Semi-Hyperbolicity Implies Existence of ACIPs for Real Multimodal Maps

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Framework

In this talk we denote by I a compact interval in $\mathbb{R}$, and we are going to work with differentiable functions $f : I \to I$.

Definition

A function $f : I \to I$ is said to be *multimodal* if it has a finite non-zero number of critical points, all non-flat. f is said to be *unimodal* if the critical point is unique.

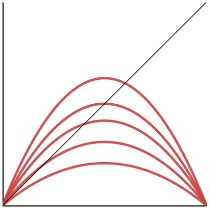


Figure: Logistic Family
$q_a(x) = ax(1-x)$

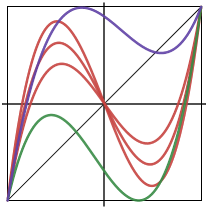


Figure: Bicritical Cubic Polynomials
$f(x) = ax^3 + bx^2 + (1-a)x - b$

The Overarching Conjecture

Conjecture (Palis Conjecture)

Typical systems in finite dimensional Riemann manifolds posses a finite number of measures (physical measures) which describe the time averages of almost all orbits with respect to the Lebesgue (volume) measure.

In one dimension the conjecture is much more specific

Conjecture (Palis Conjecture for one-dimensional systems)

For generic families of one-dimensional dynamics, with total probability in parameter space, the attractors are either **periodic sinks** *(hyperbolic) or carry an* **absolutely continuous invariant probability measure** *(stochastic).*

Why is the Palis Conjecture important?

Definition

A measure μ is said to be *physical* or *Sinai-Ruelle-Bowen measure* if there exists a set B of positive Lebesgue measure such that for any $x \in B$ and any C^0 function ϕ the following holds

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \phi(f^i(x)) = \int \phi \, d\mu.$$

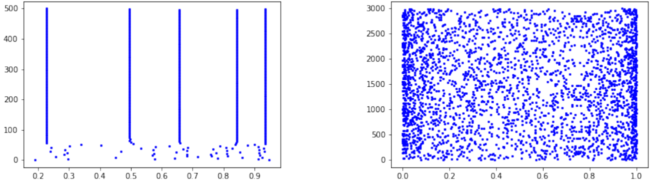


Figure: Computer simulation of hyperbolic and stochastic behaviours.

Density of Hyperbolicity

Theorem (Density of hyperbolicity)

The space of hyperbolic maps is C^r open and dense in the space of multimodal maps, for any $r \geq 1$

Proving this conjecture was a collective effort:

- Jakobson [Jak71] proved density for $r = 1$;
- Graczyk, Swiatek [GS97] and Lyubich [Lyu97]: the set of hyperbolic parameters for $q_a(x) = ax(1-x)$ is dense in $(0, 4]$;
- Kozlovski [Koz03]: the set of hyperbolic maps is C^r dense in the space of unimodal maps for any $r \geq 2$;
- Kozlovski, Shen, van Strien [KSvS07]: the set of hyperbolic maps is C^r dense in the space of multimodal maps.

How Many Stochastic Parameters do We Have?

Theorem (Jakobson [Jak81])

Let q_a be the logistic family, there exists a subset of stochastic parameters $\mathscr{C} \subset (0,4]$ of positive Lebesgue measure. The parameter $a = 4$ is a Lebesgue density point of $\mathscr{C}$.

This prompted solving the Palis conjecture (or one of its variants) in the following contexts:

- Lyubich [Lyu02] proved it for the logistic family;
- Avila, Lyubich, de Melo [ALdM03] proved it for non-trivial analytic family of quasiquadratic maps;
- Bruin, Shen, van Strien [BSvS06] showed almost every $f_{\ell,c}(x) = x^\ell + c$, for $\ell \geq 2$ even, supports a physical measure;
- Avila, Lyubich, Shen [ALS11] proved it for $f_{\ell,c}(x) = x^\ell + c$ with $\ell \geq 2$ and even;
- Clark [Cla14] proved it for non-trivial analytic family of unimodal maps (higher criticality).

When do We Have ACIPs? Metric Conditions

Philosophy

An ACIP describes the average behaviour of typical orbits. The existence of an ACIP is related to the rate of expansion along the orbit of the critical values.

- **Collet-Eckmann property** [CE83, Now88]: there exist $C > 0$ and $\rho > 1$ s.t.

$$|Df^n(f(c))| > C\rho^n \qquad \forall c \in \mathrm{Crit}(f), \forall n \geq 1;$$

- **Nowicki-van Strien summability condition** [NvS91, BvS01]:

$$\sum_{c \in \mathrm{Crit}(f)} \sum_{n \geq 1} |Df^n(f(c))|^{-\frac{1}{\ell_{\max}}} < \infty$$

where $\ell_{\max}$ is the maximal order of critical points $\mathrm{Crit}(f)$ of f;

- **Growth of the critical derivative** [BRLSvS08]:

$$\lim_{n \to \infty} |Df^n(f(c))| = \infty \qquad \forall c \in \mathrm{Crit}(f);$$

When do We Have ACIPs? Topological Conditions

Theorem (Misiurewicz [Mis81, vS90])

Let $f : I \to I$ be a C^2 multimodal map for which all periodic points are hyperbolic repelling. If the **forward orbit of critical points does not cumulate onto critical points** *then f carries an ACIP.*

When do We Have ACIPs? Topological Conditions

Theorem (Misiurewicz [Mis81, vS90])

Let $f : I \to I$ be a C^2 multimodal map for which all periodic points are hyperbolic repelling. If the **forward orbit of critical points does not cumulate onto critical points** *then f carries an ACIP.*

We extended this result.

Definition

A multimodal function $f : I \to I$ is said to be **semi-hyperbolic** if all its periodic points are hyperbolic repelling, and for all $c \in \mathrm{Crit}(f)$ **the orbit of c does not accumulate on c**.

Theorem (Clark, T., van Strien in preparation)

Let $f : I \to I$ be a C^3 semi-hyperbolic multimodal map with $Sf < 0$, then f carries an ACIP.

When do We Have ACIPs? Topological Conditions

Although our result was already known in a more general context, our new proof may yield useful extensions towards Palis conjecture.

Theorem (Rivera-Letelier, Shen [RLS14, PRL07])

Let $f : I \to I$ be a C^3 map whose periodic orbits are all hyperbolic repelling, and which is topologically exact on its Julia set. If f is **Topologically Collet-Eckmann** *then f admits an ACIPs.*

Remark: In [RLS14] less is assumed on the regularity of f.

Definition

$f : I \to I$ is TCE if $\exists M, P, r > 0$ such that for $x \in I$ there exists an increasing sequence of positive integers $\{n_j\}_{j \in \mathbb{N}}$ with $n_j \leq P \cdot j$ and

$$\# \left\{ i : 0 \leq i < n_j \, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)}(B_r(f^{n_j}(x))) \cap \mathrm{Crit}(f) \neq \emptyset \right\} \leq M$$

It was shown in [CJY94] that TCE(P=1) is equivalent to semi-hyperbolicity.

The Two Different Approaches

Our Approach: Pull-back

- For $n \geq 0$, f^n is **quasi-polynomial** (finite decomposition of f^n in simple applications of f and maps of bounded distortion)
- For any set A of small measure

$$|f^{-n}(A)| \leq C|A|^{1/\ell_{\max}};$$

- Foguel's Theorem implies ACIP existence.

[RLS14] Approach: Inducing

- Define an induced "Markov" map using return maps;
- Control the geometry of the domains of such a map (Shrinking of Components);
- Apply Young's result on tail estimates [You99, PRL07] to obtain existence of ACIP together with statistical properties.

Topological VS Metric Conditions: An Example

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a continuous family of bicritical cubic polynomials. There exists a cubic polynomial $f_{\lambda_*} : [-1,1] \to [-1,1]$ with two distinct critical points c_0 and c_1 such that

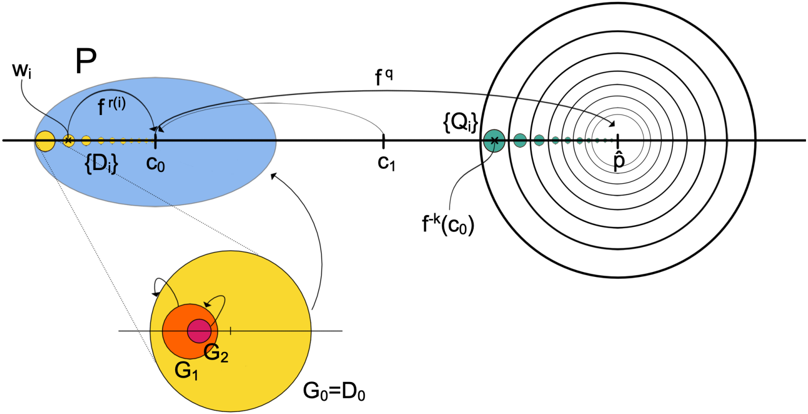
1. c_0 is preperiodic;
2. $c_i \notin \omega(c_i)$ for $i = 1,2$ and $c_0 \in \omega(c_1) \Rightarrow \exists$ **an ACIP** $\boldsymbol{\mu}$;
3. all cycles are hyperbolic repelling;
4. $\mathbf{\liminf_{n\to\infty}|Df^n_{\lambda_*}(f_{\lambda_*}(c_1))| = 0}$ (and $\chi_-(c_1(\lambda_*)) = -\infty$)

Remark

f_{λ_*} does not satisfy any metric condition, but it admits an ACIP!

Sketch of the Dynamical Construction

Let f_{λ_0} be non-renorm., with a preperiodic critical point c_0, and with c_1 landing on c_0. All periodic points are hyperbolic repelling.



Thank you for the attention!

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