Egalitarian Equivalent Capital Allocation*

By

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Abstract

We apply Moulin's notion of egalitarian equivalent cost sharing of a public good to the problem of insurance capitalization and capital allocation where the liability portfolio is fixed. We show that this approach yields overall capitalization and cost allocations that are Pareto efficient, individually rational, and, unlike other mechanisms, stable in the sense of adhering to cost monotonicity.

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I. Introduction

Mainstream risk capital allocation methods in financial institutions are grounded in the concept of the marginal cost of risk. Specifically, capital is allocated to each risk in a portfolio based on how much capital is consumed when that risk is expanded at the margin. Such approaches have obvious merit in the context of portfolio optimization, where correct pricing of marginal units of exposure is essential.

Other applications of capital allocation, however, may require fundamentally different methods. One such application is the case where capital must be allocated but the portfolio of risk is fixed. This can occur in insurance markets when a closed block of insurance business is reinsured, or when a runoff company is capitalized or recapitalized.[1] In such cases, allocating capital based on how it is consumed when risk is expanded at the margin is no longer economically relevant. Instead of devising allocation rules to prices which guarantee that the right amount of risk is taken conditional on capitalization, as existing methods are designed to do, we must devise rules to guarantee equitable treatment of participants in the course of choosing optimal capitalization.

Existing methodologies, which allocate the total cost of capital based on how the marginal unit is consumed, can introduce a wedge between individual and collective interests. In particular, individual policyholders which are intensive consumers of the marginal capital unit at the social optimum may be better off with lower levels of capitalization when the total capital cost allocation is being keyed to the consumption of the marginal unit. This relates closely to the notion of stability in allocations: If allocations are unstable with respect to small perturbations, then small changes in risk, capitalization, or in risk measure thresholds can produce large swings in policyholder welfare---which can cause individual policyholders to disagree on the optimal level of capitalization.

This paper is concerned with the latter problem. Specifically, we study how capital cost sharing rules can be designed to guarantee Pareto optimal capitalization acceptable to the various policyholders whose exposures make up the portfolio. We resolve the problem of instability by appealing to the concept of cost monotonicity used in the economic theory of public goods---where cost sharing rules are restricted to produce allocations in a way that no agent will object to the introduction of an improvement to the cost technology.

Background and Motivation

Many capital allocation methods ultimately boil down to the gradient of a risk measure. Examples include Myers and Read (2001), Denault (2001), Tsanakas and Barnett (2003), Tasche (2004), Kalkbrener (2005), and Powers (2007).[2] Economic justification for the gradient method can be recovered in profit maximization problems where the risk measure serves to constrain risk taking (e.g., Zanjani, 2002; Meyers, 2003; Stoughton and Zechner, 2007). In the latter papers, the gradient of the risk measure

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[1] A particularly vivid example of a runoff capitalization is provided by Equitas---which is in the process of discharging the liabilities of multiple Lloyd's syndicates following the market restructuring of 1993.

[2] Bauer and Zanjani (2013) provide a review of gradient methods as well as alternative approaches to capital allocation.

accurately reflects the marginal cost of risk, so allocating capital according to the risk measure gradient is consistent with marginal cost pricing.

Consistency with marginal cost seems desirable, but it is important to understand that such consistency does not grant universal application of the allocation method. Merton and Perold (1993) proved that risk capital allocation would generally fail to provide accurate pricing of inframarginal or supramarginal changes to a risk portfolio. Hence, allocating capital according to the gradient method can yield accurate pricing of marginal changes to a risk portfolio, but no more.[3] Applications where the risk portfolio is fixed may require a different approach. Rather than asking how much risk to take, given fixed capital, sometimes one is confronted with the question of how much capital to hold, given fixed risk.

The latter problem fits squarely within the public goods literature, and in particular those papers concerned with cost-sharing mechanisms for providing the optimal amount of the public good. In the case of the insurance company, the public good is capital. The policyholders of the company are the consumers, who all enjoy access to the protection afforded by the capital of the firm.

The classic solution to the problem of public good cost sharing is provided by Lindahl (1958), whose basic idea was to derive "personalized prices" that each consumer could pay for the good. These prices were based on each consumer's marginal utility associated with the public good at the optimum. This idea was subsequently refined and extended by Foley (1970), Kaneko (1977), and Mas-Colell and Silvestre (1989)---who established, among other things, various conditions to guarantee that the solution was Pareto optimal and part of the core.

Cost-sharing based on valuation of the marginal unit, however, can lead to unappealing and unstable outcomes. In particular, one can construct examples (one of which is presented in a later section) where small changes in the level of public good production yield large changes in the cost allocation. In other words, while the Lindahl solution yields a Pareto optimal outcome, the mechanics of the cost sharing can lead some consumers to prefer super-optimal or sub-optimal production levels in cases where a deviation significantly alters the cost allocation. A similar problem surfaces in the capital allocation literature. Previous research has recognized the possibility that allocations might not be stable (e.g., Myers and Read, 2001; Zanjani, 2010) to small perturbations of the portfolio or capitalization level.

In the context of the general public goods literature, Moulin (1987) introduced an additional restriction on cost sharing dubbed *cost monotonicity* aimed at this problem. He argued that a cost sharing mechanism should satisfy the property that all consumers would benefit from a technological improvement in the cost function. This additional restriction, in conjunction with some other conditions on preferences and technology, leads to a unique solution: Specifically, the cost sharing mechanism ends up adhering to what Moulin dubbed *egalitarian equivalent cost sharing of a public good.* This

[3] Although this paper is concerned with an economic approach to capital allocation, it should also be acknowledged the economic approach---in the sense of taking profit or welfare maximization as the guiding objective---is by no means the only approach to capital allocation. Examples of optimization approaches with different objectives can be found in Dhaene, Goovaerts, and Kaas (2003), Laeven and Goovaerts (2004), and Dhaene, Tsanakas, Valdez, and Vanduffel (2012).

essentially means that each consumer is allocated cost so that her resulting utility matches that which she would receive at the maximum level of public good production that results in a feasible utility distribution, *if the public good were being given away for free*.

We adapt this idea to the context of the capital allocation problem in insurance, showing that the egalitarian equivalent approach to cost sharing yields stable capital allocations. The capitalization solution, moreover, is Pareto optimal, and participation in the scheme is individually rational.

The rest of this paper is organized as follows. Section II sets up the insurance capital allocation problem, defines the notion of *egalitarian equivalent* capital allocations, and shows that the resulting allocations are Pareto optimal, cost monotonic, and individually rational. Section III provides a numerical example demonstrating the stability of the egalitarian equivalent allocations in a situation where traditional methods yield unstable allocations. Section IV concludes.

II. Insurance Capitalization and Cost Allocation

We consider a set of N consumers. Each consumer is endowed with w^i and exposed to a random loss variable L^i. Each consumer has a contract with the same insurance company promising full indemnification[4] in the event of loss. The recovery from the insurance company may turn out to be less than promised. The company has non-negative assets a which could be less than total claims, so the consumer's recovery is:

$$R^i = \min\left[L^i, \frac{a}{\sum_{j=1}^{N} L^j} L^i\right]$$

The premium paid by the consumer is denoted by p^i, and we require premiums to cover costs associated with capitalizing the firm. Total costs are assumed to consist of actuarial costs plus a frictional cost $c(a)$, so that in aggregate:

$$\sum_{i=1}^{N} p^i = \sum_{i=1}^{N} \mathbb{E}R^i + c(a)$$

Consumer utility is determined by von Neumann-Morgenstern expected utility, which we will take to be continuous with risk aversion:

$$\mathbb{E}u^i(w^i - p^i - L^i + R^i)$$

The premium paid by the consumer, p^i, can be decomposed further into the actuarial loss and an amount to cover the frictional costs of assets:

[4] Note that the contracted indemnity here is taken as a given. For analysis of the optimal level of indemnity, see Zanjani (2010) and Bauer and Zanjani (forthcoming).

$$p^i = \mathbb{E}R^i + z^i$$

so that we may write utility as a function of the asset level (the public good) and a cost share

$$V^i(a, z^i) = \mathbb{E}u^i(w^i - L^i + R^i - \mathbb{E}R^i - z^i)$$

with the restriction that the cost allocations pay for the (frictional) cost of public good production:

$$\sum_i z^i = c(a) \tag{1}$$

where we take the frictional cost function to be increasing and continuous.[5]

We write the set of feasible allocations as:

$$\Omega = \left\{ (a, z^1, \dots, z^N) \mid a \geq 0, \sum_i z^i = c(a) \right\}$$

Asset Level Selection and Cost Sharing Mechanisms

A *mechanism M* assigns to each cost function a level of public good production and a set of cost shares satisfying (1):

$$M(c, V^1, \dots, V^N) = (a_m, z_m^1, \dots, z_m^N)$$

We are interested here in two key properties of a mechanism:

1. *Pareto optimality* – Does the mechanism select a Pareto optimal allocation for every cost function?
2. *Cost Monotonicity* – A mechanism satisfies *cost monotonicity* if, for any two cost functions c_1 and c_2 we have:

$$c_1(a) \leq c_2(a) \ \ \forall a \geq 0 \ \ \Rightarrow u^i\big(M(c_1)\big) \geq u^i\big(M(c_2)\big) \ \forall i$$

These properties are less than typically required in the public good literature on cost sharing (e.g., Kaneko, 1977; Mas-Colell and Silvestre, 1989), which usually also requires allocations to be in the "core" (as described by Foley, 1970) of allocations that can't be improved upon by coalitions of consumers. The core concept is natural in cases where the public good is non-rival, but in the insurance case all consumers are rival claimants on the same assets. Contemplating sub-coalitions of consumers in this case would thus involve alteration of preferences over the public good, as removal of potential claimants affects prospective consumption by the remaining consumers. Consistent with our motivating

[5] We have expressed frictional costs as a function of assets rather than capital, but note that this form is flexible enough to capture frictional capital costs. For example, consider: $c(a) = \hat{c}(a - \mathbb{E}R^i)$, where $\hat{c}$ is a continuous and increasing function and capital is the difference between assets and expected liabilities $(a - \mathbb{E}R^i)$. Notice that capital is a continuous and increasing function of assets, so that $c(a)$ will inherit continuity and monotonicity as well.

examples, we restrict our attention to the case where consumers cannot leave the company and thus do not require the allocation to be coalition-proof.

Cost monotonicity was introduced by Moulin (1987), motivated by requiring any cost sharing mechanism to allocate responsibility in such a manner that "no agent will oppose the implementation of a technological advance." As will become clear in the example of the next section, this requirement is intimately related to the notion of stability in capital allocations: If a mechanism has a tendency to produce allocations that are unstable with respect to small changes in capitalization, it is not likely to be cost monotonic.

Egalitarian Equivalent Capital Cost Allocation

Moulin (1987) also introduced *egalitarian equivalent* cost allocation, an approach he showed to be consistent with cost monotonicity. His idea was to allocate cost responsibility so that the resulting distribution of utility would match the distribution associated with the egalitarian equivalent level of public good production---which he defined as the highest possible level of the public good that, if it were provided for free to consumers, would yield a feasible utility distribution.

In our case, the egalitarian equivalent level of assets a^* is the highest level of assets such that, if the policyholders did not have to pay for the frictional costs associated with those assets, that a feasible utility distribution would result:

$$a^* = \sup\left\{\hat{a} \geq 0 \,|\exists(a, z^1, \dots, z^N) \in \Omega\colon V^i(\hat{a}, 0) \leq V^i\left(a, z^i\right) \ \forall i \in N\right\}$$

Moreover, given an egalitarian equivalent level of assets a^*, we call any feasible allocation $(\bar{a}, \bar{z}^1, \dots, \bar{z}^N)$ satisfying:

$$V^i(a^*, 0) \leq V^i\left(\bar{a}, \bar{z}^i\right) \ \forall i \in N$$

an *egalitarian equivalent allocation*.

The following theorems establish existence, individual rationality, Pareto efficiency, and cost monotonicity of egalitarian equivalent allocations. They are essentially slight modifications of portions of Moulin's results, adapted to the problem at hand and in particular sidestepping the issue of the core property.

Theorem 1 Suppose the loss distributions are bounded and nontrivial and the cost function is strictly increasing and weakly convex. The egalitarian equivalent level of public good production a^* is finite and any egalitarian-equivalent allocation $(\bar{a}, \bar{z}^1, \dots, \bar{z}^N)$ satisfies:

$$V^i(a^*, 0) = V^i\left(\bar{a}, \bar{z}^i\right) \ \forall i \in N$$

Proof: See Appendix.

Note that a consequence of Theorem 1 is that egalitarian equivalent allocations satisfy individual rationality. The egalitarian equivalent level of assets must be nonnegative, so any egalitarian equivalent allocation at least weakly dominates the zero allocation which is the relevant one for assessing individual rationality.

$$V^i(a_m, z_m^i) \geq V^i(0,0) \quad \forall i$$

Theorem 2 An egalitarian equivalent allocation is Pareto efficient.

Proof: See Appendix.

Theorem 3 - An egalitarian equivalent mechanism is cost monotonic.

Proof: See Appendix.

III. A Numerical Example

We now consider a simple example of a company with two consumers, both with exponential utility functions with coefficients of absolute risk aversion equal to unity and facing binary loss distributions. Specifically, let $w^1 = L^1 = 5$ and $w^2 = L^2 = 5.2$, and suppose further that the probability of loss is 5%, so that each consumer faces a 5% chance of losing all of her wealth. Finally, suppose that the losses are perfectly negatively correlated, so that consumer 1 never loses when consumer 2 loses, and vice versa.

We can then express the utility functions for consumers 1 and 2 as:

$$\begin{aligned} V^1 &= -.05\exp\{-[w^1 - L^1 + \min(L^1, a) - p^1]\} - .95\exp[-(w^1 - p^1)] \\ &= -.05\exp[-\min(5, a) + p^1] - .95\exp(-5 + p^1) \end{aligned}$$

and

$$\begin{aligned} V^2 &= -.05\exp\{-[w^2 - L^2 + \min(L^2, a) - p^2]\} - .95\exp[-(w^2 - p^2)] \\ &= -.05\exp[-\min(5.2, a) + p^2] - .95\exp(-5.2 + p^2), \end{aligned}$$

respectively, where p^i is the premium paid by consumer *i* and a is the asset level of the company.

We assume that the cost of holding assets is a linear function of the assets held, as in:

$$c(a) = .011a \quad \text{for } a > 0, \quad 0 \text{ otherwise}$$

so that premiums follow:

$$p^1 = .05 \times \min(5, a) + z^1$$

$$p^2 = .05 \times \min(5.2, a) + z^2$$

with $z^1 + z^2 = .011a.$

We can then specify the Pareto problem as:

$$max_{a,z^1,z^2}\{V^1 + \lambda V^2\}$$

subject to

$$z^1 + z^2 = .011a$$

with λ being the relative weight on the second consumer.

The Pareto-optimal level of assets in this case is independent of the Pareto weight and turns out to be approximately 4.994, with the cost shares depending on the Pareto weights.

Any particular cost allocation method at the optimum can be justified by a particular weighting scheme (i.e., a particular level of λ). For example, equal shares, in which both consumers pay $z^1 = z^2 = \left(\frac{1}{2}\right) \times .011 \times 4.994$, can be justified by setting λ equal to approximately 1.2080.

Lowering λ to approximately 1.2067, on the other hand, would yield a slightly higher cost allocation $\left(z^2 = \left(\frac{5.2}{5.2+5}\right) \times .011 \times 4.994\right)$ to the second consumer---the same result that would be obtained with allocating costs according to the respective contributions to tail value-at-risk with a threshold corresponding to the point of default:

$$E[L^1 + L^2 \mid L^1 + L^2 > a\,]$$

A Lindahlian solution (see Kaneko, 1977), on the other hand, would feature cost shares based on the value placed by each consumer on the marginal unit of company assets. In other words,

$$z^i = -\left(\frac{\frac{\partial V^i}{\partial a}}{\frac{\partial V^i}{\partial z^i}}\right) \times 4.994$$

which in this case works out to

$$z^1 = .0014$$

$$z^2 = .0536$$

Our problem of interest concerns the stability of the allocations, and this problem is clearly illustrated by the latter two approaches---allocation based on the gradient of TVaR, and the Lindahl allocation. Both approaches in our example are keyed to the default threshold (asset level of 4.994). But a small increase in the threshold to, say, 5.001, would yield a discontinuous change in implied allocations under either method. Specifically, the first consumer would no longer receive any capital cost allocation.

Once assets move beyond 5.000, the loss of the first consumer is always paid in full and is thus no longer connected to company default. Thus, only the loss of the second consumer matters when calculating TVaR with a threshold of 5.001 instead of 4.994. Similarly, once assets are at 5.001, only the second consumer benefits from further increases in the asset level, so the Lindahl solution will allocate all cost to the second consumer when the optimal level of capital is 5.001.

This instability in both the TVaR gradient allocation and the Lindahl allocation flows from the fact that neither is cost monotonic. To see this, consider a slight modification to the frictional cost function:

$$\hat{c}(a) = \begin{cases} .011a & 0 \le a \le 4.994 \\ .011 \times 4.994 & 4.994 < a \le 5.001 \\ .011 \times 4.994 + \dfrac{.008}{.001} \times .011 \times (a - 5.001) & 5.001 < a \le 5.002 \\ .011a & a > 5.002 \end{cases}$$

This modification essentially makes additional units of assets free between 4.994 and 5.001, and then catches up to the original cost line by the point of *a*=5.002. Thus, this change represents a strict improvement in the cost function. As might be guessed, the Pareto optimal choice under this change would be to increase assets from 4.994 (the optimum under $c(a)$) to 5.001, as this yields more protection at the same frictional cost.

However, the increase in assets will not necessarily be welcomed by both consumers, depending on the allocation rule being used. Under both the Lindahl and the TVaR gradient allocations (using a default threshold), consumer 2 will become solely responsible for all frictional costs when assets are increased from 4.994 to 5.001. For example, under TVaR, consumer 2's utility drops from:

$$\begin{aligned} -.05 \exp\left\{-\min(5.2\,, 4.994) + \left[.05 + \left(\frac{5.2}{10.2}\right) \times .011\right] \times 4.994\right\} \\ - .95 \exp\left\{-5.2 + \left[.05 + \left(\frac{5.2}{10.2}\right) \times .011\right] \times 4.994\right\} \\ = -.07366 \end{aligned}$$

to

$$\begin{aligned} -.05 \exp[-\min(5.2\,, 5.001) + (.05 \times 5.001 + .011 \times 4.994)] \\ - .95 \exp[-5.2 + (.05 \times 5.001 + .011 \times 4.994)] \\ = -.07566 \end{aligned}$$

This illustrates that an allocation mechanism which assigns cost allocations based on the TVaR gradient at the default threshold fails cost monotonicity: Even though the transition from $c(a)$ to $\hat{c}(a)$ involves a cost improvement, consumer 2 ends up being worse off because of the reallocation of frictional cost. A similar characterization holds for Lindahl mechanism in this case, although the utility loss for consumer 2 is less dramatic since the cost allocation was already skewed heavily in her direction before the cost improvement.

An egalitarian equivalent mechanism, on the other hand, guarantees that both consumers will welcome the improved cost structure when transitioning from $c(a)$ to $\hat{c}(a)$. In this particular example, numerical methods can be applied to show that the egalitarian equivalent level of assets when operating under the original cost structure $c(a)$ is 4.1691. The associated egalitarian equivalent cost allocation features 61.39% of the frictional cost responsibility being allocated to consumer 2. In contrast to the jumps observed with the TVaR and Lindahl mechanisms, the cost allocation changes very little when the cost structure drops to $\hat{c}(a)$---with consumer 2 now having responsibility for 61.44% of the frictional cost. Also in contrast to the results observed under the TVaR and Lindahl mechanisms, consumer 2 is left slightly better off after the cost drop. The egalitarian equivalent level of assets rises to 4.1696, so utility is slightly higher for both consumer 1 and consumer 2 in the new regime.

IV. Conclusion

This paper explored egalitarian equivalence as a capital allocation concept, and argued that it is suitable for situations where the level of capital is variable but the risk portfolio is fixed. In such circumstances, the capital cost allocation problem is isomorphic with the much-studied economic problem of how to share the cost of a public good. The egalitarian equivalent allocation approach has the significant advantage of cost monotonicity, which delivers stability.

However, it must be stressed that this advantage is context-dependent: Egalitarian equivalent allocation methods are not appropriate for pricing applications where the risk portfolio is not fixed. When the problem is one of portfolio optimization, marginal cost pricing dictates the use of allocation methods, such as the Euler method in the case of risk-measure constrained portfolio optimization, even if the method produces unstable allocations. Indeed, the Euler method is likely to yield unstable allocations unless one is willing to select the risk measure specifically for stability properties.

Moulin showed that egalitarian equivalent mechanisms are the only ones that can be guaranteed to be cost monotonic in all situations, but it is possible that other methods might be admissible if further restrictions are added to nature of the cost functions. Additional restrictions might be worth exploring because the egalitarian equivalent mechanism may not be intuitive for everyone. Moulin's terminology was evidently intended to parallel egalitarian equivalence for private good allocations (Pazner and Schmeidler, 1978). There are similarities in process: Egalitarian equivalent cost shares are found by calculating amounts that yield a particular utility distribution, while egalitarian equivalent private good allocations are found by identifying Pareto optimal allocations that match a particular utility distribution. However, the relevance of the reference point in the private case (the utility distribution associated with an economy in which all goods are shared equally) is easily and intuitively grasped, while the relevance of the corresponding reference point for the public case is less obvious. Future research may uncover other cost monotonic mechanisms in the context of more restricted settings.

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APPENDIX

Proof of Theorem 1 (Moulin (1987))

Prove finiteness of a^*:

Risk aversion and non-trivial loss distributions implies that

$$V^i(a,0) > V^i(0,0) \quad \forall\, i, a > 0$$

We take an increasing sequence $\hat{a}_t$ such that

$$\lim_{t\to\infty} \hat{a}_t = a^* \tag{A.1}$$

By way of contradiction, suppose

$$\lim_{t\to\infty} \hat{a}_t = a^* = \infty \tag{A.2}$$

By definition of a^*, we know there is an associated sequence of feasible allocations $(a_t, z_t^1, \dots, z_t^N)$ satisfying:

$$V^i(\hat{a}_t, 0) \le V^i\left(a_t, z_t^i\right) \quad \forall i \tag{A.3}$$

(which exist by definition of a^*). Suppose this sequence a_t is also unbounded. Then we can find a subsequence, denoted $\dot{a}_t$, that converges to infinity.

Denote the upper bounds of the loss distributions by $\bar{L}_i$. An intermediate value argument, which we can apply due to the observation that $V^i(a_t, 0) \ge V^i(0,0) \ge V^i(a_t, \bar{L}_i) \quad \forall i$ and the continuity of the utility functions, imply that for each i there exists a function $z_i(.)$ satisfying:

$$V^i\left(\dot{a}_t, \dot{z}_t^i\right) \ge V^i(\hat{a}_t, 0) \ge V^i(0,0) = V^i\left(\dot{a}_t, z_i(\dot{a}_t)\right) \quad \forall i, t \tag{A.4}$$

Because the loss distributions are bounded, assets have no value beyond a certain point, so $z_i(.)$ is bounded from above, so (A.4) implies that $\dot{z}_t^i$ must be bounded from above as well.

It follows that

$$\limsup_{t\to\infty} \left\{\frac{\dot{z}_t^i}{\dot{a}_t}\right\} \le 0 \ \ \forall i$$

and moreover that

$$\limsup_{t\to\infty} \left\{\frac{\sum \dot{z}_t^i}{\dot{a}_t}\right\} \le 0$$

However, notice that feasibility and the convexity of the cost function implies that:

$$\limsup_{t\to\infty}\left\{\frac{\sum \dot{z}_t^i}{\dot{a}_t}\right\} = \limsup_{t\to\infty}\left\{\frac{c(\dot{a}_t)}{\dot{a}_t}\right\} > 0$$

which contradicts the previous result. Thus the sequence a_t must be bounded, meaning that

$$\lim_{\mathrm{t}\to\infty} a_t = q < \infty$$

By assumption of unboundedness, $\hat{a}_t > q$ for large enough t, so (A.3) then implies that $z_t^i \leq 0$ for all i, which violates feasibility, a contradiction indicating that $a^* < \infty$.

Prove $V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i)\ \forall i \in N$:

First, we prove that a feasible allocation $(a, z^1, \dots, z^N)$ satisfies

$$V^i(a^*, 0) \leq V^i(a, z^i)\ \forall i \qquad \text{(A.5)}$$

We take a bounded increasing sequence $\hat{a}_t$ such that

$$\lim_{\mathrm{t}\to\infty} \hat{a}_t = a^* < \infty$$

associated sequence of feasible allocations $(a_t, z_t^1, \dots, z_t^N)$ satisfying (A.3). We know from the previous step that all elements of this sequence are bounded, so the associated sequence must have a convergent subsequence. Define $\widehat{\Omega} \subset \Omega$ as:

$$\widehat{\Omega} = \left\{(a, z^1, \dots, z^N) \mid Q \times \sum_i \bar{L}^i \geq a \geq 0, \sum_i z^i = c(a), z^i \in [-Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}, Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}]\ \forall i\right\}$$

where Q is any number greater than 1. Notice that $(a_t, z_t^1, \dots, z_t^N) \in \widehat{\Omega}\ \forall t$, since $\hat{a}_t \geq 0$ and any feasible allocation lying outside $\widehat{\Omega}$ would involve a violation of (A.3). Moreover, since $\widehat{\Omega}$ is closed, any convergent subsequence of $(a_t, z_t^1, \dots, z_t^N)$ converges to a limit point of $\widehat{\Omega}$, so (A.5) is satisfied.

Given $a^* \geq 0$, note that egalitarian equivalence implies that $\bar{z}^i \notin \{-Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}, Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}\}$ for all *i*. To see why, consider the case where $\bar{z}^i = Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}$ for some i. Then it follows that:

$$V^i(a^*, 0) \geq V^i(0,0) \geq \mathbb{E}u^i(w^i - \bar{L}^i) > V^i(\bar{a}, \bar{z}^i)$$

which is inconsistent with egalitarian equivalence. Now suppose that for some *i*, $\bar{z}^i = -Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}$. Note that, since $\bar{a} \geq 0$, this implies there must be at least one $j \neq i$ such that $\bar{z}^j > \max\{\bar{L}^1, \dots, \bar{L}^N\}$. But then

$$V^j(a^*, 0) \geq V^j(0,0) \geq \mathbb{E}u^j(w^j - \bar{L}^j) > V^j(\bar{a}, \bar{z}^j)$$

which is inconsistent with egalitarian equivalence. Thus it follows that $\bar{z}^i$ will always lie in the interior of $\{-Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}, Q \times \max\{\bar{L}^1, \dots, \bar{L}^N\}\}$.

Moving on, by way of contradiction, suppose that $V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i)$ is not satisfied $\forall i \in N$. This implies that there exists some nonempty subset of N (which we will denote by M) such that:

$$V^k(a^*, 0) < V^k(\bar{a}, \bar{z}^k) \quad \forall k \in M$$

Note that M cannot be equivalent to N (i.e., $V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i)$ must hold for some i), as this would contradict the egalitarian equivalence of a^* since we could increase a^* by some amount if $V^k(a^*, 0) < V^k(\bar{a}, \bar{z}^k) \quad \forall k \in N$. So we consider a complementary set $L = N/M$ with

$$V^j(a^*, 0) = V^j(\bar{a}, \bar{z}^j) \; \forall j \in L$$

But this is also incompatible with the egalitarian equivalence of a^* since, given that all cost shares are interior to the choice set, we could form a new feasible allocation, $(\tilde{a}, \tilde{z}^1, \dots, \tilde{z}^N)$, where $\tilde{a} = \bar{a}$ and we have subtracted some small amount from each of cost shares of all agents in L and divided the sum total of those deductions among the agents in M so that:

$$V^k(a^*, 0) < V^k(\tilde{a}, \tilde{z}^k) \quad \forall k \in N$$

This contradiction implies that $V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i)$ must be satisfied for all i.

Proof of Theorem 2

Proof: Denote an egalitarian equivalent allocation as $(\hat{a}, \hat{z}^1, \dots, \hat{z}^N)$ and the associated egalitarian equivalent level of public good production as a^*. Suppose it is not Pareto efficient. Then there exists a feasible alternative allocation $(\bar{a}, \bar{z}^1, \dots, \bar{z}^N)$ satisfying:

$$V^i(a^*, 0) = V^i(\hat{a}, \hat{z}^i) \leq V^i(\bar{a}, \bar{z}^i) \; \forall i \in N$$

with strict equality for at least one of the *i*'s. Let *k* index one of the agents for whom the inequality is strict. Then there exists some $\varepsilon > 0$ such that

$$V^k(\hat{a}, \hat{z}^k) < V^k(\bar{a}, \bar{z}^k + \varepsilon) < V^k(\bar{a}, \bar{z}^k)$$

Let

$$\tilde{z}^i = \bar{z}^i - \frac{\varepsilon}{N-1} \quad \forall i \neq k$$

$$\tilde{z}^k = \bar{z}^k + \varepsilon$$

Note that

$$(\bar{a}, \tilde{z}^1, \dots, \tilde{z}^N) \in \Omega$$

But since utility is strictly decreasing in the second argument

$$V^i(a^*, 0) = V^i(\hat{a}, \hat{z}^i) < V^i(\bar{a}, \tilde{z}^i) \;\; \forall i \in N$$

which is inconsistent with a^* being the egalitarian equivalent level of public good production.

Proof of Theorem 3

Suppose not. Then there exist two cost functions $c_1(.)$ and $c_2(.)$, with $c_1 \leq c_2$, but where the associated egalitarian equivalent levels of assets, a_1^* and a_2^*, satisfy $a_1^* < a_2^*$. Let $(a_2, z_2^1, \dots, z_2^N)$ be an egalitarian equivalent allocation assigned by the mechanism under $c_2(.)$ and $(a_1, z_1^1, \dots, z_1^N)$ an egalitarian equivalent allocation assigned under $c_1(.)$. Note that:

$$\left(a_2, z_2^1 - \frac{c_2(a_2) - c_1(a_2)}{N}, \dots, z_2^N - \frac{c_2(a_2) - c_1(a_2)}{N}\right)$$

is a feasible allocation under $c_1(.)$. Egalitarian equivalence, together with $c_1 \leq c_2$, implies that

$$V^i(a_2^*, 0) = V^i(a_2, z_2^i) \leq V^i\left(a_2, z_2^i - \frac{c_2(a_2) - c_1(a_2)}{N}\right) \;\; \forall i \in N$$

But egalitarian equivalence, together with $a_1^* < a_2^*$, implies that

$$V^i(a_2^*, 0) > V^i(a_1^*, 0) = V^i(a_1, z_1^i) \;\; \forall i \in N$$

Putting these together yields

$$V^i(a_1, z_1^i) = V^i(a_1^*, 0) < V^i\left(a_2, z_2^i - \frac{c_2(a_2) - c_1(a_2)}{N}\right) \;\; \forall i \in N$$

which contradicts the supposition that a_1^* is egalitarian equivalent under $c_1(.)$.