Borel-Cantelli lemma(s) and complex dynamics

Note Title 21/06/2022

An expository talk on the Borel-Cantelli lemma(s), with applications from joint work with: Gwyneth Stallard, Anna Miriam Benini, Vasso Evdoridou & Nuria Fagella.

Borel-Cantelli Lemma(s) Let $E_n \subset [0,1]$, $n \in \mathbb{N}$, be m'ble.

BC1 If $\sum_{n=1}^{\infty} |E_n| < \infty$, then $|\{x \in [0,1] : x \in E_n \text{ i.o.}\}| = 0$.

BC2 If $\sum_{n=1}^{\infty} |E_n| = \infty$ and (E_n) are 'independent' i.e.

$$|E_{n_1} \cap \cdots \cap E_{n_j}| = |E_{n_1}| \cdots |E_{n_j}|, \text{ for all } n_1, \dots, n_j \in \mathbb{N},$$

then

$$|\{x \in [0,1] : x \in E_n \text{ i.o.}\}| = 1.$$

i.o. infinitely often

$$\{x \in [0,1] : x \in E_n \text{ i.o.}\} = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} E_n =: \overline{\lim_{n\to\infty}}\, E_n. \qquad (1)$$

__Proof__ First recall

$$\{x\in[0,1]: x\in E_n \text{ i.o.}\} = \bigcap_{N=1}^{\infty} \bigcup_{n\geq N} E_n =: \overline{\lim_{n\to\infty}}\, E_n. \qquad (1)$$

__BC1__ $\left|\bigcup_{n\geq N} E_n\right| \leq \sum_{n\geq N} |E_n| \to 0$ as $N\to\infty$.

__BC2__ By (1), want

$$\left|\bigcup_{n\geq N} E_n\right| = 1, \qquad \text{for all } N.$$

Independence of (E_n) gives independence of (E_n^c), so

$$\left|\bigcap_{n>N}^{N'} E_n^c\right| = \prod_{n=N}^{N'} |E_n^c| = \prod_{n=N}^{N'} (1-|E_n|)$$

$$\leq \prod_{n=N}^{N'} \exp(-|E_n|)$$

$$= \exp\Big(-\sum_{n=N}^{N'} |E_n|\Big) \to 0 \quad \text{as } N'\to\infty. \qquad \blacksquare$$

Application of BC1 to wandering domains

R + Stallard 2011

Theorem 1 Let f be a tef with an escaping WD U. Then w.r.t. harmonic measure in U, a.e. point of ∂U is escaping.

'Proof' Put

$$E_n = E_n(R) = \{\zeta \in \partial U : |f^n(\zeta)| \le R\}, \quad n \ge 1.$$

Want

$$\omega(z_0, \overline{\lim_{n\to\infty}} E_n, U) = 0, \qquad \text{for all } R > 0$$

harmonic measure

Löwner's lemma $f^n(U_0) \subset U_n$, $\tilde{E}_n = \partial U_n \cap \{z : |z| \le R\}$

$$\omega(z_0, E_n, U) \le \omega(f^n(z_0), \tilde{E}_n, U_n).$$

Estimates of RHS + U_n disjoint:

$$\sum_{n=1}^{\infty} \omega(z_0, E_n, U) < \infty \Rightarrow \omega(z_0, \overline{\lim_{n\to\infty}} E_n, U) = 0. \quad \blacksquare$$

Osborne + Sixsmith 2016: replace ∞ by $a \in \mathbb{C}$, n by $n_k \to \infty$.

Erdős + Renyi 1959 $\sum_{n=1}^{\infty}|E_n| = \infty$ and

$$\lim_{n\to\infty} \frac{\sum_{1\le i,j\le n}|E_i\cap E_j|}{\left(\sum_{k=1}^{n}|E_k|\right)^2} = 1 \quad (2) \Rightarrow \left|\overline{\lim_{n\to\infty}} E_n\right| = 1.$$

eg $|E_i\cap E_j| = |E_i|\times|E_j|$ — pairwise independence

'Proof' Let

$$f_n(x) = \sum_{k=1}^{n}\chi_{E_k}, \text{ so } \frac{\sum_{i,j}|E_i\cap E_j|}{\left(\sum|E_k|\right)^2} = \frac{\int_0^1 f_n^2}{\left(\int_0^1 f_n\right)^2}.$$

By (2),

$$\lim_{n\to\infty}\frac{\int_0^1 f_n^2 - \left(\int_0^1 f_n\right)^2}{\left(\int_0^1 f_n\right)^2} = \lim_{n\to\infty}\frac{D(f_n)^2}{M(f_n)^2} = 0.$$

Choose n_k s.t. $\displaystyle\sum_{k=1}^{\infty}\frac{D(f_{n_k})^2}{M(f_{n_k})^2} < \infty.$

Chebyshev's inequality $\left|\underbrace{\{x : f_n(x) \le (1-\varepsilon)M(f_n)\}}_{A_n}\right| \le \left(\frac{D(f_n)}{\varepsilon M(f_n)}\right)^2$

Thus $\displaystyle\sum_{k=1}^{\infty}|A_{n_k}| < \infty$, so $|\{x : x\in A_{n_k} \text{ i.o.}\}| = 0$, BC1

$\Rightarrow f_{n_k}(x) \to \infty$ as $k\to\infty$ for a.e. $x\in[0,1]$. ■

Weaken the independence hypothesis further

Markov chains

Lamperti 1963 Suppose $\sum_{n=1}^{\infty} |E_n| = \infty$ and $C > 1$. Then

$$|E_m \cap E_n| \leq C\,|E_m|\,|E_n|, \text{ all } m,n \Rightarrow |\overline{\lim_{n\to\infty}}\, E_n| \geq \frac{1}{2C}.$$

Ciesielski + Taylor 1962

Brownian motion

Proof Given C, choose $I \subset (0, 2/C)$ s.t.

$$x - \tfrac{1}{2}Cx^2 \geq \delta > 0, \quad \text{for } x \in I.$$

Then, for all N $\exists\, n_2 > n_1 > N$ s.t. $\sum_{n_1}^{n_2} |E_n| \in I$, so

Wlog $|E_n| \to 0$.

$$\begin{aligned}
\Big|\bigcup_{n\geq N} E_n\Big| &\geq \Big|\bigcup_{n_1}^{n_2} E_n\Big| \\
&\geq \sum_{n_1}^{n_2} |E_n| - \sum_{n_1 \leq i < j \leq n_2} |E_i \cap E_j| \\
&\geq \sum_{n_1}^{n_2} |E_n| - \tfrac{1}{2}C \sum_{n_1 \leq i \neq j \leq n_2} |E_i|\,|E_j| \\
&= x - \tfrac{1}{2}Cx^2 > \delta.
\end{aligned}$$

$x = \sum_{n_1}^{n_2} |E_n|$

Hence $|\{x : x \in E_n \text{ i.o.}\}| \geq \delta$, δ arb. close to $1/(2C)$. ∎

Lemma Let $g(z) = z^2$ and $\varepsilon_n \downarrow 0$ s.t. $\sum_{n=1}^{\infty} \varepsilon_n = \infty$. Put

$$I_n = \{ e^{i\theta} : |\theta - \pi| \le \varepsilon_n \}, \qquad n \ge 1.$$

shrinking target(s)

Then

$$\left| \{ \zeta \in \partial\mathbb{D} : g^n(\zeta) \in I_n \text{ i.o.} \} \right| > 0.$$

$(= 2\pi)$

'**Proof**' Put

$$A_n = g^{-n}\left(\{ e^{i\theta} : |\theta - \pi| \le \tfrac{1}{2}\varepsilon_n \} \right), \quad |A_n| = \varepsilon_n.$$

Claim

$$|A_m \cap A_n| \le 3\,|A_m|\,|A_n|.$$

Geometry

- if m, n fairly close, then $A_m \cap A_n = \emptyset$
- if $n \gg m$, then A_n-arcs much shorter than A_m-arcs, so $A_m \cap A_n$ relatively short. ∎

Philipp 1967 Holds for z^p, $p \ge 2$, by very precise BC2.

Question Which polynomials have 'shrinking target' property?

Inner functions dichotomy

Aaronson 1978

Doering + Mañé 1991

Theorem 2 Let $f : \mathbb{D} \to \mathbb{D}$ be inner, with Denjoy–Wolff point p.

(a) If
$$\sum_{n=1}^{\infty} (1 - |f^n(0)|) < \infty,$$
then
$$f^n(\zeta) \to p \text{ as } n \to \infty, \qquad \text{for a.e. } \zeta \in \partial\mathbb{D}.$$

(b) If
$$\sum_{n=1}^{\infty} (1 - |f^n(0)|) = \infty,$$
then
$$f^n(\zeta) \not\to p \text{ as } n \to \infty, \qquad \text{for a.e. } \zeta \in \partial\mathbb{D}.$$

Proof ideas Part (a) uses Löwner's lemma + BC1.

Part (b) uses Poisson integrals + Poincaré recurrence theorem. ∎

Bourdon + Matache + Shapiro 2005: Part (a) for $f : \mathbb{D} \to \mathbb{D}$.

Generalisation of part (a) of dichotomy

BEFRS

Theorem 3 Let $f_n : \mathbb{D} \to \mathbb{D}$ and $F_n = f_n \circ \cdots \circ f_1$. If

$$\sum_{n=1}^{\infty} (1 - |F_n(0)|) < \infty, \qquad (3)$$

then

$$|F_n(\zeta) - F_n(0)| \to 0 \text{ as } n \to \infty, \text{ for a.e. } \zeta \in \partial\mathbb{D}. \qquad (4)$$

Proof Again, Löwner's lemma + BC1. ∎

Remark Actually works when $F_n : \mathbb{D} \to \mathbb{D}$!

Part (b) does not generalise when

- f_n are Möbius maps: can have dvgce of (3) and (4) holds a.e. on $\partial\mathbb{D}$.
- f_n are Blaschke products of degree 2: can have dvgce of (3) and (4) holds for ζ in an arc.

Question Is there a version of Thm 2(b) for
$F_n = f_n \circ \cdots \circ f_1$, where f_n are inner fns,
when (F_n) is contracting i.e.
$$\text{hyp-dist}\,(F_n(z), F_n(z')) \to 0 \text{ as } n \to \infty, \quad z, z' \in \mathbb{D}?$$

The Möbius and Blaschke product examples are not.

Example $\exists\, F_n = f_n \circ \cdots \circ f_1$, f_n different, contracting,
$$\lim_{n\to\infty} F_n(0) = 1, \qquad \sum_{n=1}^{\infty} (1 - |F_n(0)|) = \infty,$$
and $\quad F_n(\zeta) \not\to 1$ as $n \to \infty$, for a.e. $\zeta \in \partial\mathbb{D}$.

Proof Take $g(z) = z^2$, $a_n \uparrow 1$, $\sum (1 - a_n) = \infty$, and put
$$f_n(z) = M_n \circ g \circ M_{n-1}^{-1}, \qquad M_n(z) = \frac{z + a_n}{1 + a_n z}, \quad a_0 = 0.$$
Then
$$F_n(z) = M_n \circ g^n(z) \quad \text{satisfies} \quad F_n(0) = a_n, \; n \geq 1,$$
and the shrinking target lemma implies
$$\text{Re}\, F_n(\zeta) \leq 0 \quad \text{i.o., for a.e. } \zeta \in \partial\mathbb{D}. \qquad \blacksquare$$