Maverick points on the boundary of wandering domains

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Joint work with David Martí-Pete and Lasse Rempe

Basic definitions

- Let $f : \mathbb{C} \to \mathbb{C}$ be analytic.
- Denote by f^n the nth iterate of f.

Definition

The **Fatou set** is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighborhood of } z\}.$$

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The **Julia set** is

$$J(f) = \mathbb{C} \setminus F(f).$$

Components of the Fatou set

If U is a Fatou component (connected component of the Fatou set) and U_n is the Fatou component containing $f^n(U)$, then either:

- U is **periodic** with period p if $U_p = U$ and $U_n \neq U_p$ for $1 \leq n < p$

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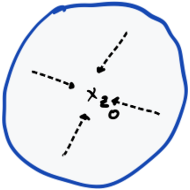
- U is **pre-periodic** if U_j is periodic for some $j \in \mathbb{N}$.
- U is wandering or a **wandering domain** if $U_m \neq U_n$ for all $m \neq n$.

Classification of periodic Fatou components

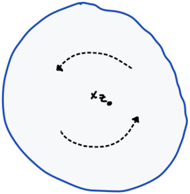
If U is a component of the Fatou set and $f^p(U) \subset U$, then we have the following four possibilities:

- **Attracting basin** U contains an attracting p-periodic point z_0. For all $z \in U$, $f^{np}(z) \to z_0$ as $n \to \infty$.
- **Parabolic basin** ∂U contains a parabolic p-periodic point z_0. For all $z \in U$, $f^{np}(z) \to z_0$ as $n \to \infty$.
- **Siegel disk** There exists a conformal map $\phi : U \to \mathbb{D}$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\theta}z$, where θ is irrational.
- **Baker domain** For all $z \in U$, $f^{np}(z) \to \infty$ as $n \to \infty$.

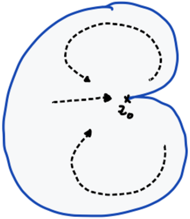
Classification of periodic Fatou components



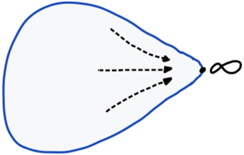
Attracting basin



Siegel disk



Parabolic basin



Baker domain

Wandering domains for polynomials?

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, then a Fatou component U of f is a **wandering domain** if $f^m(U) \cap f^n(U) = \emptyset$ for all $m \neq n$.

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Theorem (Sullivan, 1985)

A rational function whose degree is at least 2 has no wandering domains.

The escaping set of a polynomial

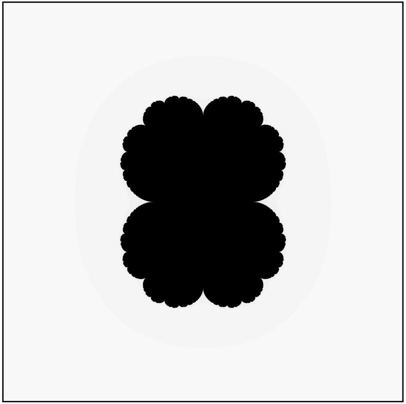
Definition

The **escaping set** is

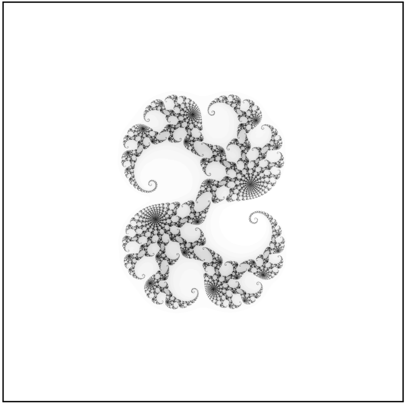
$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

- $I(f)$ is a neighborhood of ∞.
- $\partial I(f) = J(f)$.
- $I(f) \subset F(f)$.
- Points in $I(f)$ all have the same rate of escape.

Examples of the escaping set of some polynomials (in white)



$z^2 + 0.25$



$z^2 + .28 + .008i$

The escaping set of a transcendental entire function

Definition

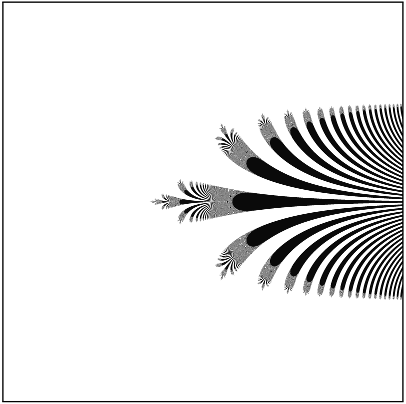
The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

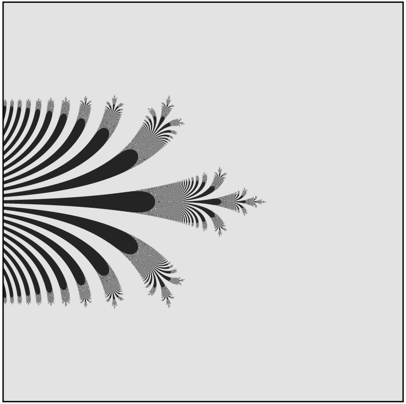
- $I(f)$ is not a neighborhood of ∞.
- $I(f)$ can meet $F(f)$ and $J(f)$.
- Points in $I(f)$ have different rates of escape.

Let $BU(f)$ be the points that neither escape nor remain bounded.

Examples of the escaping set of some transcendental entire functions (in black and gray)

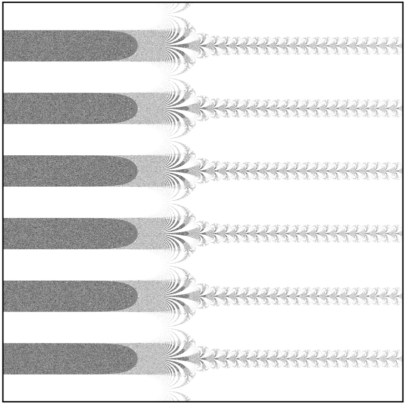


$\frac{1}{4}\exp(z)$



$z + 1 + \exp(-z)$

Examples of some transcendental entire functions with wandering domains



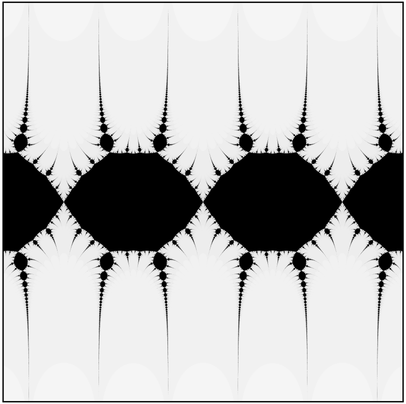
$z - 1 + e^{-z} + 2\pi i$

- Herman (1984) gave a simple example of a transcendental entire function with a simply connected wandering domain.
- These wandering domains are unbounded and are contained in $I(f)$.

Examples of some transcendental entire functions with wandering domains

- Baker (1984) also gave a family of simple example of a transcendental entire function with a bounded simply connected wandering domain.

- $z + 2n\pi + \lambda \sin z$ for λ chosen so that $z + \lambda \sin z$ has an attracting fixed point at 0.



$z + \sin(z) + 2\pi$

Wandering domains of transcendental entire functions

The first example of a wandering domain for transcendental entire functions was given by Baker.

- In 1963, he constructed a function with a sequence of annuli tending to infinity, each mapping into the next.
- In 1976, he showed that these annuli were each contained in distinct multiply connected Fatou components.

Multiply connected wandering domains

Theorem (Baker, 1984)

If U is a multiply connected Fatou component, then

- U is a wandering domain,
- U_{n+1} surrounds U_n for sufficiently large n,
- $U_n \to \infty$ as $n \to \infty$.

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Moreover, if U is a multiply connected wandering domain then there exists a large absorbing annulus contained inside each U_n (Bergweiler, Rippon, Stallard, 2013).

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Connectivity eventually becomes either 2 or ∞ (Kisaka, Shishikura, 2010).

See results of Baker, Bergweiler–Rippon–Stallard, Zheng, Kisaka–Shishikura, and Burkart–Lazebnik...

Simply connected wandering domains

Three different possible types of orbit of a simply connected wandering domain U containing a point z.

- **Escaping** $f^n(z) \to \infty$
  - Largest number of known examples.
  - Can not occur in the class $\mathcal{B}$.

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- **Oscillating** $f^n(z)$ has a bounded and unbounded subsequence.
  - Eremenko and Lyubich (1987) constructed the first known example using approximation theory.
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- **Bounded** $f^n(z)$ is bounded
  - Not known if these exist.

See results of Herman, Baker, Eremenko-Lyubich, Bishop, Martí-Pete–Shishikura, Lazebnik, Fagella–Jarque–Lazebnik, Benini–Evdoridou–Fagella–Rippon–Stallard, Evdoridou–Rippon–Stallard, and Boc Thaler...

A question of Rippon

Theorem (Rippon, Stallard) 2011

Let f be a transcendental entire function and let U be a wandering domain of f such that $U \subset I(f)$. Then $\partial U \cap I(f) \neq \emptyset$. Moreover, the set $\partial U \cap I(f)^c$ has zero harmonic measure relative to U.

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Rippon, Problem 2.94(b) in the Hayman–Lingham problem book

If U is a bounded escaping wandering domain of f, must all points on ∂U be escaping?

Maverick points!

Theorem (Martí-Pete, Rempe, W)

There exists a transcendental entire function f with a bounded escaping wandering domain U such that $\partial U \setminus I(f) \neq \emptyset$.

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Definition

Let f be a transcendental entire function and suppose that U is a wandering domain of f. We say that a point $z \in \partial U$ is *maverick* if there is a sequence (n_k) such that $f^{n_k}(z) \to w \in \hat{\mathbb{C}}$ as $k \to \infty$, but w is not a limit function of $(f^{n_k}(U))$.

Maverick points exist!

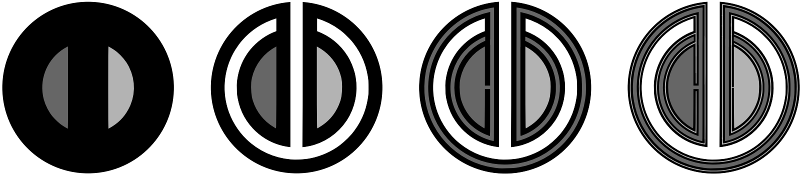
Main construction result

Theorem (Martí-Pete, Rempe, W)

Let $K \subseteq \mathbb{C}$ be a compact set with connected complement. Let $Z_I, Z_{BU} \subset K$ be disjoint finite or countably infinite sets such that no connected component of $\text{int}(K)$ intersects both Z_I and Z_{BU}. Then there exists a transcendental entire function f such that

1. $\partial K \subseteq J(f)$;
2. $f^n(K) \cap f^m(K) = \emptyset$ for $n \neq m$;
3. every connected component of $\text{int}(K)$ is a wandering domain of f;
4. $Z_I \subset I(f)$ and $Z_{BU} \subset BU(f)$.

Proof



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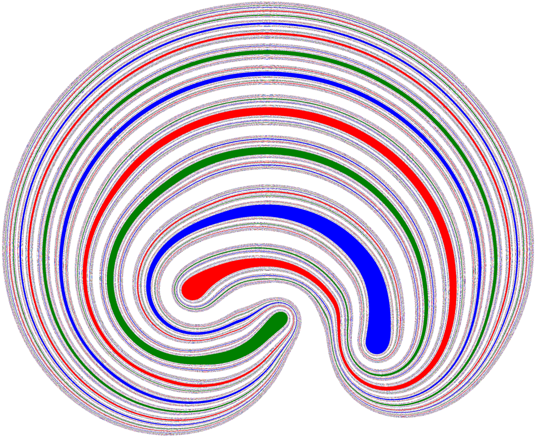

How many mavericks is too many?

Theorem (Martí-Pete, Rempe, W)

Let f be a transcendental entire function and suppose that U is a wandering domain of f. The set of maverick points in ∂U has harmonic measure zero with respect to U.

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Let U be a simply connected wandering domain. Does the set of maverick points of U have zero logarithmic capacity?

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Theorem (Martí-Pete, Rempe, W)

Let $E \subset \partial \mathbb{D}$ be a compact set of zero logarithmic capacity. Then there exist transcendental entire functions f_I and f_{BU} such that

- (i) $\mathbb{D}$ is an escaping wandering domain of f_I;
- (ii) $\mathbb{D}$ is an oscillating wandering domain of f_{BU};
- (iii) every point in E is a maverick point for both f_I and f_{BU}.

Even more mavericks!

Theorem (Martí-Pete, Rempe, W)

There exists a transcendental entire function f with an escaping or oscillating wandering domain U such that the set of non-maverick points on ∂U has Hausdorff dimension 1, while the set of maverick points contains a continuum of positive Lebesgue measure.

Thank you for your attention!