Short time existence and smoothness of the nonlocal mean curvature flow of graphs

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Plan

1. Mean curvature flow

2. Nonlocal mean curvature

3. Short time existence and smoothness of the nonlocal mean curvature flow of graphs

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Mean curvature flow

- *W.W. Mullins (1956). Two-dimensional motion of idealized grain boundaries. Journal of Applied Physics, 27(8), 900-904.*
- Progrès ▶ 1980.
- *K. A. Brakke, (1978). The motion of a surface by its mean curvature, in Math. Notes, Princeton Univ. Press, Princeton, NJ.*

Mean curvature flow

- Differential geometry,
- Partial diefferential equations,
- Stochastic control,
- Mathematical finance ...

Some applications

- Industrial transformation of metals,
- Crystal growth,
- Image processing ...

Mean curvature flow

Definition 1.1

We will say that the boundary of E_0 is moving by mean curvature if $\{E_t\}_{0\le t\le T}$ of $\mathbb{R}^N$ such that

$$\begin{cases} \partial_t X(t)\cdot\nu(X(t)) = -H(X(t)), \forall X(t)\in\partial E_t,\ t\in[0,T] \\ X(0)=X_0\in\partial E_0, \end{cases} \tag{1}$$

where

- $\nu(X(t))$ *is the unit normal vector to* ∂E_t *at* $X(t)$*,*
- $H(X(t))$ *is the mean curvature of* ∂E_t *at* $X(t)$*, and*
- $v := \partial_t X(t)\cdot\nu(X(t))$ *is the normal velocity at* $X(t)$*.*

Mean curvature flow

In two dimensional case $\Longrightarrow$ curve shortening flow.

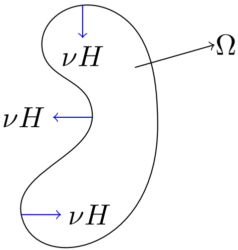

Figure: A domain Ω with its mean curvature vector νH.

Formation of singularities

Example 1.1 (Evolution of the circle S^1 $(N = 2)$)

The evolution of the circle $S^1_{r(t)}$ *is given by* $r(t) = \sqrt{r_0^2 - 2t}$, *where* $t \in (-\infty, \frac{r_0^2}{2})$.

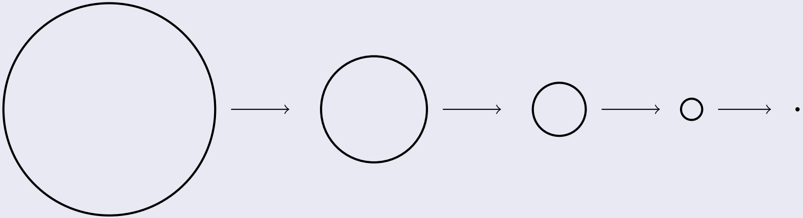

Figure: *Shrinking circle* $S^1 \subset \mathbb{R}^2$.

Formation of singularities

Example 1.2 (Evolution of the sphere S^2 ($N = 3$))

Similarly, the evolution of the sphere $S^2_{r(t)}$ is given by $r(t) = \sqrt{r_0^2 - 4t}$, where $t \in (-\infty, \frac{r_0^2}{4})$.

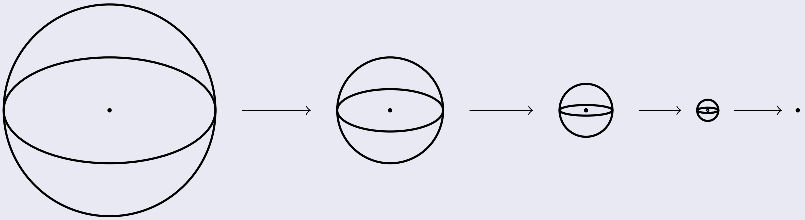

Figure: *Shrinking sphere $S^2 \subset \mathbb{R}^3$.*

Formation of singularities

- Convex closed hypersurfaces, Gerhard Huisken ($N \geq 3$) and Gage Michael and S. Richard Hamilton ($N = 2$).
- Nonconvex hypersurfaces, *M. A. Grayson, (1987). The heat equation shrinks embedded plane curves to round points. Journal of Differential geometry, 26(2), 285-314.*
- Compact hypersurfaces, *R. Alessandroni, (2008). Introduction to mean curvature flow. Séminaire de théorie spectrale et géométrie, 27, 1-9.*

Evolution of graphs

For $\forall t \geq 0$, $\partial E_t = \mathrm{graph}(u(t,\cdot))$, $u(t,\cdot) : \Omega \subseteq \mathbb{R}^{N-1} \to \mathbb{R}$. Then,

$$\partial_t u = \sqrt{1+|\nabla u|^2}\, \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \qquad u(0,\cdot) = u_0 \tag{2}$$

where $H = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$.

Theorem 1.1 (Klaus Ecker and Gerhard Huisken)

Let ∂E_0 be a locally Lipschitz continuous graph. Then, the initial value problem (2) has a smooth solution ∂E_t for all $t > 0$. Moreover, each ∂E_t is a graph.

Evolution of graphs

Example 1.3 (Grim reaper)

For $N = 2$, $u(t,\cdot) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ *given by* $u(t,x) = t - \log(\cos(x))$ *is an explicit solution of the equation* (2).

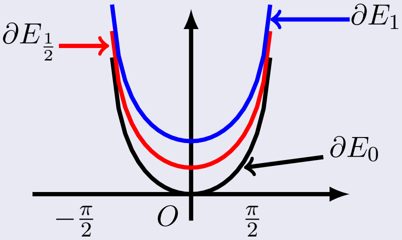

Figure: *Evolution of the graph of the function* $u(t,x) = t - \log(\cos(x))$.

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Nonlocal mean curvature

$M \subset \mathbb{R}^N$ ia a hypersurface of class $C^{1,\beta}$ for some $\beta > s$. $\forall x \in M$

$$H^s(x) = \frac{2}{s} \int_M \frac{(y-x)\cdot \nu_M(y)}{|y-x|^{N+s}} d\sigma_M(y), \quad s \in (0,1). \tag{3}$$

The integral in (3) is absolutely convergent in the Lebesgue sense if

$$\int_M \frac{1}{(1+|y|)^{N+s-1}} d\sigma_M(y) < \infty. \tag{4}$$

Nonlocal mean curvature

Convergence to the classical mean curvature as $s \to 1$

- The nonlocal mean curvature of $M = \partial E \subset \mathbb{R}^N$

$$H^s(x) := (1-s)\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|y-x|^{N+s}} \mathrm{d}y. \tag{5}$$

  where $\mathbb{1}_A$ is characteristic function of A.

- The classical mean curvature of $M = \partial E$ (of class C^2)

$$H(x) = C_N \lim_{r \longrightarrow 0} \frac{-1}{r|B_r(x)|} \int_{B_r(x)} \Big(\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)\Big) dy. \tag{6}$$

- $$H^s \longrightarrow H, \quad s \nearrow 1.$$

Nonlocal mean curvature

Convergence as $s \to 0$

If $M = \partial E$ where E is of class C^2, then $\forall x \in M$, $H^s(x)$ converges as $s \to 0$ to

$$H^0(x) := \begin{cases} \lim\limits_{R \to +\infty} \lim\limits_{r \to 0^+} H_R^r(x) - N\omega_N \log R, & \text{if } E \subset\subset \mathbb{R}^N \\ \lim\limits_{R \to +\infty} \lim\limits_{r \to 0^+} H_R^r(x) + N\omega_N \log R, & \text{if } E^c \subset\subset \mathbb{R}^N, \end{cases} \tag{7}$$

$$H_R^r(x) := \int_{B_R(x) \setminus B_r(x)} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^N} dy. \tag{8}$$

Example of Nonlocal (fractional) mean curvature

Example 2.1 (Nonlocal (fractional) mean curvature of sphere)

Let $x \in S_R^{N-1}(0)$. *We have*

$$H^s_{S_R^{N-1}(0)}(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{\mathbb{1}_{\mathbb{R}^N \setminus S_R^{N-1}(0)}(y) - \mathbb{1}_{S_R^{N-1}(0)}(y)}{|x-y|^{N+s}} dy$$
$$= R^{-s} H^s_{S_1^{N-1}(0)}(\overline{x}),$$

where

$$H^s_{S_1^{N-1}(0)}(\overline{x}) = \frac{1}{s} \int_{S_1^{N-1}(0)} \frac{1}{|\overline{x}-y|^{N+s-2}} d\sigma_{S_1^{N-1}(0)}(y) < +\infty. \tag{9}$$

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Nonlocal mean curvature of graphs

Let $s \in (0,1)$ and $u : [0,T] \times \mathbb{R}^{N-1} \to \mathbb{R}$, such that for all $t \geq 0$, $u(t,\cdot) \in C^{1+\beta}_{loc}(\mathbb{R}^{N-1})$, where $\beta > s$. Consider

$$\begin{cases} E_u(t) = \{(x(t), y(t)) \in \mathbb{R}^{N-1} \times \mathbb{R} : y(t) < u(t, x(t))\}, \ t > 0 \\ E_u(0) = E_{u_0} = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} : y < u_0(x)\}. \end{cases} \tag{10}$$

By change of variables,

$$H(u)(t,x) := H^s(x(t), u(x(t))) = P.V. \int_{\mathbb{R}^{N-1}} \frac{\mathcal{G}(p_u(t,x,y))}{|x-y|^{N-1+s}} dy, \tag{11}$$

$$p_u(t,x,y) = \frac{u(t,y) - u(t,x)}{|x-y|}, \qquad \mathcal{G}(p) = -\int_{-p}^{p} \frac{d\tau}{(1+\tau^2)^{\frac{N+s}{2}}}. \tag{12}$$

The associated quasilinear equation

$$\nu(X(t)) = \frac{(-\nabla u(t,x(t)),1)}{\sqrt{1+|\nabla u(t,x(t))|^2}}, \quad \forall X(t) \in \partial E_u(t). \tag{13}$$

$$\partial_t X(t) \cdot \nu(X(t)) = \frac{\partial_t u(t,x(t))}{\sqrt{1+|\nabla u(t,x(t))|^2}}. \tag{14}$$

Therefore, the evolution of u is

$$\partial_t u = -\sqrt{1+|\nabla u|^2} H(u), \quad t \in (0,T], \qquad u(0) = u_0. \tag{15}$$

(15) was recently considered by Julin and La Manna (2020) $\Longrightarrow$ starting from a bounded $C^{1,1}$ initial set.

Main results

Soient $\beta \in (s, 1)$, $\rho \in (0, 1)$ et $\gamma_\rho := \beta + \rho(1 + s)$.

Theorem 3.1 (Attiogbé-Fall-Weth (2022))

Let $\nu > 0$. $\forall u_0 \in C^{1+\gamma_\rho}_{loc}(\mathbb{R}^{N-1})$ *with* $\|\nabla u_0\|_{C^{\gamma_\rho}(\mathbb{R}^{N-1})} \leq \nu$, *there exists* $T = T(\rho, s, \beta, \gamma, N, \nu) > 0$ *and* $C_0 = C_0(\rho, s, \beta, \gamma, N, \nu) > 0$ *such that*

$$\begin{cases} \partial_t u + \sqrt{1 + |\nabla u|^2} H(u) = 0 & \text{in } [0, T] \times \mathbb{R}^{N-1} \\ u(0) = u_0 & \text{in } \mathbb{R}^{N-1} \end{cases} \tag{16}$$

admets a unique solution
$u \in C^\rho([0, T], C^{1+\beta}_{loc}(\mathbb{R}^{N-1})) \cap C^{1+\rho}([0, T], C^{\beta-s}_{loc}(\mathbb{R}^{N-1}))$ *satisfying*

$$\|u - u_0\|_{C^\rho([0,T], C^{1+\beta}(\mathbb{R}^{N-1})) \cap C^{1+\rho}([0,T], C^{\beta-s}(\mathbb{R}^{N-1}))} \leq C_0. \tag{17}$$

Main results

Moreover, if $\nabla u_0 \in C^{1+\gamma_\rho}(\mathbb{R}^{N-1})$ then, $\forall \beta' \in (s, \beta)$ there exists $C = (\rho, s, \beta, \gamma, N, \nu, T, \beta') > 0$ such that

$$\|\nabla u\|_{C^\rho([0,T],C^{1+\beta'}(\mathbb{R}^{N-1}))} \leq C\|\nabla u_0\|_{C^{1+\gamma_\rho}(\mathbb{R}^{N-1})}. \tag{18}$$

Theorem 3.2 (Attiogbé-Fall-Weth (2022))

Under the assumptions of Theorem 3.1, we have $u(t, \cdot) \in C^\infty(\mathbb{R}^{N-1})$ for all $t \in (0, T]$. Moreover, for all $\beta' \in (s, \beta)$, $\rho \in (0, \frac{s}{1+s}]$ and $k \in \mathbb{N} \setminus \{0\}$, there exists $C_k = C_k(\rho, s, \beta, \gamma, N, \nu, \beta', T, k) > 0$ such that

$$\|t^k \nabla u\|_{C^\rho([0,T],C^{k+\beta'}(\mathbb{R}^{N-1}))} \leq C_k. \tag{19}$$

Main results

Theorem 3.3 (Attiogbé-Fall-Weth (2022))

Under the assumptions of Theorem 3.1, we have

$$\|\nabla u\|_{L^\infty((0,T)\times\mathbb{R}^{N-1})} \le \|\nabla u_0\|_{L^\infty(\mathbb{R}^{N-1})} \tag{20}$$

and

$$\|\partial_t u\|_{L^\infty((0,T)\times\mathbb{R}^{N-1})} \le \|\sqrt{1+|\nabla u_0|^2}H(u_0)\|_{L^\infty(\mathbb{R}^{N-1})}. \tag{21}$$

Moreover, if $u_0 \in L^\infty(\mathbb{R}^{N-1})$*, then* $\|u\|_{L^\infty((0,T)\times\mathbb{R}^{N-1})} \le \|u_0\|_{L^\infty(\mathbb{R}^{N-1})}$.

Proof of Theorems 3.1, 3.2 and 3.3.

The proof of Theorems 3.1, 3.2 and 3.3 are based on the strongly coutinuous analytic semigroups theory and the maximum principle. □

Main results

Next, we consider the Banach space defined by

$$\mathcal{C}_0^\theta(\mathbb{R}^{N-1}) = \overline{C_c^\infty(\mathbb{R}^{N-1})}^{\|\cdot\|_{C^\theta(\mathbb{R}^{N-1})}} \qquad \text{for } \theta \in \mathbb{R}_+ \setminus \mathbb{N}, \tag{22}$$

endowed with $C^\theta(\mathbb{R}^{N-1})$ norm.
Set

$$E_T = C^\rho([0,T], \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})) \cap C^{1+\rho}([0,T], \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})), \tag{23}$$

endowed with the norm

$$\|\cdot\|_{E_T} = \|\cdot\|_{C^\rho([0,T],C^{1+\beta})} + \|\cdot\|_{C^{1+\rho}([0,T],C^{\beta-s})}. \tag{24}$$

Main results (general case)

Theorem 3.4 (Attiogbé-Fall-Weth (2022))

Let $\nu > 0$ *et* $\gamma_\rho := \beta + \rho(1+s)$. *Then,* $\forall u_0 \in \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$ *with* $\|\nabla u_0\|_{C^{\gamma_\rho}(\mathbb{R}^{N-1})} \le \nu$, *there exists* $T = T(\rho, s, \beta, N, \nu) > 0$, *such that*

$$\begin{cases} \partial_t u + \sqrt{1+|\nabla u|^2} H(u) &= 0 \quad \text{in } [0,T] \times \mathbb{R}^{N-1} \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^{N-1} \end{cases} \tag{25}$$

admets a unique solution $u \in E_T$.
Moreover, there exists $C_0 = C_0(\rho, s, \beta, N, \nu) > 0$ *such that*

$$\|u - u_0\|_{E_T} \le C_0. \tag{26}$$

Sketch of the proof

$$\begin{cases} \partial_t u + \sqrt{1+|\nabla u|^2} H(u) = 0 & \text{dans } [0,T] \times \mathbb{R}^{N-1} \\ u(0) = u_0 & \text{dans } \mathbb{R}^{N-1} \end{cases} \tag{27}$$

The problem (27) becomes

$$\begin{cases} \partial_t u - \mathcal{L}_0 u = F(u) & \text{in } [0,T] \times \mathbb{R}^{N-1} \\ u(0) = u_0 & \text{in } \mathbb{R}^{N-1}, \end{cases} \tag{28}$$

where $\mathcal{L}_0 := -D\mathcal{H}(u_0) : \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$

$$u \mapsto \mathcal{H}(u) := \sqrt{1+|\nabla u|^2} H(u)$$

at u_0 and the nonlinear function F is defined form $\mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$ to $\mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$ by $u \mapsto F(u) = -\mathcal{H}(u) - \mathcal{L}_0 u$.

Sketch of the proof

For all $u_0 \in \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$, our strategy is

1. to prove that the nonlinear function $F : \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$ is of class C^∞.
2. to prove that the linear operator $\mathcal{L}_0 := -D\mathcal{H}(u_0) : \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$ generates a strongly continuous analytic semigroup on $\mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$.
3. to apply Banach's fixed point theorem.

Sketch of the proof: Step 1

We have the following lemma.

Lemma 3.1

For all $u_0 \in \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$,

$$F : \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1}), \qquad F(u) = D\mathcal{H}(u_0)[u] - \mathcal{H}(u)$$

is of class C^∞.

Sketch of the proof: Step 2

On the strongly continuous analytic semigroup generated by $\mathcal{L}_0$

We start by the decomposition of $\mathcal{L}_0$ as $\mathcal{L}_0 = L_1 + L_2 + L_3$, where

$$L_1 u(x) = -\frac{Q(u_0)(x)}{2} \int_{\mathbb{R}^{N-1}} \frac{(2u(x) - u(x+y) - u(x-y))}{|y|^{N+s}} A(x,y)\, dy, \tag{29}$$

$$L_2 u(x) = -Q(u_0)(x) \int_{\mathbb{R}^{N-1}} \frac{(u(x) - u(x+y))}{|y|^{N+s}} B(x,y)\, dy, \tag{30}$$

$$L_3 w(x) = H(u_0)(x) \frac{\nabla u_0(x) \cdot \nabla w(x)}{Q(u_0)(x)} \tag{31}$$

where

Sketch of the proof: Step 2

$$Q(u_0)(x) = \sqrt{1 + |\nabla u_0(x)|^2} \qquad (32)$$

and the kernels $A(x, y)$ and $B(x, y)$ are such that

1. $L_1 : \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$ falls into a class of integrodifferential operators studied by Abels-Kassmann (2009). In particular, they proved that each element of this class is generator of a strongly continuous analytic semigroup on $\mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$.
2. $\forall \epsilon > 0$, $\forall u \in \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$, there exists $C_\epsilon > 0$ such that

$$\|(L_2 + L_3)u\|_{C_0^{\beta-s}(\mathbb{R}^{N-1})} \leq \epsilon \|L_1 u\|_{C_0^{\beta-s}(\mathbb{R}^{N-1})} + C_\epsilon \|u\|_{C_0^{\beta-s}(\mathbb{R}^{N-1})}.$$

Sketch of the proof: Step 2 (end)

By the perturbations of infinitesimal generators of analytic aemigroups theorem by Pazy (1983), we can conclude that $\mathcal{L}_0 = L_1 + L_2 + L_3$ generates a strongly continuous analytic semigroup on $\mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1})$.

The space $\mathcal{D}_{\mathcal{L}_0}(\rho,\infty) = (\mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1}), \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1}))_{\rho,\infty}$

$L := -(-\Delta)^{\frac{1+\sigma}{2}} : \mathcal{C}_0^{1+\alpha}(\mathbb{R}^{N-1}) \to \mathcal{C}_0^{\alpha-\sigma}(\mathbb{R}^{N-1})$ generates a strongly continuous analytic semigroup on $E = \mathcal{C}_0^{\alpha-\sigma}(\mathbb{R}^{N-1})$ with $\mathcal{D}(L) = \mathcal{C}_0^{1+\alpha}(\mathbb{R}^{N-1})$, where $\sigma \in (-1,1)$ and $\alpha \in (\sigma, 1+\sigma)$. Define

$$\mathcal{D}_L(\rho,\infty) = \{f \in E : [f]_{\mathcal{D}_L(\rho,\infty)} = \sup_{0<t\leq 1} \|t^{1-\rho} L e^{Lt} f\|_E < \infty\}. \tag{33}$$

Proposition 1

$$\mathcal{D}_L(\rho,\infty) = \mathcal{C}_0^{\alpha-\sigma}(\mathbb{R}^{N-1}) \cap C^{\alpha+\rho(1+\sigma)-\sigma}(\mathbb{R}^{N-1}). \tag{34}$$

In the paricular case where $\sigma = s \in (0,1)$ and $\alpha = \beta \in (s,1)$,

$$\mathcal{D}_{\mathcal{L}_0}(\rho,\infty) = \mathcal{D}_L(\rho,\infty) = \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1}) \cap C^{\beta+\rho(1+s)-s}(\mathbb{R}^{N-1}). \tag{35}$$

Sketch of the proof: Step 3

For all $u_0 \in \mathcal{C}_0^{1+\beta}(\mathbb{R}^{N-1})$, we have

1. $F(u) \in C^{\rho}([0,T], \mathcal{C}_0^{\beta-s}(\mathbb{R}^{N-1}))$, $\forall u \in E_T$.
2. $\mathcal{L}_0 u_0 + F(u_0) = -\mathcal{H}(u_0) \in \mathcal{D}_{\mathcal{L}_0}(\rho, \infty)$.

Then, there exists a unique function $\Phi(u) \in E_T$ satisfying

$$\begin{cases} \partial_t \Phi(u) - \mathcal{L}_0 \Phi(u) &= F(u) \quad \text{in } [0,T] \times \mathbb{R}^{N-1} \\ \Phi(u)(0) &= u_0 \quad \text{in } \mathbb{R}^{N-1}. \end{cases} \tag{36}$$

$$\mathcal{E}_{T,R} := \{u \in E_T \, : \, u(0) = u_0 \, , \, \|u - u_0\|_{E_T} \leq R\} \, . \tag{37}$$

A fixed point of function $\Phi : E_T \to E_T$ in $\mathcal{E}_{T,R}$ will be a solution of

$$\begin{cases} \partial_t u - \mathcal{L}_0 u &= F(u) \quad \text{in } [0,T] \times \mathbb{R}^{N-1} \\ u(0) &= u_0 \quad \text{in } \mathbb{R}^{N-1}, \end{cases} \tag{38}$$

Sketch of the proof: Step 3 (end)

Provided $\|\nabla u_0\|_{C^{\beta+\rho(1+s)}(\mathbb{R}^{N-1})} \leq \nu$, there exists $R = R(N, s, \beta, \rho, \gamma, \nu) > 0$ and $T = T(N, s, \beta, \rho, \gamma, \nu) > 0$ such that

1. $\Phi(\mathcal{E}_{T,R}) \subset \mathcal{E}_{T,R}$.
2. Φ is a contraction on $\mathcal{E}_{T,R}$.

We can apply the Banach fixed point on $\mathcal{E}_{T,R}$ to obtain a unique fixed point $u \in \mathcal{E}_{T,R}$ of Φ.


Thank you