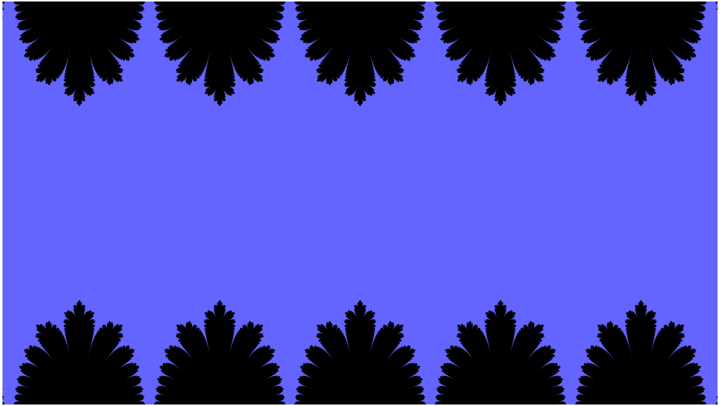
The basin of 0

$$f_0(z) = \sin(z)/2$$



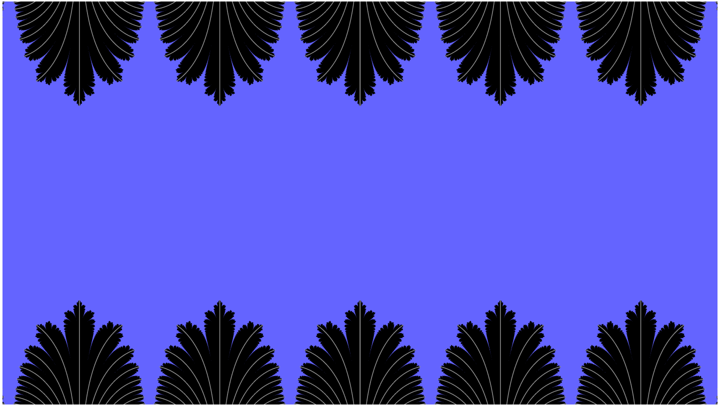
The basin of 0

$$f_0(z) = \sin(z)/2$$



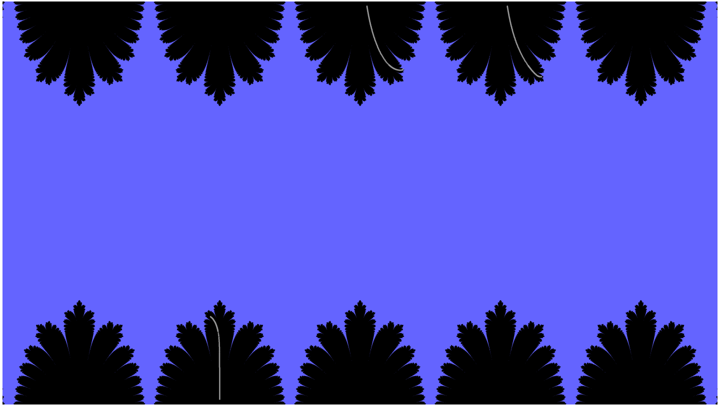
The basin of 0

$$f_0(z) = \sin(z)/2$$



The basin of 0

$$f_0(z) = \sin(z)/2$$



Eremenko's conjecture

Eremenko's conjecture (1989): Every *connected component* of the *escaping set*

$$I(f) := \{z \in \mathbb{C} \colon f^n(z) \to \infty$$

of an analytic function $f \colon \mathbb{C} \to \mathbb{C}$ is *unbounded*.

- *Central problem* in transcendental dynamics.
- Rippon–Stallard 2011: $I(f) \cup \{\infty\}$ is *connected*.
- Rottenfußer-Rückert-R.-Schleicher 2011: There exists f such that $I(f)$ *contains no curve*.

Eremenko's conjecture

Eremenko's conjecture (1989): Every *connected component* of the *escaping set*

$$I(f) := \{z \in \mathbb{C} \colon f^n(z) \to \infty$$

of an analytic function $f \colon \mathbb{C} \to \mathbb{C}$ is *unbounded*.

- *Central problem* in transcendental dynamics.
- Rippon–Stallard 2011: $I(f) \cup \{\infty\}$ is *connected*.
- Rottenfußer-Rückert-R.-Schleicher 2011: There exists f such that $I(f)$ *contains no curve*.

Eremenko's conjecture

Eremenko's conjecture (1989): Every *connected component* of the *escaping set*

$$I(f) := \{z \in \mathbb{C} \colon f^n(z) \to \infty$$

of an analytic function $f \colon \mathbb{C} \to \mathbb{C}$ is *unbounded*.

- *Central problem* in transcendental dynamics.
- Rippon–Stallard 2011: $I(f) \cup \{\infty\}$ is *connected*.
- Rottenfußer-Rückert-R.-Schleicher 2011: There exists f such that $I(f)$ *contains no curve*.

Eremenko's conjecture

Eremenko's conjecture (1989): Every *connected component* of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty\}$$

of an analytic function $f : \mathbb{C} \to \mathbb{C}$ is *unbounded*.

- *Central problem* in transcendental dynamics.
- Rippon–Stallard 2011: $I(f) \cup \{\infty\}$ is *connected*.
- Rottenfußer-Rückert-R.-Schleicher 2011: There exists f such that $I(f)$ *contains no curve*.

Resolution of Eremenko's conjecture

Theorem (Martí-Pete, R., Waterman, 2022)

There exists a transcendental entire function such that $\{0\}$ is a connected component of $I(f)$.

- The proof uses tools of classical complex function theory (approximation theory).
- Gives a general procedure for constructing entire functions with *interesting properties*.
- This method resolves a number of other *long-standing open problems*.

Resolution of Eremenko's conjecture

Theorem (Martí-Pete, R., Waterman, 2022)

There exists a transcendental entire function such that $\{0\}$ *is a connected component of* $I(f)$.

- The proof uses tools of classical complex function theory (approximation theory).
- Gives a general procedure for constructing entire functions with *interesting properties*.
- This method resolves a number of other *long-standing open problems*.

Resolution of Eremenko's conjecture

Theorem (Martí-Pete, R., Waterman, 2022)

There exists a transcendental entire function such that $\{0\}$ *is a connected component of* $I(f)$.

- The proof uses tools of classical complex function theory (approximation theory).
- Gives a general procedure for constructing entire functions with *interesting properties*.
- This method resolves a number of other *long-standing open problems*.

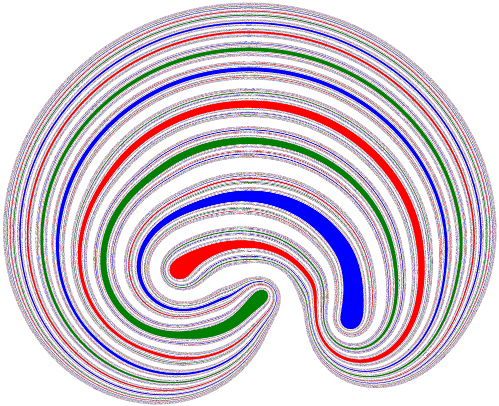
Resolution of Eremenko's conjecture

Theorem (Martí-Pete, R., Waterman, 2022)

There exists a transcendental entire function such that $\{0\}$ is a connected component of $I(f)$.

- The proof uses tools of classical complex function theory (approximation theory).
- Gives a general procedure for constructing entire functions with *interesting properties*.
- This method resolves a number of other *long-standing open problems*.

Resolution of Eremenko's conjecture



Thank you!