Neutrino Mixing Models For Non-Hermitian Quantum Mechanics

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Project Aim

Aim: Consider the oscillation probability for a non-Hermitian quantum mechanics (QM) model of neutrinos with Dirac type mass.

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What is is a neutrino?

Problem (1930): e^- released in beta decay with a continuous energy spectrum. But in QM: e^- energy quantised.

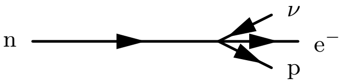
Solution:

- **Bohr** Abandon conservation laws on a fundamental level - only true statistically.
- **Pauli** Add a new particle - only currently three fundamental particles p, e and γ.

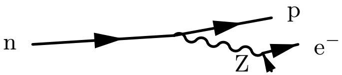
(Abridged) History of Neutrinos

Theoretical Advances:

- **Fermi (1933)** : Beta decay process of neutron decaying to proton to stabilise atomic nuclei



- **Electro-Weak (196X):** Left-handed neutrinos are charged under electro-weak gauge group with anti-neutrinos as distinct objects



- **Standard Model (197X):** Three massless neutrino flavours only charged under the electro-weak interaction.

Why am I talking about neutrino mass?

If neutrinos are massless why am I talking about their mass?

Problem: Measurements of solar or atmospheric neutrinos result in too few neutrinos of the expected type and too many of the others.

Solution:

- Pontecorvo (1957): Neutrinos oscillate between types.

But this requires massive neutrinos.

Oscillation Probability Calculation 1

- Suppose we have two neutrino flavours $\{|\nu_e\rangle\,,\ |\nu_\mu\rangle\}$.
- Solar neutrinos = electron-neutrino state $|\nu_e\rangle$
- States evolve according to equation:

$$|\nu(t)\rangle = e^{i\bar{H}t}\,|\nu(0)\rangle\,, \tag{1}$$

  $\bar{H}$ is Hamiltonian operator whose e-vals are possible energy levels.
- Define mixing matrix U by

$$|\nu_i\rangle = U^*_{ia}\,|\nu_a\rangle \tag{2}$$

  where $i = \{e,\ \mu\}$ and $i = \{1,\ 2\}$ are energy eigenstate labels.
- Such that

$$|\nu_a(t)\rangle = e^{iE_a t}\,|\nu_a(0)\rangle\,. \tag{3}$$

$(\hbar = c = 1)$

Oscillation Probability Calculation 2

- Probability calculated by

$$P_{k\to l}(t,0) = |\langle \nu_l(t)|\nu_k(0)\rangle|^2. \quad (4)$$

and so expect $P_{l\to l}(t,t) = 1$. Dirac inner product with conjugate basis $\langle \nu| = (|\nu\rangle)^\dagger$.

- Neutrinos (if massive) are extremely light → use relativisitic approximation ($E_i \approx p + \frac{m_i}{2E}$, $t \approx L$):

$$|\nu_e(t)\rangle = e^{-iE_a t} U^*_{ea} |\nu_a\rangle \approx |\nu_e(L)\rangle \approx e^{-iE_1 t}\Big[U^*_{ea} \exp\Big(-i\frac{m_a^2 - m_1^2}{2E} L\Big)$$

- Mixing probability

$$P_{e\to\mu}(L) \approx \sum_{a,b} |U_{ea}U^*_{\mu a}U^*_{eb}U\mu b|^2 \cos\Big[\frac{L}{2}(m_a^2 - m_b^2) - arg(U_{ea}U^*_{\mu a}U^*_{eb}U\mu b)\Big]$$

- For Dirac neutrinos where U is a rotation matrix with two flavours

$$P_{e\to\mu}(L) \approx \sin^2(2\theta)\sin^2\Big(\frac{|m_1^2 - m_2^2|}{4E}L\Big) \quad (6)$$

Neutrino Mass Problem

Naturalness Problem: Neutrino mass is too small

Cosmological Measurements: $\sum_\nu m_\nu < 0.23$ eV

Heaviest Fermion: 172 GeV, Lightest Fermion: 0.5 MeV

Questions:

- What mechanism can naturally lead to such a small mass?
- What about the neutrino leads to this small mass?

Mass terms

- System dynamics described by a functional called the Lagrangian $\mathcal{L}$.
- $\mathcal{L}$ obeys all symmetries of the system.
- Classical dynamics give stationary action: $S = \int d^4x\mathcal{L}$
- Quantum dynamics from path integral: $Z(J) = \int D\phi \exp(i(S[\phi] + \int d^4xJ(x)\phi(x))$
- Mass term is non-derivative bi-linear term in Lagrangian - skirting Higgs mechanism.

Dirac Mass

- Dirac Equation : $(i\gamma_\mu \partial^\mu - m)\psi = 0$ where $\gamma^\mu = \left\{ \begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & -\sigma_i \\ \sigma_i & \mathbf{0}_2 \end{pmatrix} \right\}$
- Two-component spinor notation:$\psi = \begin{bmatrix} \xi_{\alpha,1} \\ \xi_2^{\dagger\dot{\alpha}} \end{bmatrix}$.
- Free Dirac mass Lagrangian (i=flavour index:

$$\mathcal{L}_D = i\xi^{\dagger i}_{\dot{\alpha},1}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \xi_{\alpha,1i} + i\xi^{\dagger}_{\dot{\alpha},2i}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \xi^i_{\alpha,2} - m_D^{ij}(\xi^\alpha_{1i}\xi_{\alpha,2j} + \xi^\dagger_{\dot{\alpha},1i}\xi^{\dagger\dot{\alpha}}_{2j}), \tag{7}$$

- Write as mass matrix: $M = \begin{bmatrix} \mathbf{0}_N & \boldsymbol{m}_D \\ \boldsymbol{m}_D^\dagger & \mathbf{0}_N \end{bmatrix}$
- All massive Fermions have Dirac mass.

Majorana Mass

- Majorana Field: $\psi = \gamma_0 C \psi^*$, C=Unitary transformation.
- Majorana spinor field: $\psi = \begin{bmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{bmatrix}$.
- Free Majorana field Lagrangian:

$$\mathcal{L}_M = i\xi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \xi_\alpha - \frac{m_M}{2}(\xi^\alpha \xi_\alpha + \xi^\dagger_{\dot{\alpha}} \xi^{\dagger\dot{\alpha}}) \tag{8}$$

- Only connects a single Weyl spinor to itself - Dirac mass connects two Weyl spinors to each other.
- Weak force is $\mathcal{CP}$ violating and only involves LH Weyl spinors $\rightarrow$ could give a right-handed neutrino a Majorana mass term.

Seesaw Mechanism

- Give Dirac mass term a natural mass scale through Higgs-Yukawa mechanism.
- Give right-handed neutrino a Majorana mass term on the scale of some new physics $m_M \gg m_D$.
- Gives mass matrix $\begin{bmatrix} 0 & m_D \\ m_D & m_M \end{bmatrix}$.
- Mass eigenvalues given potential mass measurements:

$$\lambda_\pm = \frac{m_M \pm m_M \sqrt{1 + 4\frac{m_D^2}{m_M}}}{2} \tag{9}$$

and thus $\lambda_+ \approx m_M$ and $\lambda_- \approx -\frac{m_D^2}{m_M}$.

- Regular neutrinos dominated by light mass eigenstate. Heavy eigenstate gives sterile neutrinos.

Mixing Matrix

- Measurable quantities are given by Hermitian operators e.g. $M = M^\dagger$
- Diagonalised by unitary transformations $\hat{M} = U^\dagger M U$ where $UU^\dagger = \mathbf{1}$.
- $N \times N$ unitary matrix has N^2 components, $\frac{N(N-1)}{2}$ real d.o.f and $\frac{N(N+1)}{2}$ complex phases.
- Dirac mass: Can rephase $2N-1$ complex phases.
- Majorana mass: Can rephase N complex phases.
- $\mathbf{U} = \mathbf{VK}$, where

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{10}$$

$$\mathbf{K} = \mathrm{diag}(1, e^{i\phi_1}, e^{i(\phi_2+\delta)}). \tag{11}$$

Testing Majorana Mass

Majorana Mass: Can explain small mass and provide additional $\mathcal{CP}$ violation as well as providing a candidate for part of dark matter contribution.

How do we test this? Neutrinoless double beta decay

Sterile neutrino is its own anti-particle allowing for lepton flavour violating interactions.
Bounds on similar interaction:
$T^{0\nu}_{\frac{1}{2}}(^{130}\mathrm{Te}) > 1.5 \times 10^{25}\mathrm{yrs}$

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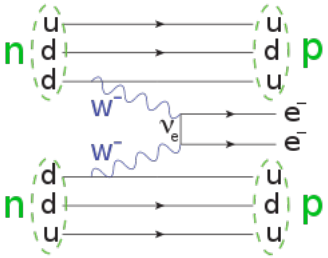

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Project Aim

Aim: Consider the oscillation probability for a **non-Hermitian quantum mechanics** (QM) model of neutrinos with Dirac type mass.

Non-Hermtian Quantum Mechanics

Some of the assumptions of QM:

- Locality = Causality $\Rightarrow$ Physical
- Lorentz invariant = special relativity $\Rightarrow$ Physical
- Hermitian operators = real e-vals + unitarity $\Rightarrow$ both individually physical

Hermiticity is sufficient but not neccessary for this.
Real e-vals exist for matrices with $\mathcal{PT}$-symmetry.
Call region where e-vals are real unbroken $\mathcal{PT}$-symmetric regime.
Unitarity regained using $\mathcal{C}'\mathcal{PT}$ inner product.

Why consider non-Hermitian mass?

- No mechanism for small neutrino mass is verified.
- Non-Hermitian quantum mechanics field of growing interest in other areas, e.g. optics
- Neutrino physics could provide a testbed for understanding how non-Hermtian theories work in QFT.

The Model

We have a Lagrangian

$$\mathcal{L} = \nu^\dagger \mathbf{A} \nu - g \nu_L^\dagger \gamma^0 W_\mu \gamma^\mu e_L - g e_L^\dagger \gamma^0 \gamma^\mu W_\mu \nu_L. \tag{12}$$

where $\mathbf{A} = \begin{bmatrix} 0 & m-\mu \\ (m+\mu)^\dagger & 0 \end{bmatrix}$.

Matrix diagonalised by a similarity not a unitary transformation $\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$,

$$V = \begin{bmatrix} \mathbf{V_A} & \mathbf{V_B} \\ \mathbf{V_C} & \mathbf{V_D} \end{bmatrix}, \; V^{-1} = \begin{bmatrix} \mathbf{\Lambda_A} & \mathbf{\Lambda_B} \\ \mathbf{\Lambda_C} & \mathbf{\Lambda_D} \end{bmatrix}. \tag{13}$$

Lagrangian can be rewritten in mass eigenbasis

$$\mathcal{L} = (\tilde{\nu^\dagger}) \hat{\mathbf{A}} \tilde{\nu} - g (\tilde{\nu^\dagger})_L \mathbf{V_N}^{\mathbf{-1}} \gamma^0 W_\mu \gamma^\mu e_L - g e_L^\dagger \gamma^0 \gamma^\mu W_\mu \mathbf{V_N} \tilde{\nu}_L. \tag{14}$$

Only left chiral components in interaction thus $\mathbf{V_A}$ and $\mathbf{\Lambda_A}$ used in interaction.

$\mathcal{PT}$ Symmetry

One generation:

$$\mathbf{A} = \begin{bmatrix} 0 & m-\mu \\ m+\mu & 0 \end{bmatrix}, \tag{15}$$

for now assume m, μ are real. Matrix eigenvalues

$$\lambda_{\pm} = \pm\sqrt{m^2 - \mu^2}. \tag{16}$$

Thus our reality condition is $|m| \geq |\mu|$.
This matrix diagonalises by $\hat{\mathbf{A}} \equiv \mathbf{V_1}^{-1}\mathbf{A}\mathbf{V_1}$

$$\mathbf{V_1} = \begin{bmatrix} a & -\frac{\det(\mathbf{V_1})}{2a}\left(\frac{m-\mu}{m+\mu}\right)^{\frac{1}{2}} \\ a\left(\frac{m-\mu}{m+\mu}\right)^{\frac{1}{2}} & \frac{\det(\mathbf{V_1})}{2a} \end{bmatrix}. \tag{17}$$

$\mathcal{PT}$ Symmetry

Consider enforcing $\mathcal{PT}$ symmetry on the mass matrix.
Fermion fields transform according to:

$$\mathcal{P}: \quad \psi(x,t) \to \gamma^0 \psi(-x,t), \qquad \bar{\psi}(x,t) \to \bar{\psi}(-x,t)\gamma^0$$

$$\mathcal{T}: \quad \psi(x,t) \to i\gamma^1\gamma^3\psi^*(x,-t), \qquad \bar{\psi}(x,t) \to i\psi^T(x,-t)\gamma^1\gamma^3\gamma^0, \qquad \mathbf{A} \to \mathbf{A}^*$$

Writing mass matrix as $\mathbf{A} = \mathbb{P}_{\mathrm{L}} \otimes (\mathbf{m}^\dagger + \boldsymbol{\mu}^\dagger) + \mathbb{P}_{\mathrm{R}} \otimes (\mathbf{m} - \boldsymbol{\mu})$ and using the property $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$:

$$\begin{aligned} \mathcal{PT}: \quad & \bar{\psi}(x,t)\mathbf{A}\psi(x,t) \\ \to \quad & \mathbb{P}_{\mathrm{L}} \otimes (\mathbf{m}^\dagger + \boldsymbol{\mu}^\dagger) + \mathbb{P}_{\mathrm{R}} \otimes (\mathbf{m} - \boldsymbol{\mu}). \end{aligned}$$

i.e. $\mathcal{PT}$-symmetry requires $\mathbf{m} = \mathbf{m}^\dagger$ and $\boldsymbol{\mu} = \boldsymbol{\mu}^\dagger$.

Mass eigenvalues

Define: $\mathbf{m}_- = (\mathbf{m} - \boldsymbol{\mu})$ and $\mathbf{m}_+ = (\mathbf{m} + \boldsymbol{\mu})$, noting $\mathbf{m}_\pm = \mathbf{m}_\pm^\dagger$.
Want to find mass matrix eigenvalues

$$\det(\mathbf{A} - \lambda 1_{2N}) = \det \begin{bmatrix} -\lambda 1_N & \mathbf{m}_- = \\ \mathbf{m}_+ & -\lambda 1_N \end{bmatrix} = 0. \quad (18)$$

Determinant of block matrix found using

$$\det(\mathbf{A} - \lambda 1_{2N}) = \det(-\lambda 1_N)\det(-\lambda 1_N - \mathbf{m}_-(-\lambda 1_N)^{-\mathbf{1}}\mathbf{m}_+). \quad (19)$$

Which results in a matrix $\mathbf{M^2} = \mathbf{m}_-\mathbf{m}_+$ obeying $\det(\mathbf{M^2} - \lambda^2 1_N) = 0$.
Each mass eigenvalue will come in a pair with same magnitude and opposite sign

$$\hat{\mathbf{A}} = \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0_N} \\ \mathbf{0_N} & -\boldsymbol{\lambda} \end{bmatrix}. \quad (20)$$

Unbroken $\mathcal{PT}$-phase

Consider the reality conditions on our matrix, from characteristic equation

$$\lambda^2 = \frac{\mathrm{tr}(\mathbf{M^2})}{2} \pm \sqrt{\frac{\mathrm{tr}(\mathbf{M^2})^2}{4} - \det(\mathbf{M^2})}. \tag{21}$$

Therefore we have two conditions, which can be written as either:

1. $$0 < \det(\mathbf{M^2}) < \frac{1}{4}\mathrm{tr}(\mathbf{M^2})^2. \tag{22}$$

2. $$\mathrm{tr}(\mathbf{M^4}) < (\mathrm{tr}(\mathbf{M^2}))^2 < 2\mathrm{tr}(\mathbf{M^4}). \tag{23}$$

Since $\mathbf{m}_\pm$ are Hermitian $\det(\mathbf{M^2})$ and $\mathrm{tr}((\mathbf{M^2})^n)$ are real.

Other bases

Consider diagonalising $\mathbf{m}_-$ with similarity transformation U through Ω - similarity transformations do not change e-vals

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{U} & \mathbf{0_N} \\ \mathbf{0_N} & \mathbf{U} \end{bmatrix}, \tag{24}$$

then can write

$$\mathbf{A} = \begin{bmatrix} \mathbf{0_N} & \hat{\mathbf{m}}_- \\ \mathbf{U_+}^\dagger \hat{\mathbf{m}}_+ \mathbf{U_+} & \mathbf{0_N} \end{bmatrix}, \tag{25}$$

where U_+ is mixing matrix between $\mathbf{m}_-$ and $\mathbf{m}_+$ bases.
Writing the eigenvalues of $\mathbf{m}_-$ and $\mathbf{m}_+$ as m_{-i} and m_{+i} respectively, then

$$\mathrm{tr}(\mathbf{M^2}) = m_{-1}m_{+1} + m_{-2}m_{+2} - (m_{-1} - m_{-2})(m_{+1} - m_{+2})\mathrm{s}_+^2. \tag{26}$$

$$\det(\mathbf{M^2}) = m_{-1}m_{-2}m_{+1}m_{+2}, \tag{27}$$

Note that the complex phases have dropped out the constraint.

Using form of $\hat{\mathbf{A}}$ and block structure of mixing matrix

$$\begin{bmatrix} \mathbf{V_A} & \mathbf{V_B} \\ \mathbf{V_C} & \mathbf{V_D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} & \mathbf{0_N} \\ \mathbf{0_N} & -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0_N} & \mathbf{m_-} \\ \mathbf{m_+} & \mathbf{0_N} \end{bmatrix} \begin{bmatrix} \mathbf{V_A} & \mathbf{V_B} \\ \mathbf{V_C} & \mathbf{V_D} \end{bmatrix}, \tag{28}$$

we find that the full mixing matrix must have the form

$$\mathbf{V_N} = \begin{bmatrix} \mathbf{V_A} & \alpha \mathbf{V_A} \\ \gamma(\mathbf{V_A}^{-\mathbf{1}})^\dagger & -\alpha\gamma(\mathbf{V_A}^{-\mathbf{1}})^\dagger \end{bmatrix}. \tag{29}$$

with inverse

$$\mathbf{V_N}^{-\mathbf{1}} = \begin{bmatrix} \boldsymbol{\Lambda}_\mathbf{A} & \boldsymbol{\Lambda}_\mathbf{B} \\ \boldsymbol{\Lambda}_\mathbf{C} & \boldsymbol{\Lambda}_\mathbf{D} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{V_A}^{-\mathbf{1}} & \frac{1}{2\gamma}\mathbf{V_A}^\dagger \\ \frac{1}{2\alpha}\mathbf{V_A}^{-\mathbf{1}} & -\frac{1}{2\alpha\gamma}\mathbf{V_A}^\dagger \end{bmatrix}, \tag{30}$$

Interaction Mixing Matrix

For the interaction we are particularly interested in $\mathbf{V_A}$ where $\mathbf{V_A}\boldsymbol{\lambda} = \boldsymbol{M}^2\mathbf{V_A}$. For the two-flavour case, defining

$$\mathbf{V_A} \equiv \begin{bmatrix} |V_{A11}|\mathrm{e}^{ia_{11}} & V_{A12} \\ V_{A21} & |V_{A22}|\mathrm{e}^{ia_{22}} \end{bmatrix} \quad \text{and} \quad \mathbf{M^2} \equiv \begin{bmatrix} M_{11}^2 & M_{12}^2 \\ M_{21}^2 & M_{22}^2 \end{bmatrix}. \tag{31}$$

$$\begin{aligned} M_{11}^2 &= \mathrm{sgn}(M_{11}^2)|M_{11}^2|, \quad M_{12}^2 = \mathrm{sgn}(M_{12}^2)|M_{12}^2|\mathrm{e}^{-i(\phi_{+1}-\phi_{+2})}, \\ M_{22}^2 &= \mathrm{sgn}(M_{22}^2)|M_{22}^2|, \quad M_{21}^2 = \mathrm{sgn}(M_{21}^2)|M_{21}^2|\mathrm{e}^{i(\phi_{+1}-\phi_{+2})}, \end{aligned} \tag{32}$$

Resulting in mixing matrx

$$\mathbf{V_A} = \begin{bmatrix} |V_{A11}|\mathrm{e}^{ia_{11}} & \mathrm{sgn}(V_{A12})|V_{A12}|\mathrm{e}^{i(a_{22}-(\phi_1-\phi_2))} \\ \mathrm{sgn}(V_{A21})|V_{A21}|\mathrm{e}^{i(a_{11}+\phi_1-\phi_2)} & |V_{A22}|\mathrm{e}^{ia_{22}} \end{bmatrix}. \tag{33}$$

Rephasing

Unbroken $\mathcal{PT}$ symmetry gave no additional constraints on complex phases $\Rightarrow$ phases can be arbitrarily rephased as in the Dirac mass case
In general we have the flavour structure

$$\begin{pmatrix} e^{-i\phi_{e1}} & 0 \\ 0 & e^{-i\phi_{e2}} \end{pmatrix} \begin{pmatrix} |V_{11}|e^{i\phi_{11}} & |V_{12}|e^{i\phi_{12}} \\ |V_{21}|e^{i\phi_{21}} & |V_{22}|e^{i\phi_{22}} \end{pmatrix} \begin{pmatrix} e^{i\phi_{\nu 1}} & 0 \\ 0 & e^{i\phi_{\nu 2}} \end{pmatrix} . \tag{34}$$

We can eliminate three out of four complex phases from the mixing matrix by taking

$$\phi_{e1} = \phi_{11} + \phi_{\nu 1} \ , \qquad \phi_{e2} = \phi_{22} + \phi_{\nu 2} \ , \qquad \phi_{\nu 1} - \phi_{\nu 2} = \phi_{22} - \phi_{21} \ , \tag{35a}$$

leading to

$$\begin{pmatrix} |V_{11}| & |V_{12}|e^{i\phi} \\ |V_{21}| & |V_{22}| \end{pmatrix} , \tag{36}$$

where

$$\phi = \phi_{12} + \phi_{21} - \phi_{11} - \phi_{22} = 0 \ \text{ in our case..} \tag{37}$$

What about oscillation probability?

Problem: Simple calculations of oscillation probability are leading to unphysical probabilities.

Solution: Use complementary bases to span eigenspace.

Define flavour kets

$$\mathcal{L} \;=\; \partial_\alpha \tilde{\Phi}^\dagger \partial^\alpha \Phi - \tilde{\Phi}^\dagger M^2 \Phi \,, \tag{38}$$

where $\Phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ with ϕ_1 a scalar and ϕ_2 a pseudo-scalar.
We then have $\mathcal{PT}$-symmetric mass matrix

$$M^2 \;=\; \begin{bmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{bmatrix} \;\neq\; (M^2)^\dagger \,. \tag{39}$$

with eigenvalues

$$m_\pm^2 = \frac{1}{2}(m_1^2 + m_2^2) \pm \frac{1}{2}\sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4} \tag{40}$$

and energy eigenvectors

$$\mathbf{e}_+ \;=\; N \begin{bmatrix} \eta \\ -1 + \sqrt{1-\eta^2} \end{bmatrix},\; \mathbf{e}_- = N \begin{bmatrix} -1 + \sqrt{1-\eta^2} \\ \eta \end{bmatrix} \;. \tag{41a}$$

Scalar Model 2

We find flavour kets

$$|\phi_{1,\vec{p}}(x)\rangle \;=\; \cosh(\theta)\,\xi_{+,\vec{p}}(x)\,\mathbf{e}_{+} + \sinh(\theta)\,\xi_{-,\vec{p}}(x)\,\mathbf{e}_{-}\,, \tag{42a}$$

$$|\phi_{2,\vec{p}}(x)\rangle \;=\; \cosh(\theta)\,\xi_{-,\vec{p}}(x)\,\mathbf{e}_{-} + \sinh(\theta)\,\xi_{+,\vec{p}}(x)\,\mathbf{e}_{+}\,, \tag{42b}$$

where $\theta = \frac{1}{2}\text{arctanh}(\eta)$, $\eta = \frac{2\mu^2}{|m_1^2 - m_2^2|}$.
$\mathcal{C}'^{\mathcal{PT}}$ conjugate states are given by $\langle\tilde{\phi}_i(t)|$ for the flavour states and $\mathbf{e}^{\S}$ for the energy states, we have

$$\mathbf{e}^{\S}_{\pm}\mathbf{e}_{\pm} \;=1\,, \qquad \mathbf{e}^{\S}_{\pm}\mathbf{e}_{\mp} \;=\; 0\,, \tag{43}$$

but

$$\langle\phi_i^{\mathcal{C}'\mathcal{PT}}(t)|\phi_j(t)\rangle \;=\; \begin{cases} \cosh(2\theta), & i \;=\; j \\ \sinh(2\theta), & i \;\neq\; j. \end{cases} \tag{44}$$

Scalar Model 3

Hamiltonian is $\mathcal{C}'$ symmetric $\Rightarrow$ could span the space with either $\{|\phi_1\rangle, |\phi_2\rangle\}$ or $\{|\phi_1^{\mathcal{C}'}\rangle, |\phi_2\rangle\}$.
We find

$$\langle\phi_1^{\mathcal{C}'\mathcal{PT}}(t)|\phi_1(t)\rangle = 1\,, \quad \langle\phi_2^{\mathcal{PT}}(t)|\phi_2^{\mathcal{C}'}(t)\rangle = 1\,, \tag{45a}$$
$$\langle\phi_1^{\mathcal{C}'\mathcal{PT}}(t)|\phi_2^{\mathcal{C}'}(t)\rangle = 0\,, \quad \langle\phi_2^{\mathcal{PT}}(t)|\phi_1(t)\rangle = 0, \tag{45b}$$

which results in probabilities

$$\mathbb{P}_{1(2)\to 1(2)}(t, t_0) = 1 - \eta^2 \sin^2\left[\Delta\omega\Delta t/2\right]\,, \tag{46a}$$
$$\mathbb{P}_{1(2)\to 2(1)}(t, t_0) = \eta^2 \sin^2\left[\Delta\omega\Delta t/2\right]\,, \tag{46b}$$

where $\Delta\omega = \omega_1 - \omega_2$ and $\Delta t = t - t_0$.

Thank You for Your Attention!