Irreducibility of Gleason polynomials implies irreducibility of Per_n

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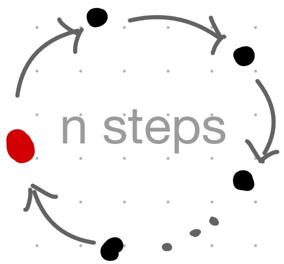
Critically periodic rational maps

$$\mathrm{Per}_n = \left\{ \begin{array}{c} f \\ \text{quadratic} \\ \text{rational} \end{array} \;\middle|\; \text{Critical} \right.$$



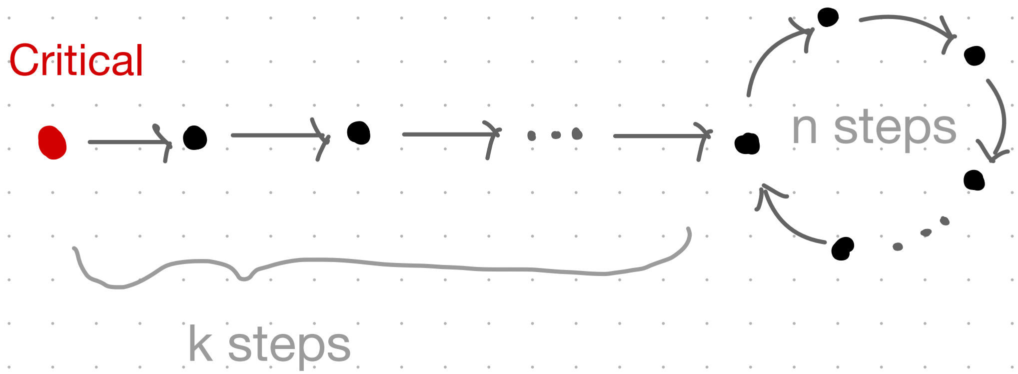
$$\left. \vphantom{\begin{array}{c} f \\ \text{quadratic} \\ \text{rational} \end{array}} \right\} \Big/ \text{Conjugation}$$

Affine algebraic curve /

Punctured (nodal) Riemann surface

Critically pre-periodic rational maps

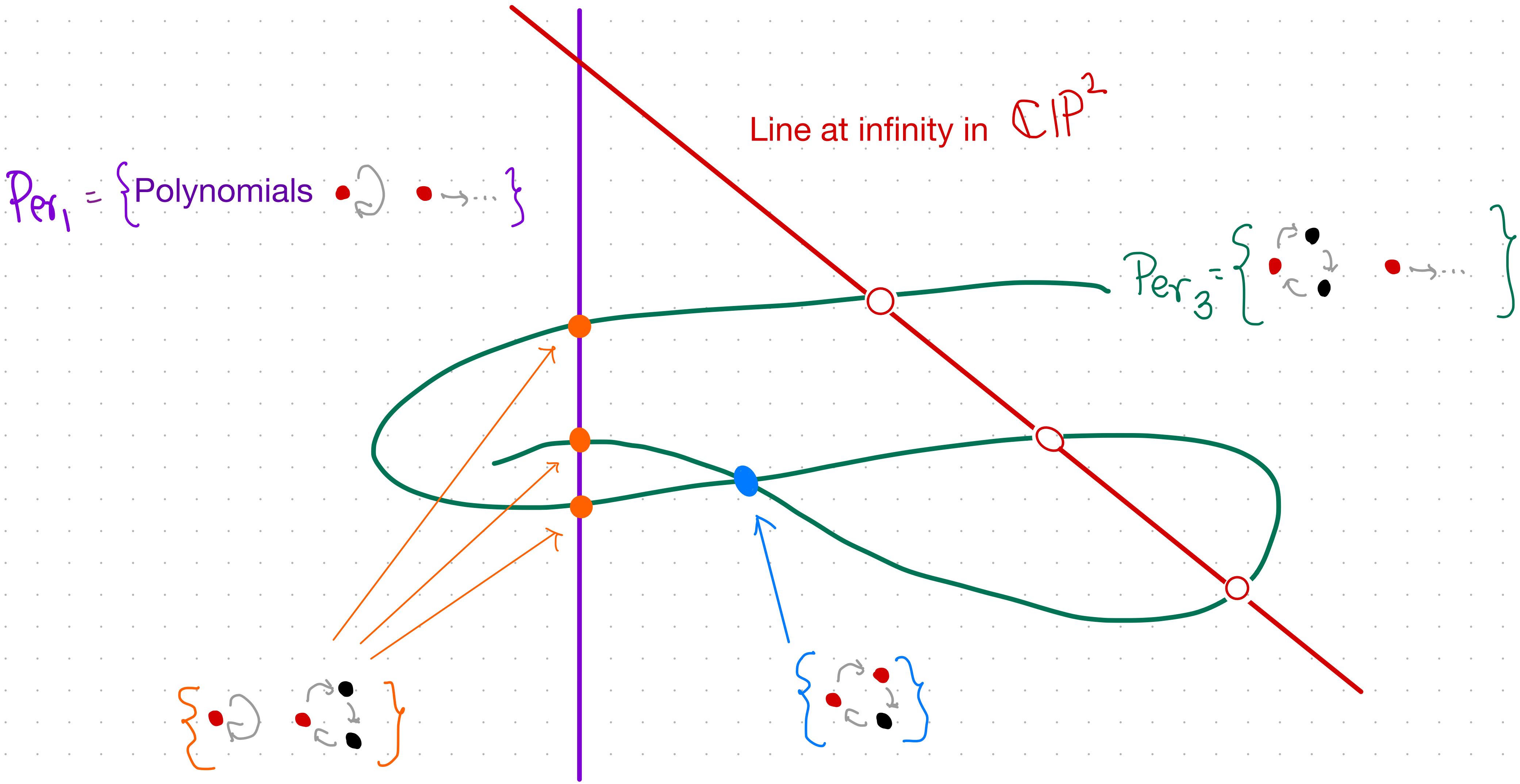
$$\mathrm{Per}_{k,n} := \left\{ f \text{ quadratic rational} \;\middle|\; \cdots \right\} \Big/ \text{Conjugation}$$



Affine algebraic curve /

Punctured (nodal) Riemann surface

$$\text{Milnor: } \left\{ \begin{matrix} f \\ \text{quadratic} \\ \text{rational} \end{matrix} \right\} \Big/ \text{Conjugation} \cong \mathbb{C}^2$$



Open question: Is Per_n irreducible (over $\mathbb{C}$)?

Irreducibility results:

Arfeux-Kiwi: Per_n in $\{$cubic polynomials$\}$ is

irreducible over $\mathbb{C}$

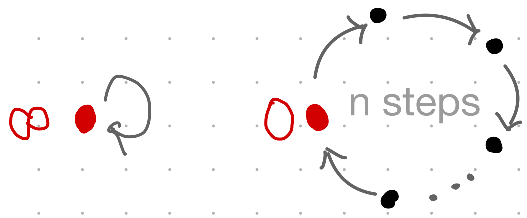
Buff-Epstein-Koch: $\mathrm{Per}_{k,1}$ in $\{$cubic polynomials$\}$

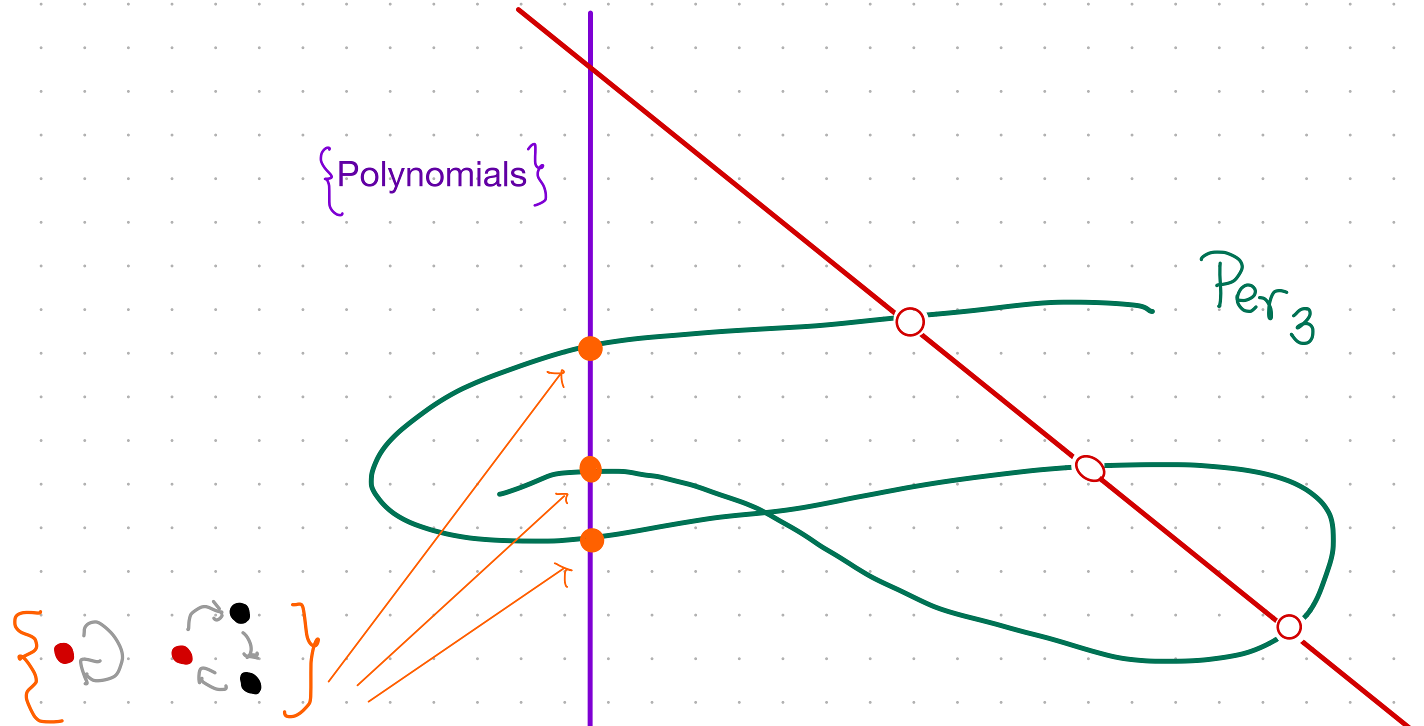
and in $\{$quadratic rational maps$\}$

are irreducible over $\mathbb{C}$

Gleason polynomials

$G_n(c)$ Polynomial with $\mathbb{Q}$-coefficients

Roots are $\left\{ c \mid z^2 + c \text{ satisfies } \right.$  $\left. \right\}$



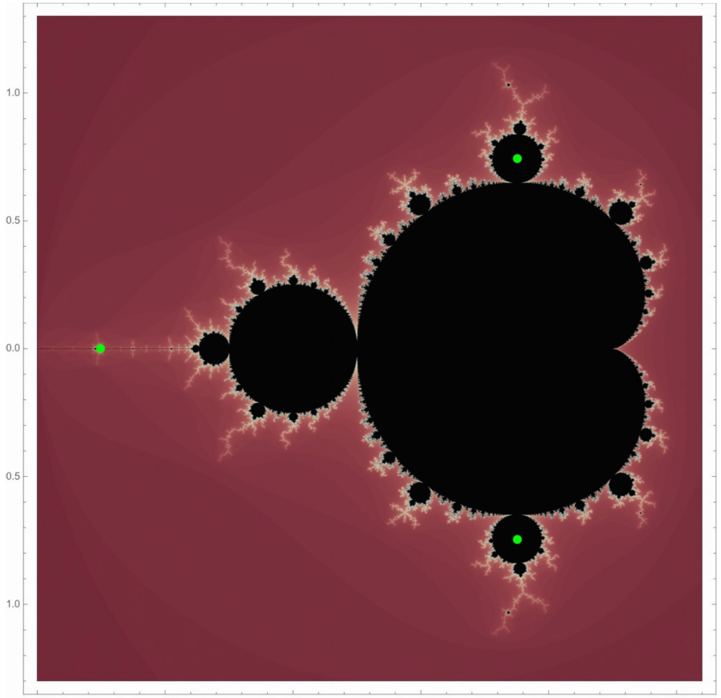
Roots of G_3

Gleason: roots simple

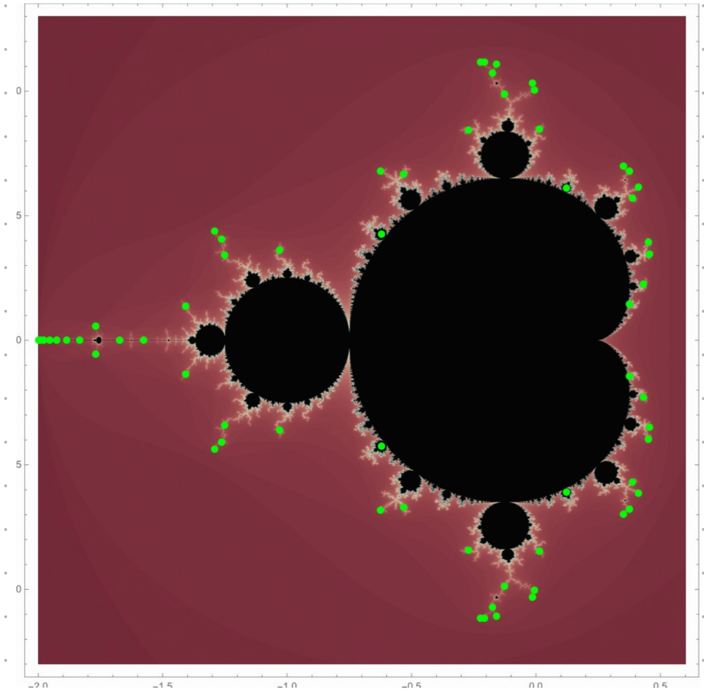
Open: Irreducible? (Over $\mathbb{Q}$)

(Goksel, Buff-Floyd-Koch-Parry)

Experiment (Doyle, Fili, Tobin): Yes for $n \leq 19$



Roots of G_3



Roots of G_7

Theorem (Ramadas)

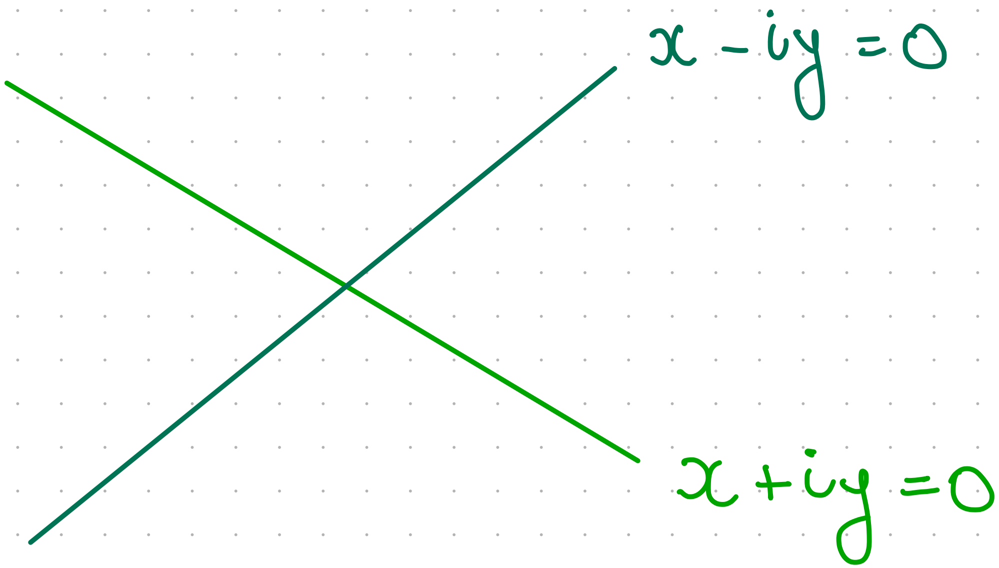
If G_n is irreducible over $\mathbb{Q}$

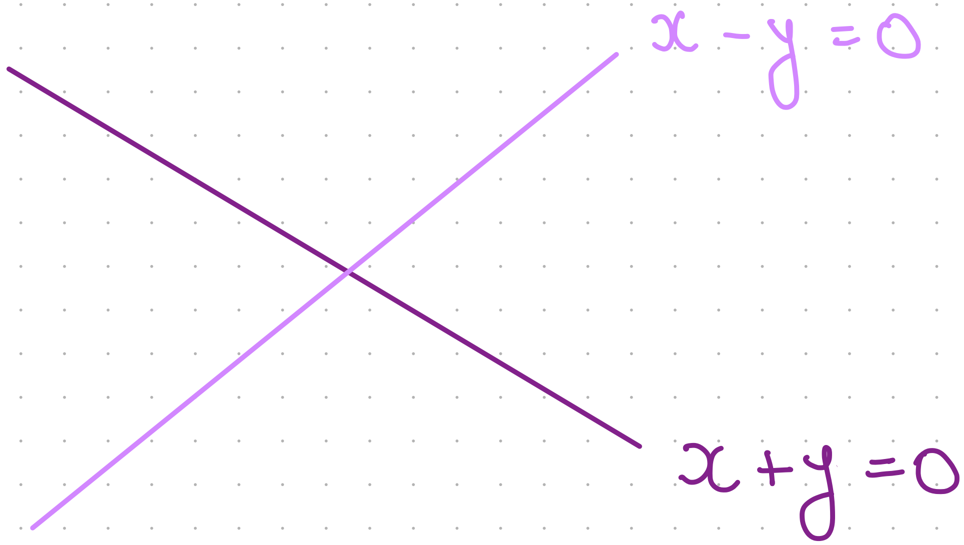
then Per_n is irreducible over $\mathbb{C}$

(and is therefore connected)

Corollary (based on Doyle-Fili-Tobin experiments):
Per_n irreducible over $\mathbb{C}$ for $n \leq 19$

Weaker concept: Is Per_n irreducible over $\mathbb{Q}$?





$$x^2 + y^2 = 0$$

$$x^2 - y^2 = 0$$

$$(x - iy)(x + iy) = 0$$

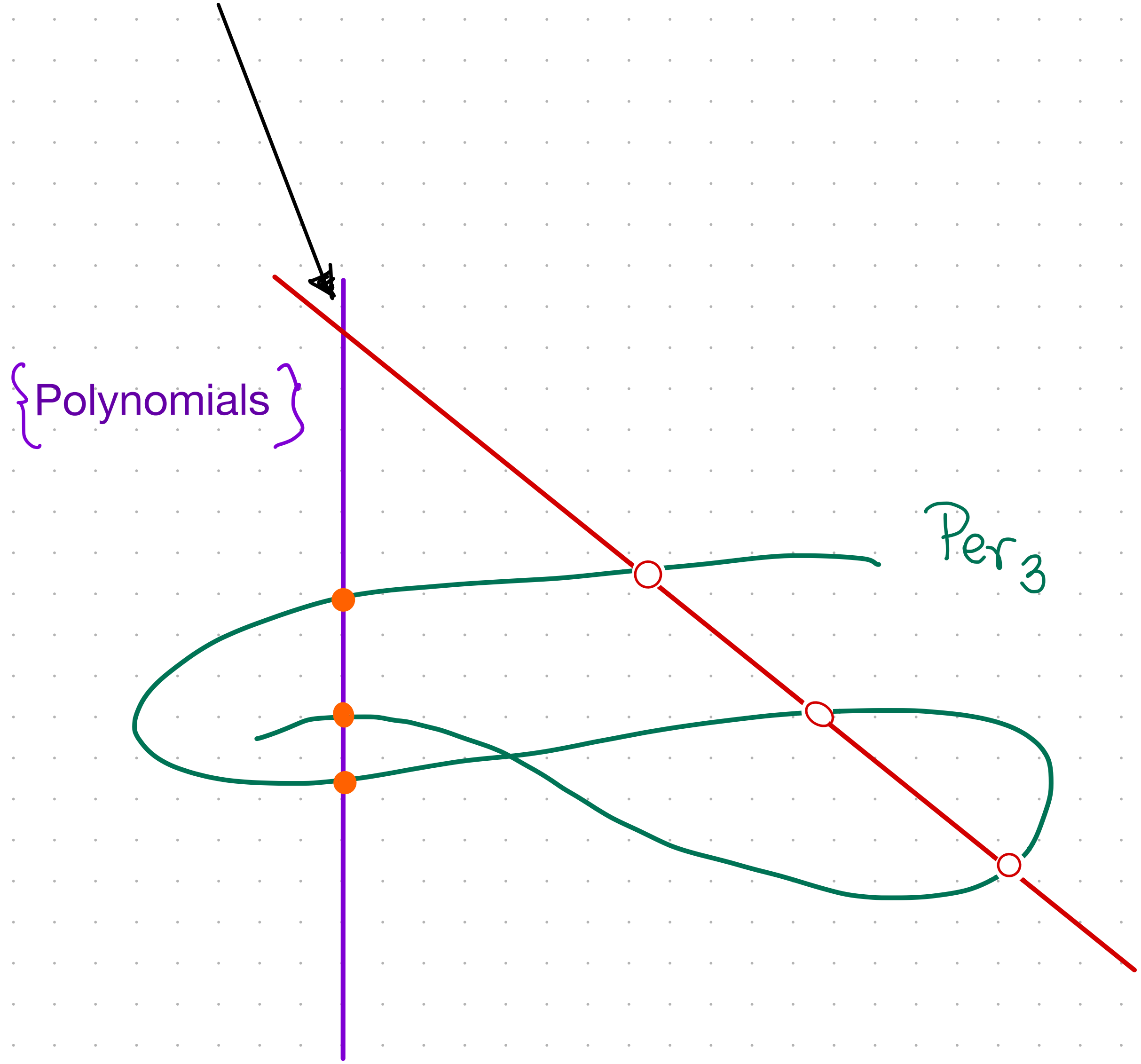
$$(x - y)(x + y) = 0$$

Reducible over $\mathbb{C}$

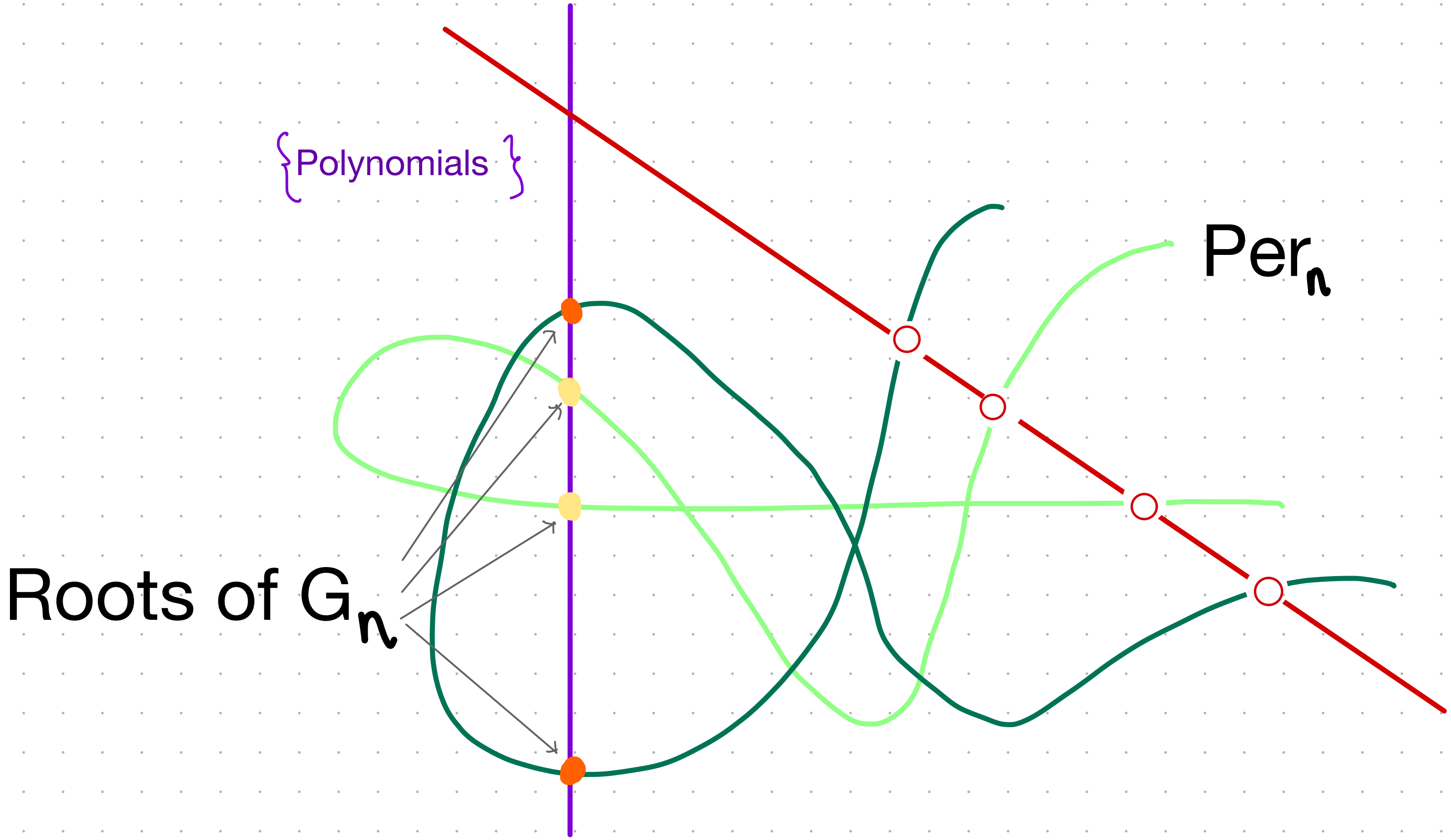
Irreducible over $\mathbb{Q}$

Reducible over $\mathbb{Q}$ and $\mathbb{C}$

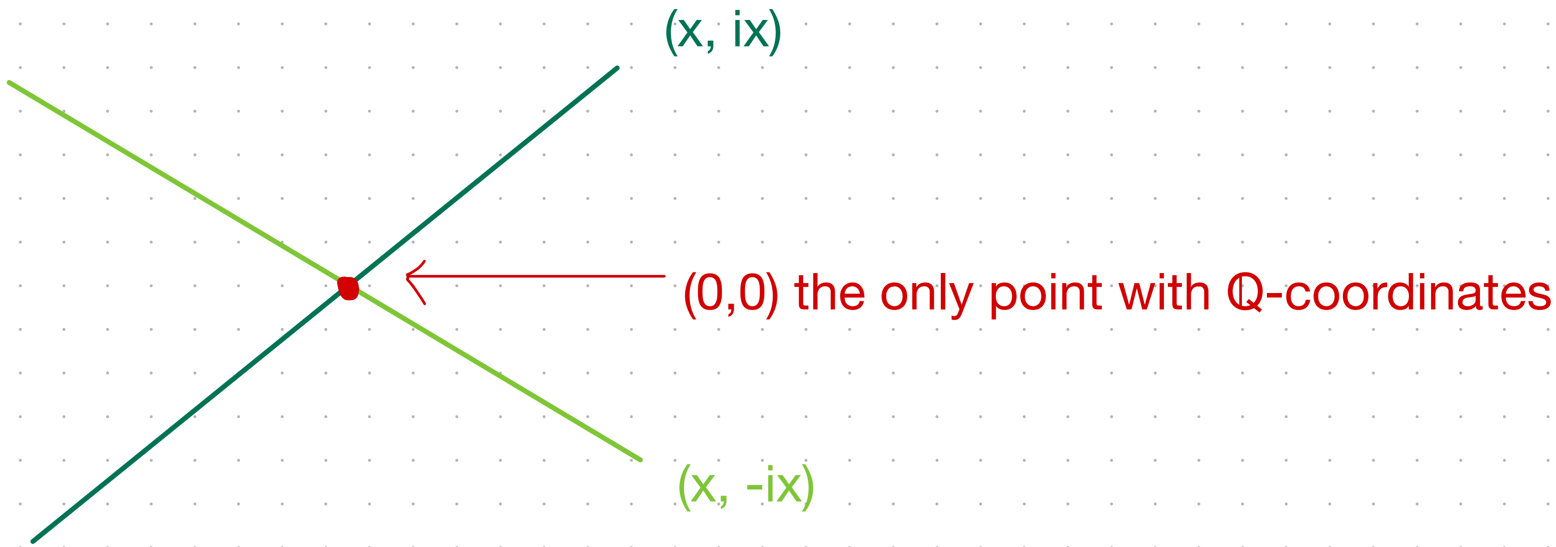
Milnor: “no intersections at infinity”



Step 1 (direct consequence of Milnor's "no intersections at infinity"):
If G_n is irreducible over $\mathbb{Q}$, then Per_n is irreducible over $\mathbb{Q}$



If X is irreducible over $\mathbb{Q}$, and has a smooth point with $\mathbb{Q}$-coordinates, then X is irreducible over $\mathbb{C}$



$x^2 + y^2 = 0$

Reducible over $\mathbb{C}$

Irreducible over $\mathbb{Q}$

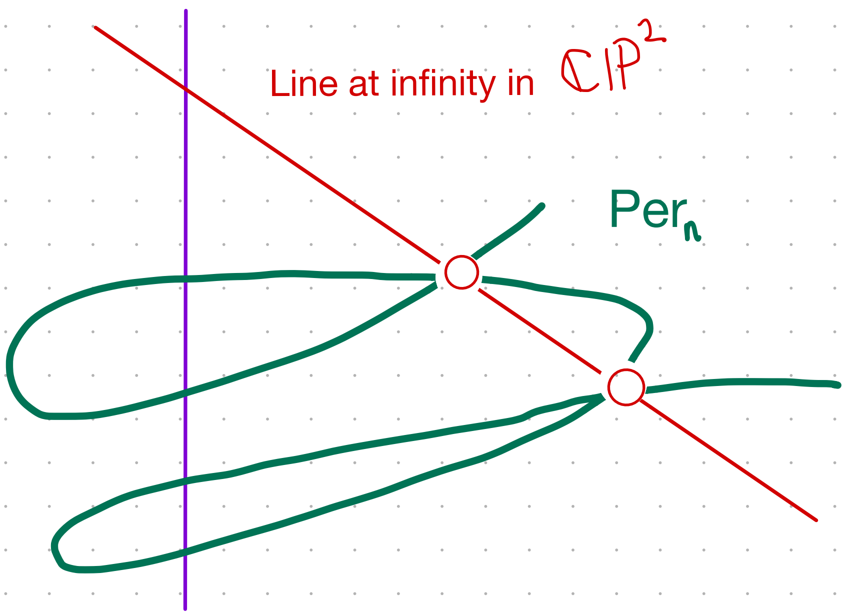
Step 2: Find a smooth point on Per_n with $\mathbb{Q}$-coordinates.

This promotes irreducibility over $\mathbb{Q}$ to irreducibility over $\mathbb{C}$

(Compare to Buff-Epstein-Koch for $\mathrm{Per}_{k,1}$)

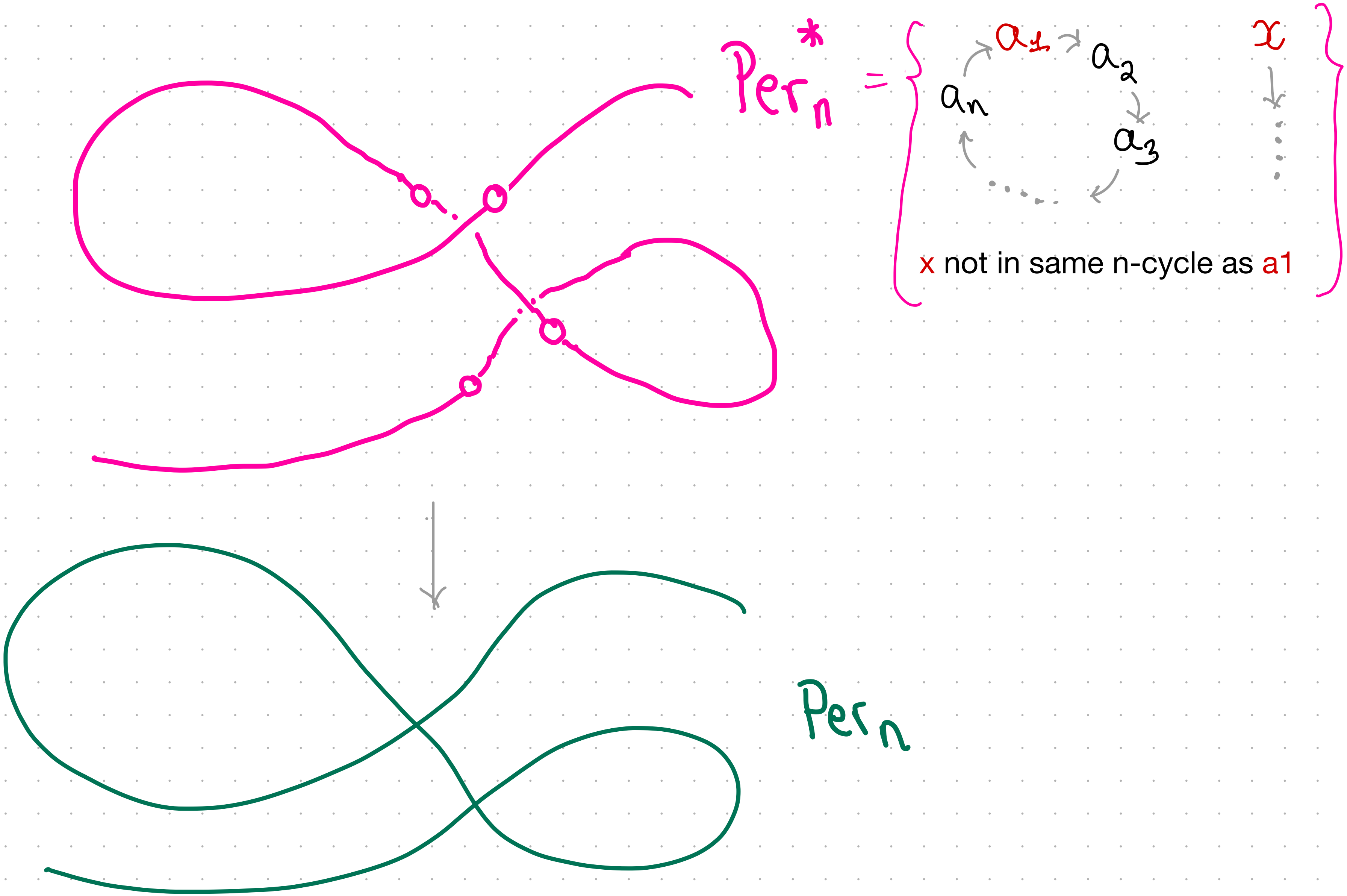
Issue: Per_n unlikely* to have points with $\mathbb{Q}$-coordinates, except at the line at infinity

Stimson: Points on Per_n at the line at infinity usually singular (many smooth branches)



* Uniform Boundedness Conjecture (Morton-Silverman)

Step 2.1: desingularize by marking the critical point. Also remove some PCF maps.



$$\mathrm{Per}_n^* \hookrightarrow \mathcal{H} = \left\{ \begin{array}{c|cccccccc} \mathbb{CP}^1 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n & x \\ f \big\downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ \mathbb{CP}^1 & a_2' & a_3' & a_4' & & a_n' & a_1' & y \end{array} \right\} \Big/ \sim$$

$$\mathrm{Per}_n^* \hookrightarrow \mathcal{H} = \left\{ \begin{array}{lccccccc} \mathbb{CP}^1 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n & x \\ f \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ \mathbb{CP}^1 & a_2' & a_3' & a_4' & & a_n' & a_1' & y \end{array} \right\} \Big/ \sim$$

$$\mathcal{M}_{0,n} = \left\{ p_1, \ldots, p_n \in \mathbb{CP}^1 \right\} \Big/ \sim$$

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$$\begin{tikzcd}
 & \mathcal{H} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{M}_{0,n} & & \mathcal{M}_{0,n}
\end{tikzcd}$$

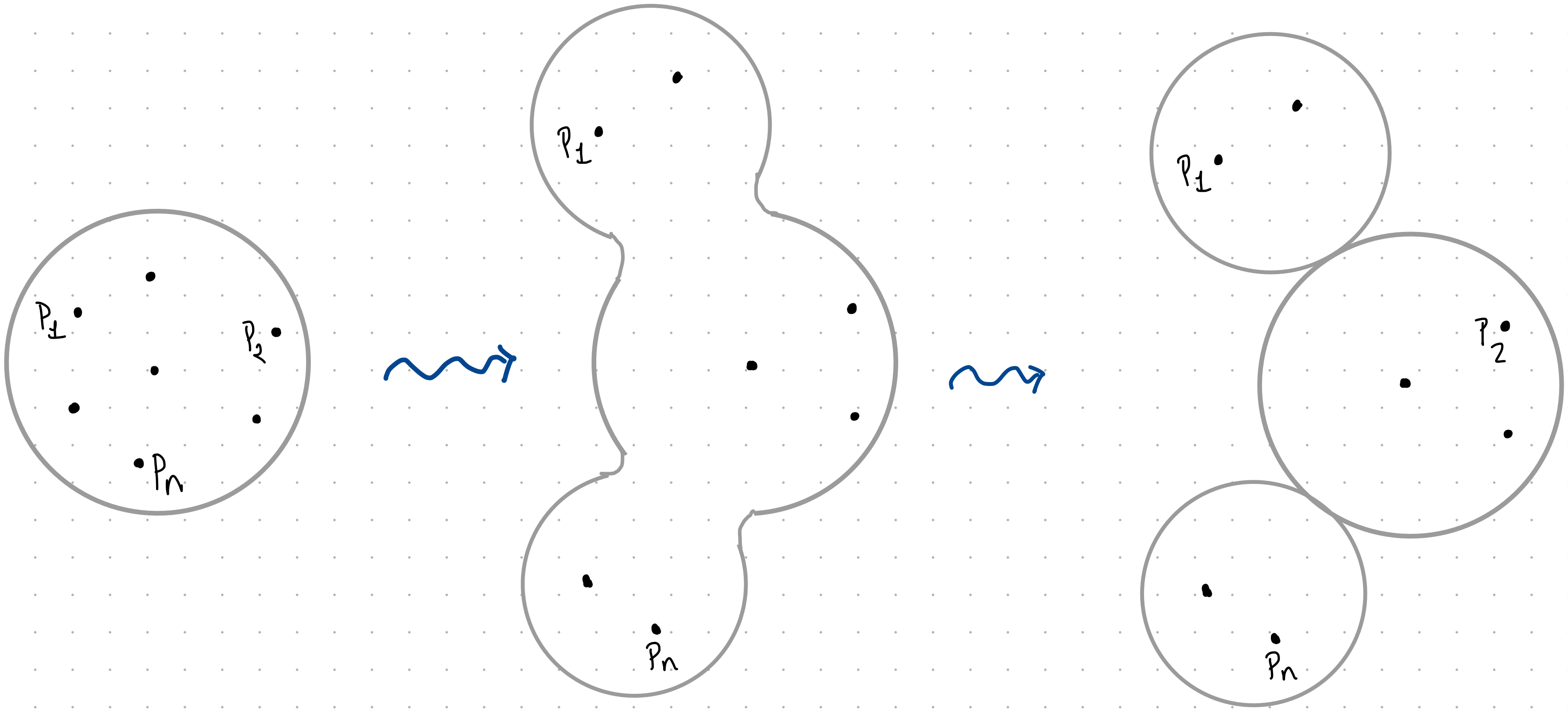
$(a_1', \dots, a_n')$ $\qquad$ $(a_1, a_2, a_3, \dots, a_n)$

$$\mathcal{M}_{0,n} = \{ p_1, \dots, p_n \in \mathbb{CP}^1 \} / \sim$$

"Equalizer"= $(\pi_1 \times \pi_2)^{-1}(\mathrm{diag}) \cong \mathrm{Per}_n^*$

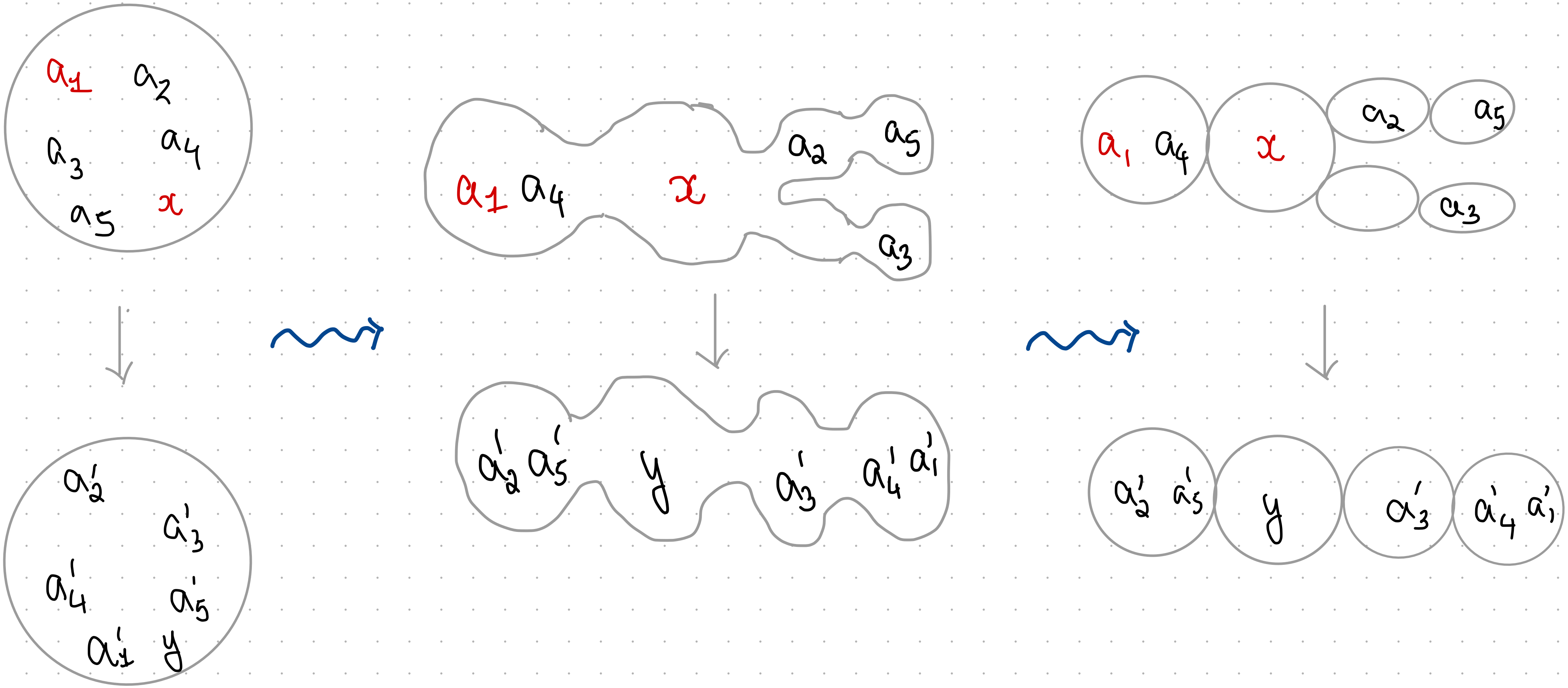
Epstein, Hironaka-Koch, Firsova-Kahn-Selinger

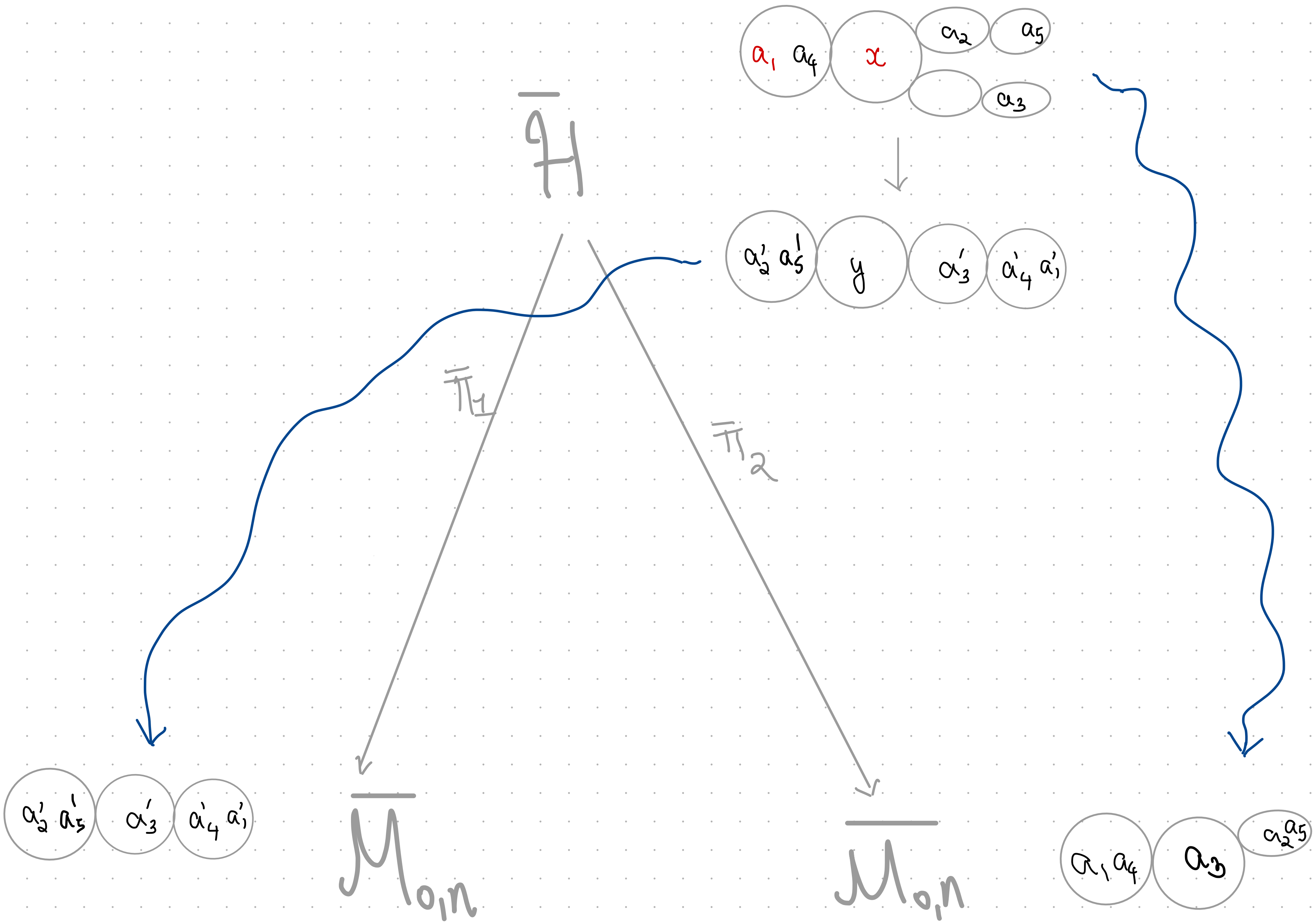
Deligne-Mumford-Knudsen: Compactify $\mathcal{M}_{0,n} \hookrightarrow \overline{\mathcal{M}}_{0,n}$



At infinity: singular surface with n distinct labeled marked points

Harris-Mumford, Abramovich-Corti-Vistoli: compactify $\mathcal{H} \hookrightarrow \overline{\mathcal{H}}$





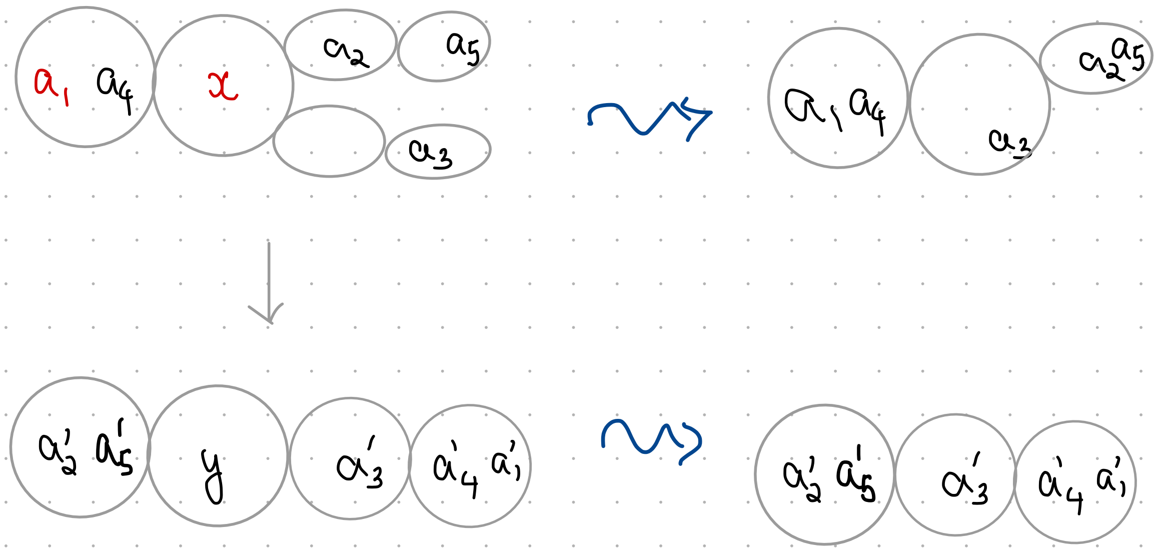
$\overline{H}$
a_1 a_4
x
a_2
a_5
a_3
a_2' a_5'
y
a_3'
a_4' a_1'
$\overline{\pi}_1$
$\overline{\pi}_2$
a_2' a_5'
a_3'
a_4' a_1'
$\overline{M}_{0,n}$
$\overline{M}_{0,n}$
a_1 a_4
a_3
a_2 a_5

$$\begin{tikzcd} & \overline{\mathcal{H}} \arrow[dl, "\overline{\pi}_1"'] \arrow[dr, "\overline{\pi}_2"] & \\ \overline{\mathcal{M}}_{0,n} & & \overline{\mathcal{M}}_{0,n} \end{tikzcd}$$

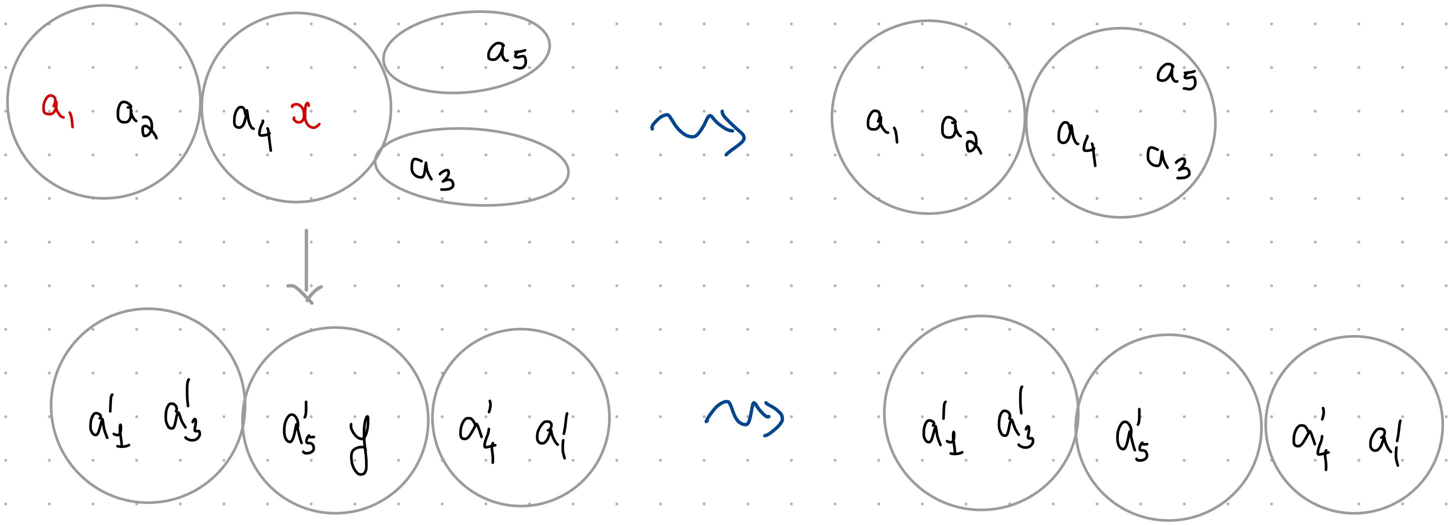
Equalizer= $\overline{\mathrm{Per}}_n^* \cup (\mathrm{Stuff})$

Ramadas-Silversmith: how to find points “at infinity” and local equations for $\overline{\mathrm{Per}}_n^*$

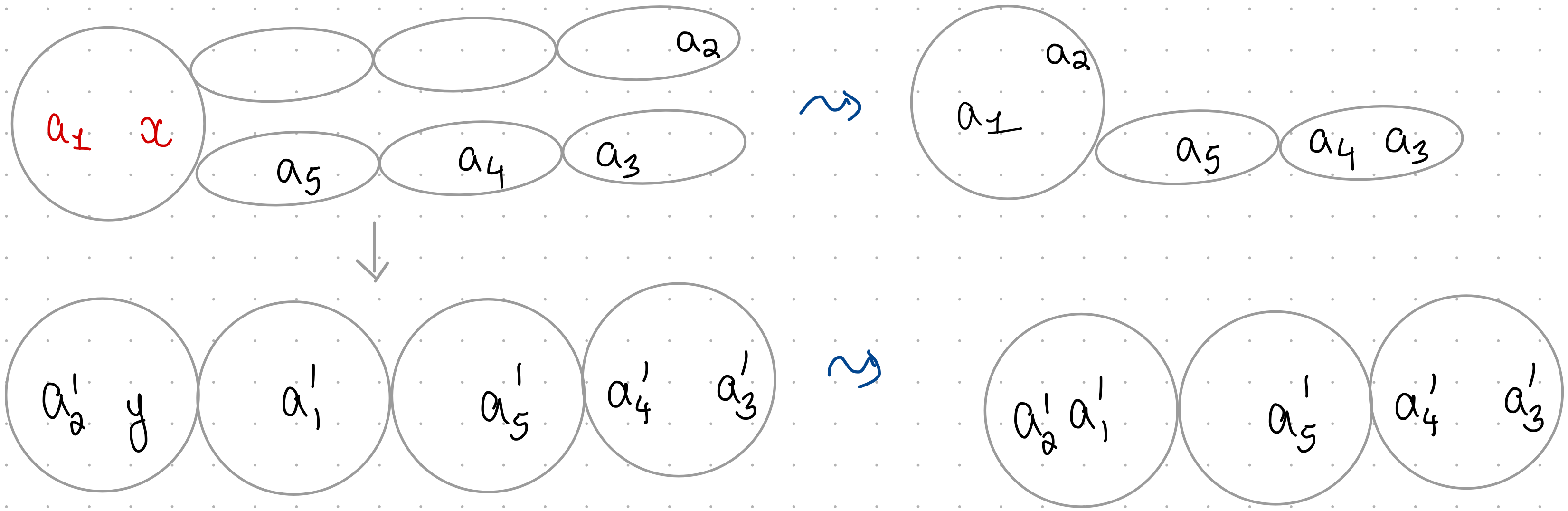
Uses Harris-Mumford local coordinates on $\overline{\mathcal{H}}$



Example:
in equalizer
but not in $\overline{\mathrm{Per}}_5^*$

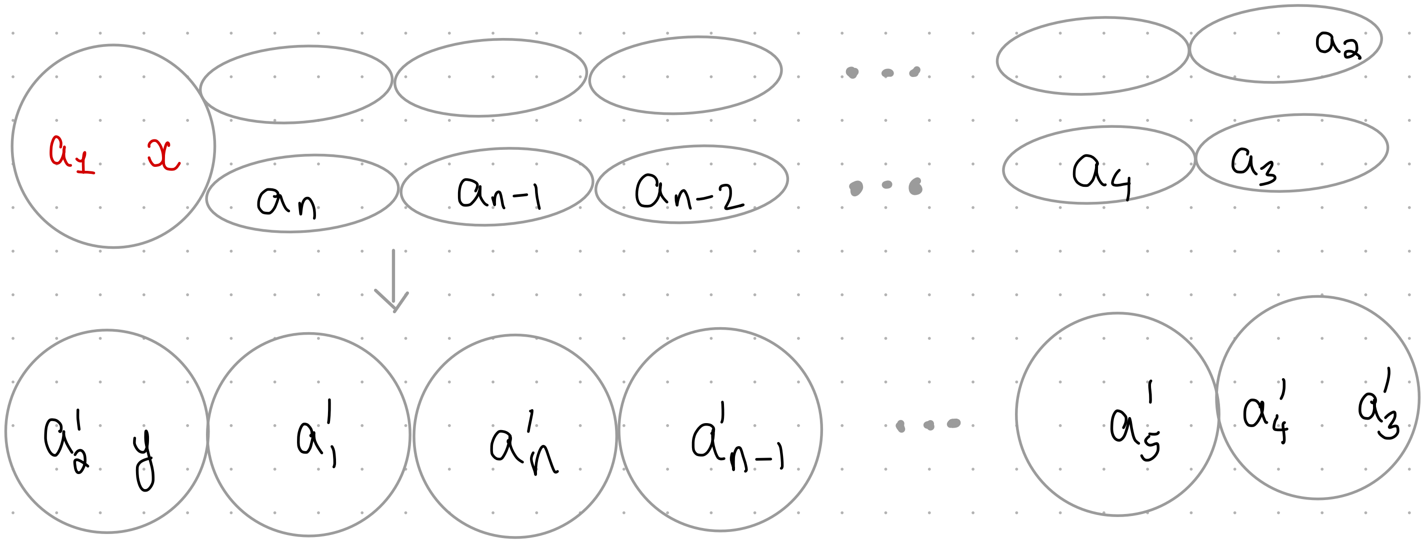


Not in equalizer



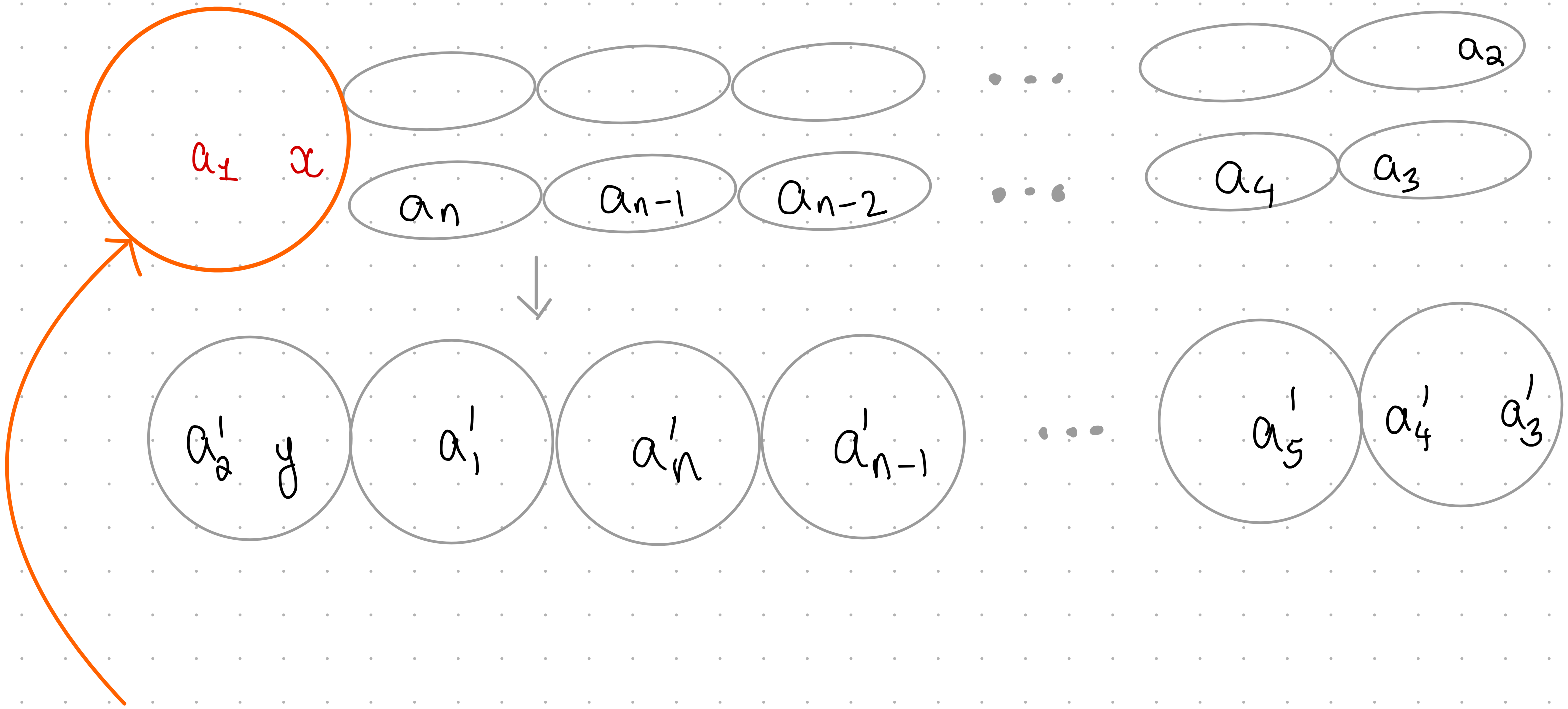
on $\overline{\mathrm{Per}}_5^{*}$

Theorem (Ramadas): for all $n>3$, the following is a smooth $\mathbb{Q}$-rational point on $\overline{\mathrm{Per}}_n^*$



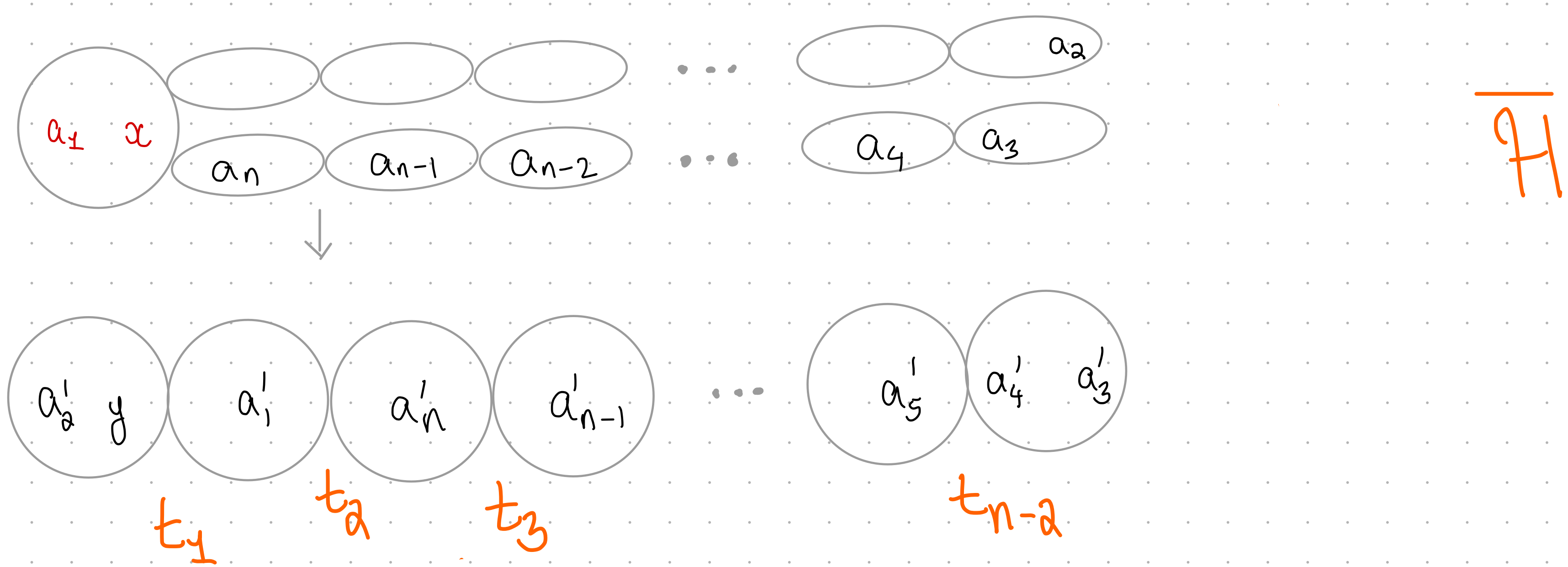
Also: a smooth $\mathbb{Q}$-rational point on $\overline{\mathrm{Per}}_{2,n}^*$

Why $\mathbb{Q}$-rational?

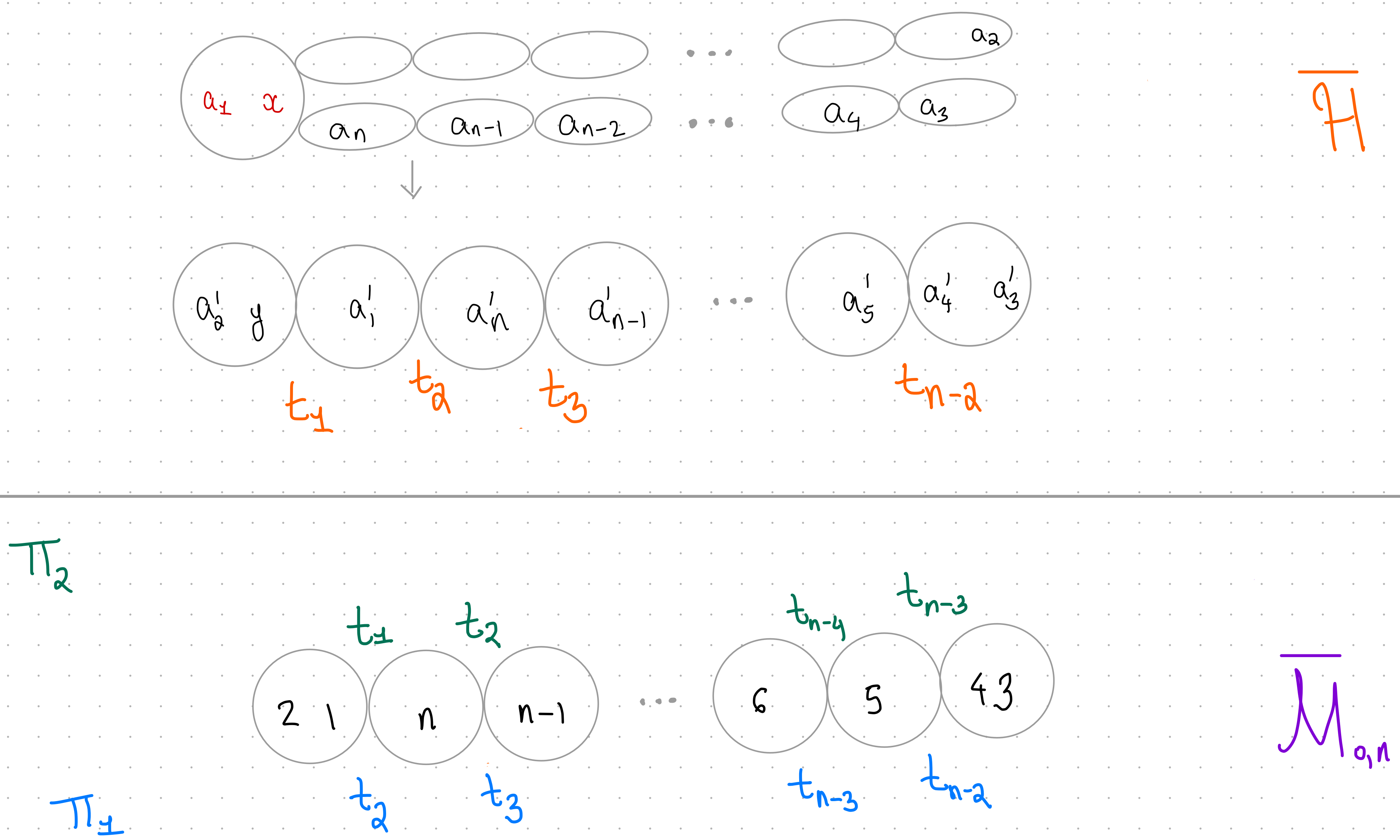


Cross-ratio = -1

Why smooth? Local coordinates

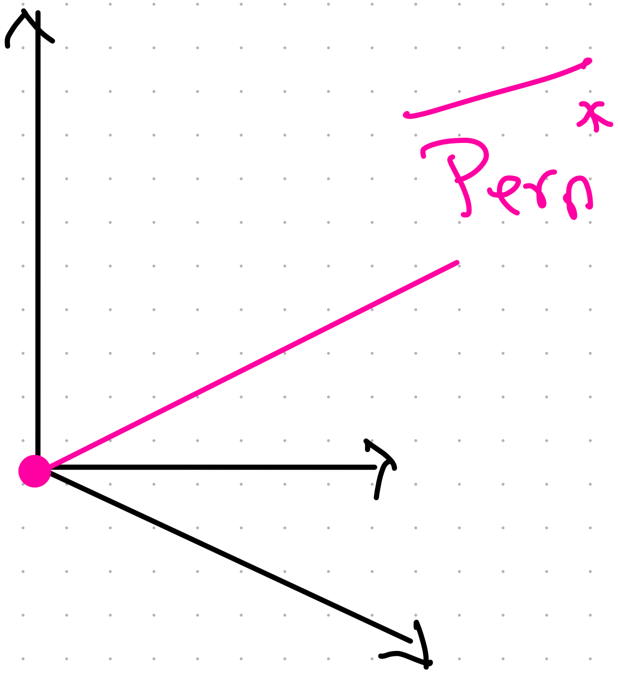


Why smooth? Local coordinates



$$\begin{tikzcd}
\overline{H}\,(t_1, t_2, \ldots, t_{n-2}) \arrow[d, bend right, "\pi_1"', "{(t_2, t_3, \ldots, t_{n-2})}"] \arrow[d, bend left, "\pi_2", "{(t_1, t_2, \ldots, t_{n-3})}"'] \\
\overline{M}_{0,n}
\end{tikzcd}$$

Equalizer: $t_1 = t_2, \quad t_2 = t_3, \ldots, \; t_{n-3} = t_{n-2}$


$\overline{\text{Pern}}^*$

Ramadas-Silversmith n=5